Implications of local integrability

- Ergodicity breaking. $\langle \Psi_0 | \tau_i^z | \Psi_0 \rangle = \langle \tau_i^z \rangle = \text{const}$.

  Memory of initial state retained, even locally.

- Dynamics is non-trivial! For generic initial conditions, information spreads, $\Rightarrow$ slow growth of entanglement.

  $\Psi_0 = \uparrow\downarrow\uparrow\downarrow\uparrow\ldots\downarrow$ (in the physical spin basis)

Since $\uparrow \neq \Uparrow$, generally Ψ_0 will be a superposition of extensively many eigenstates

So we can consider an initial state in the eff. spin basis:

$$|\alpha_1 \Uparrow + \beta_1 \Downarrow\rangle \otimes |\alpha_2 \underset{\substack{\uparrow \\ \text{eff. spins}}}{\Uparrow} + \beta_2 \Downarrow\rangle \otimes \ldots$$

Let us consider first just two eff. spins.

$$\Psi_0 = \frac{\Uparrow_1 + \Downarrow_1}{\sqrt{2}} \otimes \frac{\Uparrow_2 + \Downarrow_2}{\sqrt{2}}$$

$$H_0 = h_1 \tau_1^z + h_2 \tau_2^z + J_{12} \tau_1^z \tau_2^z.$$

$$\Psi_0(t) = \frac{1}{2}\Big[\Uparrow_1 \Uparrow_2 \cdot e^{i(h_1+h_2+J_{12})t} + \Uparrow_1 \Downarrow_2 \cdot e^{i(h_1-h_2-J_{12})t} + \Downarrow_1 \Uparrow_2 \cdot e^{i(h_1+h_2-J_{12})t} + \Downarrow_1 \Downarrow_2 \, e^{-i(h_1+h_2)t + iJ_{12}t} \Big]$$

Let us compute reduced density matrix of the first spin:

$$\rho_1(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4}\left(e^{2ih_1 t + 2iJ_{12}t} + e^{2ih_2 t - 2iJ_{12}t}\right) \\ \text{c.c.} & \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} e^{2ih_1 t} \cdot \cos 2J_{12}t \\ \frac{1}{2} e^{-2ih_1 t} \cos 2J_{12}t & \frac{1}{2} \end{bmatrix}$$

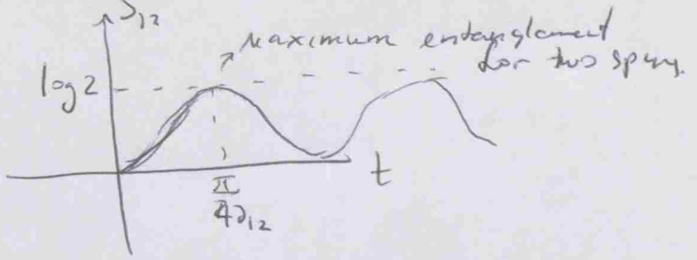
$\left(\frac{1}{2}-\lambda_{1,2}\right)^2 - \frac{1}{4}\cos^2 2J_{12}t = 0 \qquad \frac{1}{2} - \lambda_{12} = \pm\frac{1}{2}\cos 2J_{12}t$

$\boxed{\cancel{\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{2}\cos 2J_{12}t}} \qquad \lambda_{1,2} = \frac{1}{2} \pm \frac{1}{2}\cos 2J_{12}t$

If $J_{1,2}=0 \Rightarrow \lambda_1=1,\ \lambda_2=0$ (pure state, no entanglement)

But $J_{12}\neq 0 \Rightarrow \Uparrow_1 \Updownarrow_2$ get entangled!

$S = -\sum_{i=1,2}\lambda_i \log\lambda_i$;



Dephasing b/w remote effective spins $\Rightarrow$ entanglement.

Oscillations are the artefact of considering just 2 spins (just one interaction parameter)

~~If we con~~

Many-spin problem. Enough to consider

$|\Psi_0\rangle = |\alpha_1\Uparrow_1 + \beta_1\Downarrow_1\rangle \otimes |\alpha_2\Uparrow_2 + \beta_2\Downarrow_2\rangle \ldots \qquad |\alpha_i|^2 + |\beta_i|^2 = 1$

(More generally, $\Psi_0 = U|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$, conclusions the same)

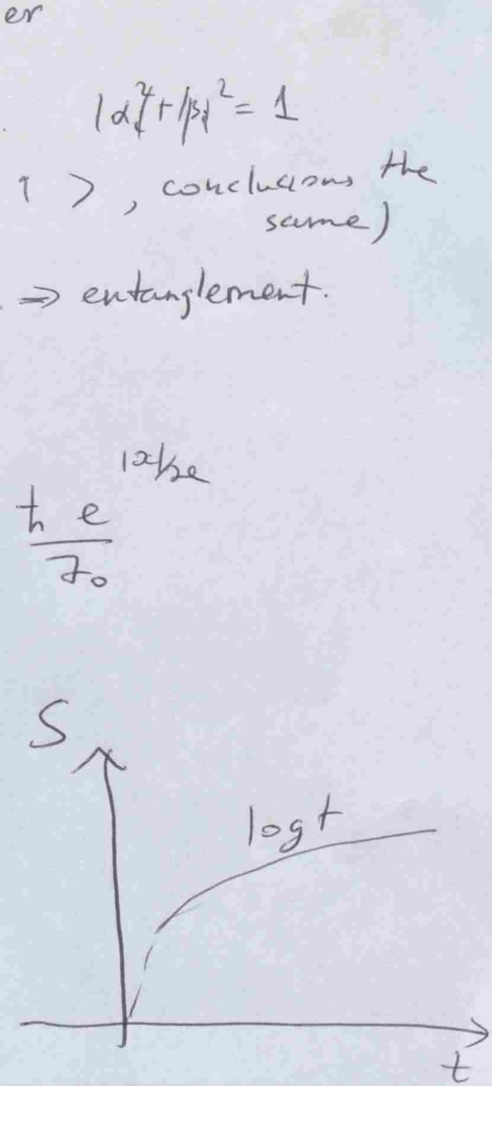
Interactions generate random phases $\Rightarrow$ entanglement.

Since $J_{ij\ldots j+y} \sim J_0 e^{-|x|/\varkappa}$,

$$t_{deph}(x) \sim \frac{\hbar}{J_0 e^{-|x|/\varkappa}} \sim \frac{\hbar e^{|x|/\varkappa}}{J_0}$$

Random phases, entanglement $\boxed{|x(t)| \sim \varkappa \cdot \log\frac{J_0 t}{\hbar}}$

$$S_{ent} \propto |x(t)| \sim \varkappa \log\frac{J_0 t}{\hbar}$$



Log.-growth of entanglement.
Characteristic of MBL phase

This dephasing can be observed experimentally.

Local observables, (quantum quench), spin echo

(quantum infor. is recovered by spin echo since dynamics is due to dephasing)

In general, various observables can be easily computed using this approach $\langle \tau_x(t) \rangle$, $\langle \tau_x(t)\, \tau_x(t) \rangle \ldots$

Observables diagonal in $\tau_i^z \tau_j^z$ remain unchanged

Off-diagonal observables decay as a power-law:

$$|\langle \tau_i^x(t) \rangle| \sim \frac{1}{t^{\alpha}}$$

α is related to scale $\underline{\varkappa}$

(Note $\varkappa$ is not ~~the~~ loc. length – does not diverge at the transition)

Important that initial state is a superposition of extensively many eigenstates.

(If $|\psi_0\rangle$ is an eigenstate, there is no dephasing)

But generic initial states are of the kind we consider.

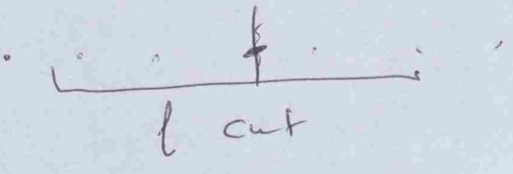
(impossible to prepare an eigenstate – 2^N of them) would take $t \sim \frac{1}{\delta E} \sim 2^N$)

<u>Area-law of eigenstates.</u>

$S_{ent} \sim$ Const for almost all eigenstates

No long-range entanglement

(There are rare states with large entanglement – – resonances. Prob. to have $S_{ent} \sim \ell$:



$$p(\ell) \sim e^{-c\ell}\,)$$

MBL eigenstates are like ground states!
(Instead of gap, localization makes remote degrees of freedom, essentially, non-entangled.)

They can be well-approximated by MPS.

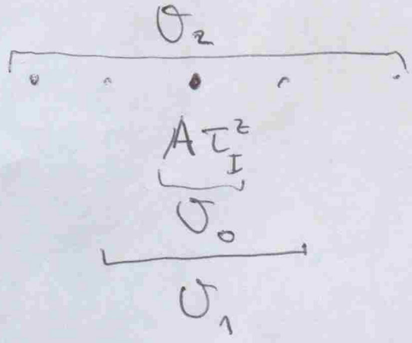
DMRG for excited states (much progress recently)

Matrix elements in the MBL phase

$\hat{O} = \sigma^z_I$. Express $\hat{O}$ via $\hat{\tau}$:

$$\hat{O} = \sigma^z_I = \underbrace{U^+ \tau^z_I U}_{\text{quasilocal}} = \sum_{k=1}^{\infty} \tilde{O}_k$$

range k; acts on spins distance k away



τ-spins

$$O_1 = C^z_I \tau^z_I$$

$$O_2 = C^{zz}_{I\,I+1} \tau^z_I \tau^z_{I+1} + C^{zz}_{I+1\,I,} \tau^z_I \tau^z_{I+1} +$$

$$+ C^{xx}_{I,I+1} (\tau^x_I \tau^x_{I+1} + \tau^y_I \tau^y_{I+1}) + \dots$$

$$\sum_k \sum_{\alpha} |C^{\alpha_{I-k} \dots \alpha_{I+k}}_{I, I+1 \dots I+k}|^2 = 1 \quad \text{(normalization)}$$

Diagonal matrix elements:

Keep diagonal part of the operator:

$$\bar{O} = C^z_I \hat{\tau}^z_I + C^{zz}_{I\,I+1} \hat{\tau}^z_I \hat{\tau}^z_{I+1} + \dots$$

$$\langle \{\tau\} | \hat{O} | \{\tau\} \rangle = C^z_I \tau^z_I + \dots$$

Not a smooth function of energy! Because we can find two nearby states

↑ ↓ ↑ ↓ ↑ ↓ ↑
↓ ↑ ↓ ↑ ↓ ↑ ↓

with different τ^z_I, and therefore different $\bar{O}$!

Off-diagonal matrix elements

$$\hat{O} = \sum_{k} \sum_{\{\alpha\}} C^{\alpha_{-k} \ldots \alpha_k}_{I-k \ldots I+k} \cdot \hat{\tau}^{\alpha_{-k}}_{I-k} \ldots \hat{\tau}^{\alpha_k}_{I+k}$$

Off-diagonal matrix elements correspond to terms which have at least one $\alpha = x,y$ (such that an eff. spin can be flipped).

Typical two states $\{\tau\}$ $\{\tau'\}$ differ everywhere in the system.

$\{\tau\}$ $\Uparrow \Downarrow \Uparrow \Downarrow \Uparrow$

$\{\tau'\}$ $\Downarrow \Uparrow \Downarrow \Uparrow \Downarrow$

$\langle \{\tau'\} | \hat{O} | \{\tau\} \rangle$ is determined by coefficient of the term

C_{I2}

$\tau^{\alpha_1}_{i_1} \ldots \tau^{\alpha_l}_{i_l}$, $\{i_k\}$ - ~~site~~ LIOMs which differ in two states

Such a coeff. decays exponentially with system size

$$\langle \{\tau'\} | \hat{O} | \{\tau\} \rangle_{typ} \sim e^{-L/\xi}$$

Compare with ETH: $\langle m | \hat{O} | n \rangle \sim \frac{1}{\sqrt{D}} \sim \frac{1}{e^{S L/2}}$

In MBL, $\langle \{\tau'\} | \hat{O} | \{\tau\} \rangle_{typ} \sim e^{-L/\xi} \ll e^{-SL/2}$.

Matrix elements decay much faster than in ETH phase

Also, their distribution is broad (log-normal)

Constructing LIOMs:

Let us construct a single IOMs (with eigenvalues different from $\tilde{\tau}$)
One simple way: take a local operator $\hat{O}$ and take infinite-time average:

$$\bar{O} = \lim_{T\to\infty} \frac{1}{T}\int_0^T O(t)\,dt = C_1^z \tau_1^z + C_{I,I+1}^{z}\, \tau_I^z \tau_{I+1}^z + \dots$$

Quasi-local I.OM but not a pseudospin operator!

MBL-ETH transition.

$(\downarrow \uparrow)\ \underbrace{\downarrow\ \uparrow\ \downarrow\ \uparrow}_{\text{perturbative}}\ (\downarrow \uparrow)$ strong disorder

rare, disjoint resonant pairs

$(\downarrow \uparrow)\ \downarrow\ \uparrow\ (\downarrow \uparrow)$ weaker disorder

resonant spots are more dense; they start to hybridise and eigenstate have more entanglement.

$(\uparrow \downarrow)(\uparrow \downarrow)(\downarrow \uparrow)$ weak disorder: A lot of entanglement
ETH.

W_c - critical value of disorder where transition occurs.
$W_c \sim 7$. Thus, MBL is very stable.
(numerically)
Philosophically, LIOMs are reminiscent of KAM result.
But they are much more stable!

Bethe ansatz → $W=0$ — ETH — $\left(\frac{W}{J}\right)_c$ — MBL → W/J

ETH	MBL
ETH holds	breaks down
Eigenstates volume-law $S_{ent} \propto L$	area-law $S_{ent} \sim$ const as $L \to \infty$
In a quench: $S(t) \sim L^a$, $a \le 1$	$S(t) \sim \log t$
Thermalized stedy state	equilibrates, but memory retained!
	Dephasing, can be detected by studying spin-echo, local observables.
Conductivity	No conductivity

→ eigenstates phase transition of a new kind

Other development. - "DMRG" for excited MBL states
- MBL-protected quantum order, role of symmetries

Experiments with cold atoms (quench experiments) in 1d, 2d

Periodically driven systems.

$$H(t+T) = H(t) \quad \text{(e.g. E-M. field)}$$

$$F = T\exp - i\int_0^T H(t)\,dt \quad \text{- Floquet operator}$$

$$F|\psi_\alpha\rangle = e^{-i\theta_\alpha}|\psi_\alpha\rangle \quad \text{- Floquet Eigenstates}$$

θ_α ↑ quasi-energies. (Analogy with quasimom. entum)

$$\psi_\alpha(t) = e^{-i\theta_\alpha t}\cdot|\varphi_\alpha(t)\rangle$$

$|\varphi_\alpha(t)\rangle$ ← periodic in time

In <u>isolated</u> Floquet systems, Floquet eigenstates determine long-time behavior of observables.

Effective Hamiltonian:

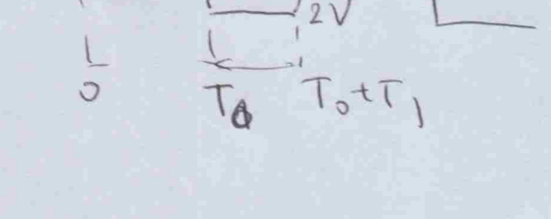
$$\hat{F} = \exp[-iH_{eff}T]$$

H_{eff} ↑ Floquet Hamiltonian

H_{eff} exists and not unique. But what are its properties? How to construct?

In many-body system, driving leads to <u>heating.</u>

Example: kick protocol.



$$F = e^{-iH_0T_0}\,e^{-2iVT_1}$$