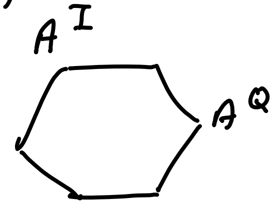
Lecture 2

29/8/06

Summary: for $N_e = 1, 2$

$$F_{a_1 a_2 a_0}(x) = A_{12}\, e^{i(k_1 x_1 + k_2 x_2)} \sum_{Q \in 3!} A^{Q}_{a_1 a_2 a_3}\, \Theta(x_Q)$$



standard ordering: 120

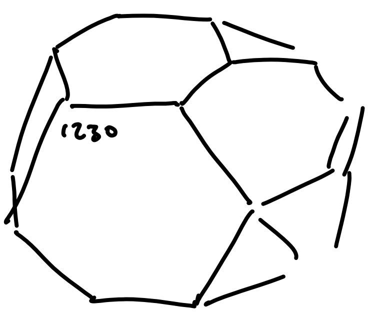
$$A^I \rightsquigarrow A^Q = (\Pi S) A^I$$

YB : $$S^{12} S^{10} S^{20} = S^{20} S^{10} S^{12}$$

$$S^{i0} = \frac{1 - icP^{i0}}{1 - ic} \quad , \quad S^{ij} = P^{ij}$$

$A^I_{a_1, \dots a_{N_e}, a_0}$ is still a challenge, but the other $(N_e+1)!$ are known.

$N_e = 3$: $N = N_e + 1 = 4$, $4! = 24$ regions:



$$F_{a_1 a_2 a_3 a_0} = e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} \sum_{Q \in 4!} A^Q \Theta(x_Q)$$

New restrictions? No!

$$\left.\begin{aligned} S^{ij} S^{ik} S^{jk} &= S^{jk} S^{ik} S^{ij} \\ S^{ij} &= (S^{ji})^{-1} \end{aligned}\right\} \text{suffice}$$

(Theorem by Brezin, Zinn-Justin)

$S^{ij} S^{ji} = 1$ ✓

$P^{ij} P^{ji} = 1$ ✓

$S^{i0} S^{i0\dagger} = 1$? $\dfrac{1 - icP^{i0}}{1 - ic} \dfrac{1 + icP}{1 - ic} = 1 \Rightarrow S$ is unitary. ✓

Unitarity: (explanation follows below)

$$S^{j0} = \frac{\overset{(1-0)}{1} - icP^{j0}}{1 - ic}$$

$$S^{0j} = \frac{(0-1) - icP^{0j}}{(0-1) - ic} = \frac{1 + icP^{0j}}{1+ic}$$

Kondo is special case of more general class of models (Gross-Neveau, Thirring).

$$-i\int\left[\psi^\dagger_{Ra}\partial\psi_{Ra} - \psi^\dagger_{La}\partial\psi_{La} + J\left(\psi^\dagger_{Ra}\sigma_{ab}\psi_{Rb}\right)\left(\psi^\dagger_{La'}\sigma_{a'b'}\psi_{Lb'}\right)\right]$$

Exactly solvable! $S^{ij} = \dfrac{(\alpha_i - \alpha_j) + icP^{ij}}{(\alpha_i - \alpha_j) + ic}$, with $\alpha_i = \pm$ for R/L

$\uparrow$ velocities $\pm v_F = \pm 1$

For impurity in Kondo model: $\alpha_i = 0$

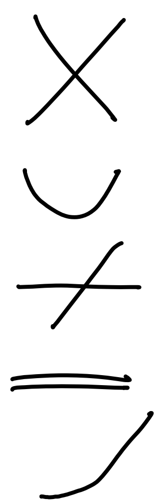
For Hubbard model: $H = -t\sum_n \left(c^\dagger_{na} c_{n+1a} + h.c.\right) + U\sum_n \hat{n}_{n\downarrow}\hat{n}_{n\uparrow}$

$$= \sum_k \cos k \; c^\dagger_{ka} c_{ka}$$

solved by Lieb, Liniger in 1967.

Here, $\alpha_j = \sin(k_j)$ for energy

Summary:
$$\alpha_j = \begin{cases} \pm \text{ for } R/L & \text{Gross-Neveau} \\ \sin k_j & \text{Hubbard} \\ 1,\ 0 & \text{Kondo} \\ 0 & \text{Heisenberg} \\ k_j & \text{Yang} \end{cases}$$



Yang solved: $h = -\sum_j \partial_j^2 + \sum_j \delta(x_i - x_j)$

Theorem by Brezin, Zinn-Justin: If 2-particle S-matrix satisfies,

$$S^{ij} S^{ik} S^{jk} = S^{jk} S^{ik} S^{ij}$$

$$S^{ij} = (S^{ji})^{-1},$$

Special feature of integrable models:

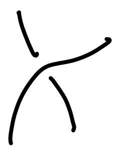
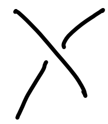
[$\{k_i\}$ is the same set in any region

A^{I} suffices to construct A^{Q}'s.

$E = \sum_j k_j$]

then the following Ansatz is consistent wave-function:

$$F_{a_1, \dots a_{N_e}, a_0}(x_1, \dots x_{N_e}) = A\, e^{i \sum_j x_j k_j} \sum_{Q \in N!} A^{Q}_{a_1 \dots a_{N_e}, a_0}\, \theta(x_Q)$$

$P =$  , S^{ij}  , S^{ji}  Group in knot theory

Periodic Boundary Conditions & Spectrum

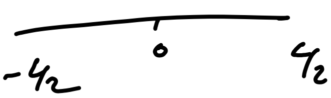
Finding k's, A's What are boundary conditions?

Given $h = -i\partial_x$, $F(x) = e^{ikx}$,

for periodic b.c.: $F(0) = F(L) \Rightarrow 1 = e^{ikL}$, $k = \frac{2\pi n}{L}$.

In thermodynamic limit, b.c. does not matter, any choice is OK.

$$F_{aa_0}(x) = e^{ikx}\left[A\, \theta(-x) + S A\, \theta(x)\right]$$



$F_{aa_0}(-L/2) = F_{aa_0}(L/2)$: $\quad e^{-ikL/2} A = S A\, e^{ikL/2}$

$$\Rightarrow S A = A\, e^{-ikL}$$

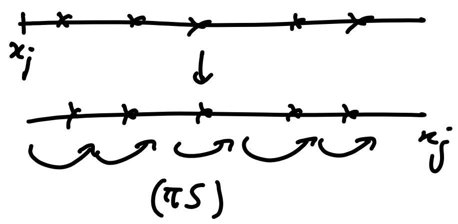
[Aside: corrections to thermodynamic limit $E = E_0 + \frac{E'}{L}$ ← interesting information]

P.B.C. for N_e particles:

$$F(x_1, \dots x_j = -\tfrac{L}{2}, \dots x_{N_e}) = F(x_1, \dots, x_j = \tfrac{L}{2}, \dots x_{N_e})$$

$$A_Q \, e^{-ik_j L/2}$$

$$A_{\tilde{Q}} \, e^{ik_j L/2}$$



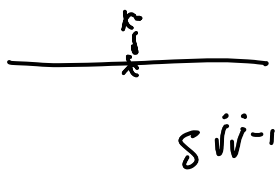
But $\quad A_{\tilde{Q}} \equiv \Pi S A^Q = A^Q e^{-ik_j L}$

More explicitly:

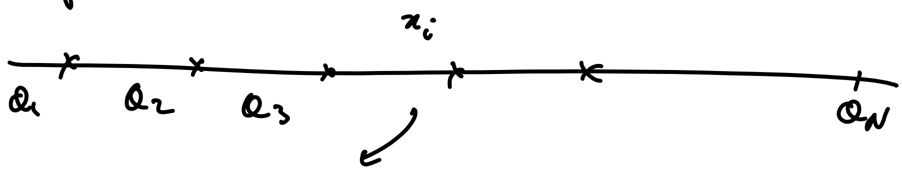
$$\boxed{(Z_j)^{b_1 \dots b_N}_{a_1 \dots a_N} A_{b_1 \dots b_N} = e^{-ik_j L} A_{a_1 \dots a_N}} \quad \circledast$$

$a_N \equiv a_{N_e} a_0$

$$Z_j \equiv S^{jj-1} \quad S^{j1} S^{jN} \dots S^{jj+1}$$



Q



$$S^{1j} \qquad S^{j-2j} S^{j-1j} A \, e^{-i\frac{L}{2}k_j} = S^{jN} \dots S^{jj+2} S^{jj+1} \; e^{i\frac{L}{2}k_j}$$

with $\quad [Z_j, Z_k] = 0.$

$\circledast$ is a eigenvalue problem for $2^N \times 2^N$ matrix.

This was solved by C.N. Yang, 1967. $\left[H = \int \psi^\dagger_A \partial^2 \psi_A + J (\psi^\dagger_A \psi_A)^2 \right].$

Similar structure emerged in work of Baxter, 1970: XYZ-Hamiltonian, 8-vertex model.

St. Petersburg School (Fadeev, Sklyanin): Inverse Scattering Problem.

Quantum Inverse Scattering Problem : (Can we find Potential from Phase Shifts?)

diagonalize: $\quad Z_j A = e^{-ik_j L} A \quad$ (details: in Andrei's notes, lecture 2).

We started from $\qquad H|0\rangle = 0 \qquad |0\rangle = \text{vacuum}.$

$N_e = 1 \qquad \int dx \; F(x) \, \psi^\dagger_a(x) |0\rangle \;, \quad F(x) = e^{ikx}$

$N_e = 2 \qquad |k_1 k_2\rangle, \quad S\text{-matrix}$

Yang did the same:

$$|F\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$$

$$|1\rangle = |\underset{n=1}{\uparrow} \ldots \underset{j}{\downarrow} \ldots \underset{N}{\uparrow}\rangle$$

$$e^{in\Lambda_j} \quad , \quad \Lambda_j = \text{spin momentum (rapidity)}$$

$$|\uparrow \ldots \overset{n_1}{\underset{\Lambda_1}{\downarrow}} \ldots \uparrow\uparrow \ldots \overset{n_2}{\underset{\Lambda_2}{\downarrow}} \ldots \uparrow\rangle$$

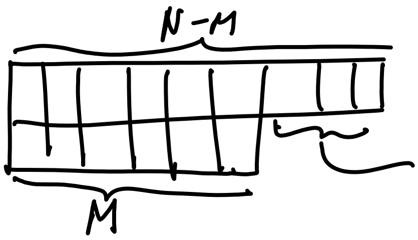
Review of Young Tableaux:

$\uparrow\ \uparrow$
$\downarrow\ \downarrow$

symmetric product of spins

$\uparrow$
$\downarrow$

antisymmetric.



for states: $\uparrow\uparrow\uparrow$
$\uparrow\uparrow\downarrow$
$\downarrow\downarrow\uparrow$
$\downarrow\downarrow\downarrow$.

, symmetrized.

Yang found eq. for how Λ's are connected:

spin wavefunction: $$g^{\Lambda_1 \ldots \Lambda_M}_{j_1 \ldots j_M} = ?$$

for $N - M \uparrow$
$M \downarrow$

k's satisfy:
$$e^{ik_j L} = \prod_{\gamma=1}^{M} \frac{\Lambda_\gamma - 1 + ic/2}{\Lambda_\gamma - 1 - ic/2} \qquad (1)$$

$$-\prod_{\delta=1}^{M}\frac{\Lambda_\gamma-\Lambda_\delta+ic}{\Lambda_\gamma-\Lambda_\delta-ic} = \left(\frac{\Lambda_\gamma-1+ic/2}{\Lambda_\gamma-1-ic/2}\right)^{N_e}\left(\frac{\Lambda_\gamma+ic/2}{\Lambda_\gamma-ic/2}\right)^{1} \quad (2)$$

would depend on k_j in Hubbard, then equations for Λ_δ, k_j couple.

(hence spin and charge are decoupled in Kondo, coupled in Hubbard.)

(2) gives $\Lambda_1 \dots \Lambda_M$

(1) gives k_j $\Rightarrow$ $E = \sum_j k_j$

[Which M gives lowest energy? Dynamic question.]

Dimension-counting: $\hbar = 1$, $c = 1$: $h = -i\int \psi^\dagger \partial \psi \, dx + J \psi^\dagger \vec{\sigma} \psi \cdot \vec{S}$

dimensions: $x = -1$, $\partial = 1$, $E = 1$, $\psi = 1/2$, $\boxed{J = 1}$

$$S = \frac{1-icP}{1-ic} \Rightarrow \text{momentum does not appear in S-matrix!}$$

[typically: $\underset{2}{-\partial^2} + \underset{1}{g}\,\underset{1}{\delta(x)} \Rightarrow S = e^{i\delta(k/g)} \Rightarrow k$ appears!! in S-matrix!]

log (1): $$k_j = \frac{-i}{L}\sum_{\delta=1}^{M} \underbrace{\ln\left(\frac{\Lambda_\gamma-1+ic/2}{\Lambda_\gamma-1-ic/2}\right)}_{\Theta(2\Lambda_\gamma-2)} + \frac{2\pi n}{L},$$

where $\Theta(x) = -2\tan^{-1} x/c$, $\tan^{-1}x = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right)$.

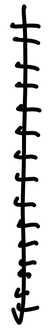
$$E = \sum_j k_j = \frac{N_e}{L}\sum \Theta(2\Lambda_\gamma-2) + \frac{2\pi}{L}\sum_{j=1}^{N_e} n_j$$

log(2): $$\sum_{\delta=1}^{M}\Theta(\Lambda_\gamma-\Lambda_\delta) = N^e\,\Theta(2\Lambda_\gamma-2) + \Theta(2\Lambda_\gamma) + 2\pi \overset{\text{integer.}}{I_\gamma} \quad (2')$$

we need to solve (2') for Λ_γ.

Quantum numbers: $\{n_j, I_\gamma\}$ infinite set.

spin QN, describes spin dynamics of $\uparrow \dots \underset{\Lambda_\delta}{\downarrow} \dots \uparrow$

Bethe basis:  $(\dots\, I_\gamma \,\dots)$ vs. conventional Fock 

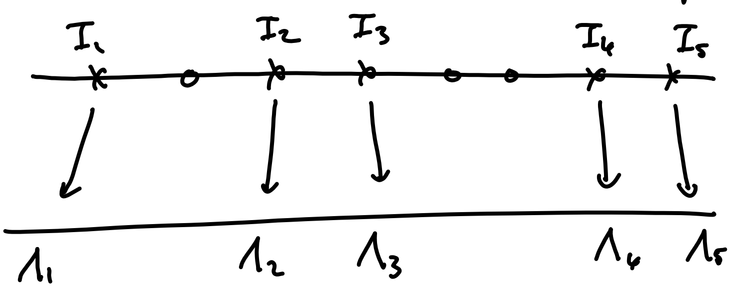
Pert. theory, level shift.

n_j

Same Hilbert space is built differently!

$|GS\rangle$: what $\{n_j^0, I_\gamma^0\}$ give lowest energy?

(later: Thermodynamics: $Z = \sum_{\{n_j, I_\gamma\}} e^{-\beta \sum_j k_j}$)



$\{I_\gamma\}$

$$\sum_{\delta=1}^{M} \overset{M\pi}{\Theta(\Lambda_\gamma-\Lambda_\delta)} = \overset{N^e\pi}{N^e\,\Theta(2\Lambda_\gamma-2)} + \overset{\pi}{\Theta(2\Lambda_\gamma)} + \overset{2\pi I^\infty \text{ integer.}}{2\pi I_\gamma}$$

↳ send $\Lambda_\gamma \to \infty$; which I_γ would do this?

Maximal $I_{max} = \frac{N-M-1}{2}$

$I_{min} = -\frac{N-M-1}{2}$

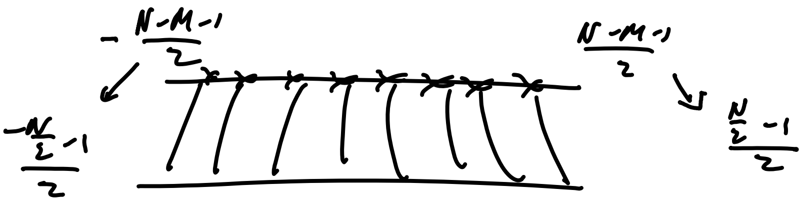
for I_δ outside this range, there are no solutions.

If we remove impurity energy should be ε_F. So, remove

$$E = \sum_j k_j = \frac{N^e}{L} \underbrace{\sum_\gamma \Theta(2\Lambda_\gamma-2)}_{\frac{1}{L}\sum_\gamma 2\pi I_\gamma} + \frac{2\pi}{L}\sum_{j=1}^{N_e} n_j = \text{Free}$$

$= \text{energy of } \vec{J}_s\cdot\vec{J}_s + J_c J_c$

$|GS\rangle$:



$M = N/2$ we'll see this is $|GS\rangle$.

we'll show any choice raises energy.

$\#$ of slots : $\left(\frac{N}{2}-1\right)+1 = \frac{N}{2}$

$\#$ of QH $= M = N/2$

$\Rightarrow$ all slots are filled.

$$I^0_{\gamma+1} = I^0_\gamma + 1$$

chosen I^0_γ, what are Λ^0_γ ?

difficult in general, but doable in thermodynamic limit:

density of solutions: $\sigma(\Lambda_\delta) = \frac{1}{\Lambda_{\gamma+1}-\Lambda_\gamma}$ $\leftarrow$ ground state density

$$E = \frac{N^e}{L}\sum_\gamma \Theta(2\Lambda_\gamma - 2) = \frac{N_e}{L}\int d\Lambda\, \sigma^0(\Lambda)\,\Theta(2\Lambda - 2)$$

$\gamma+1$: $$\sum_{\delta=1}^{M}\Theta(\Lambda_{\gamma+1}-\Lambda_\delta) = N^e\,\Theta(2\Lambda_{\gamma+1}-2) + \Theta(2\Lambda_{\gamma+1}) + 2\pi I_{\gamma+1} \qquad (*)$$

for γ $$\sum_{\delta=1}^{M}\Theta(\Lambda_\gamma-\Lambda_\delta) = N^e\,\Theta(2\Lambda_\gamma-2) + \Theta(2\Lambda_\gamma) + 2\pi I_\gamma \qquad (**)$$

$(*) - (**)$. Assume $\Lambda_{\gamma+1}-\Lambda_\gamma \sim \frac{1}{N}$ (spectrum becomes dense).

$$\cancel{(\Lambda_{\gamma+1}-\Lambda_\gamma)}\sum_{\delta=1}^{M}\Theta'(\Lambda_\gamma-\Lambda_\delta) = N^e\cancel{(\Lambda_{\gamma+1}-\Lambda_\gamma)}\,\Theta'(2\Lambda_\gamma-2)$$

$$+\cancel{(\Lambda_{\gamma+1}-\Lambda_\gamma)}\,\Theta'(2\Lambda_\gamma) + \frac{2\pi\, 1.}{\Lambda_{\gamma+1}-\Lambda_\gamma} = 2\pi\sigma(\Lambda_\gamma) \qquad (*.)$$

$$\theta(x) = -2\tan^{-1} x/c$$

$$\theta'(x) = \frac{1}{2c}\,\frac{1}{1+(x/c)^2} = \frac{1}{2}\,\frac{c}{c^2+x^2} \equiv K(x)$$

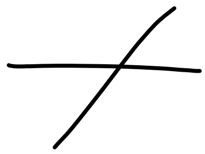
(*) $N \to \infty$, $M \to \infty$

$$\boxed{-\int_{-\infty}^{\infty} d\Lambda'\; K(\Lambda-\Lambda')\,\sigma(\Lambda) = \underbrace{N^e K(2\Lambda-2) + K(2\Lambda)}_{f(\Lambda)} + 2\pi\,\sigma(\Lambda)}$$

Solve by Fourier transform: $\tilde{K}(p)\,\tilde{\sigma}(p) = \tilde{f}(p) + \tilde{\sigma}(p)$ $\qquad \tilde{K} \to \frac{\tilde{K}}{2\pi}$, $\tilde{f} \to \frac{\tilde{f}}{2\pi}$

$$\tilde{\sigma}(p) = \frac{\tilde{f}(p)}{1+\tilde{K}(p)} \Longrightarrow \sigma(\Lambda) = \frac{N_e}{\operatorname{ch}\frac{\pi}{c}(\Lambda-1)} + \frac{1}{\operatorname{ch}\frac{\pi}{c}\Lambda}$$

$$E_0 = \underbrace{\int d\Lambda\; \sigma_0(\Lambda)\,\theta(2\Lambda-2)}_{\text{some constants}} + \sum_{j=1}^{N_e} \frac{2\pi}{L} n_j$$



no two n_j's can be equal, but can be negative

Cut-off: $\quad -N_0 \leq n_j$

$$-\frac{N_e}{L} = -D \leq \frac{2\pi}{L} n_j \;,$$