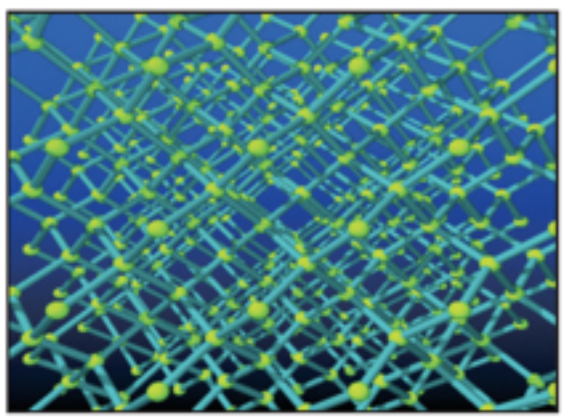
Topological insulators and Berry phases

Munich, 2015



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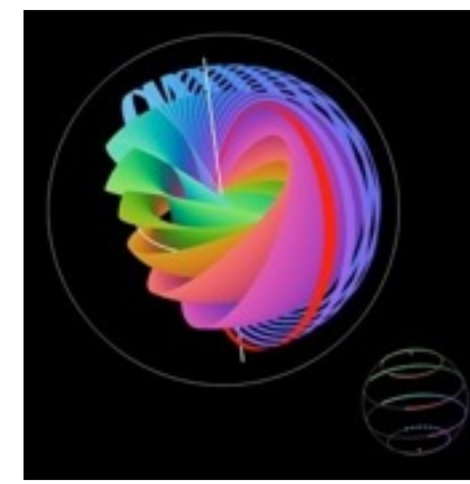

Thanks

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"An insulator's metallic side"
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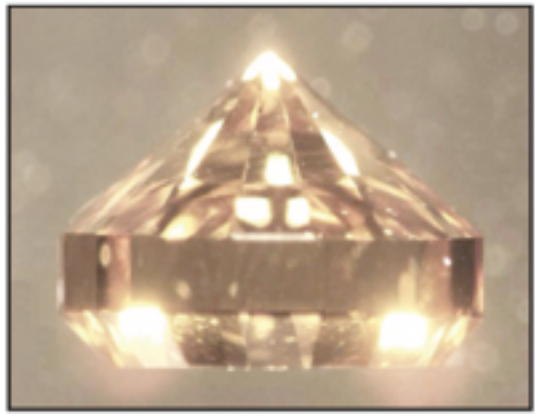
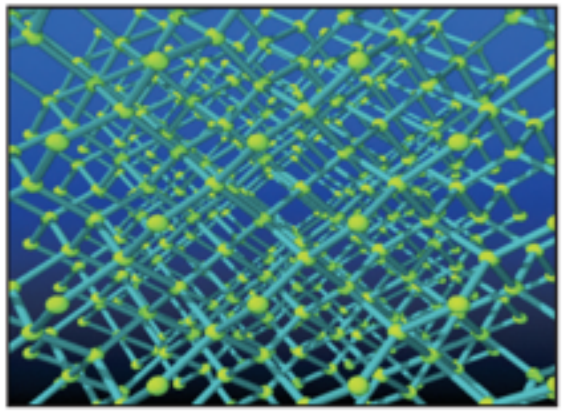
"Quasiparticles do the twist"

Types of order

Much of condensed matter is about how different kinds of order emerge from interactions between many simple constituents.





Until 1980, all ordered phases could be understood as "symmetry breaking":

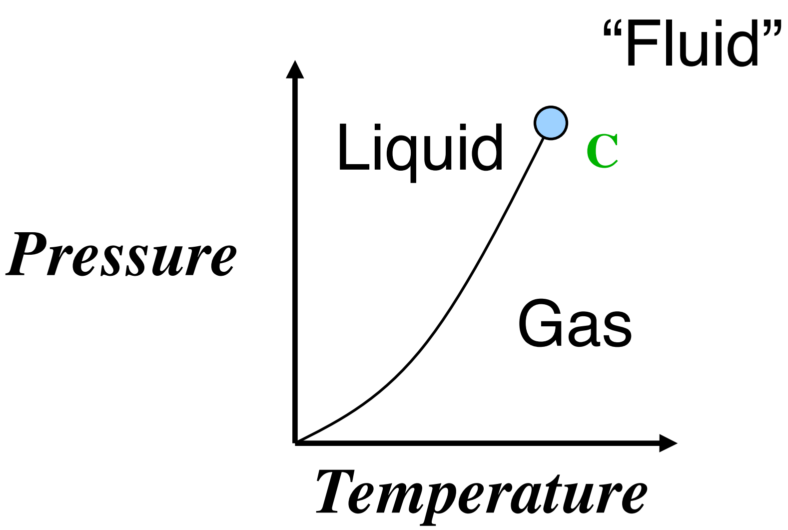
an ordered state appears at low temperature when the system spontaneously loses one of the symmetries present at high temperature.

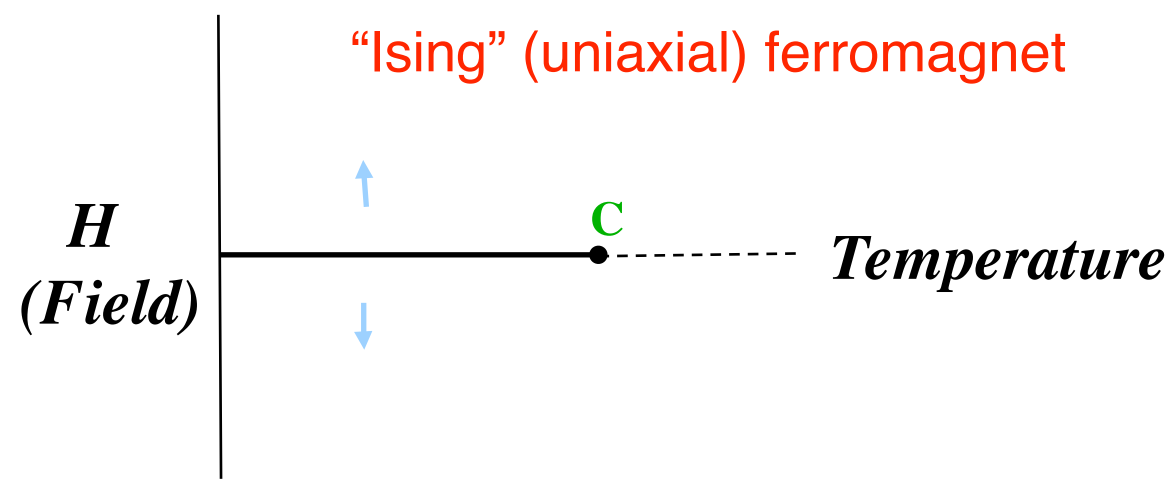
Examples:
Crystals break the *translational* and *rotational* symmetries of free space.
The "liquid crystal" in an LCD breaks *rotational* but not *translational* symmetry.
Magnets break time-reversal symmetry and the rotational symmetry of spin space.
Superfluids break an internal symmetry of quantum mechanics.

Types of order

At high temperature, entropy dominates and leads to a disordered state.
At low temperature, energy dominates and leads to an ordered state.

In case this sounds too philosophical, there are testable results that come out of the "Landau theory" of symmetry-breaking:





$$\rho_L - \rho_G \sim \left(\frac{T_C - T}{T_C} \right)^{\beta}$$

$$M_{\uparrow} - M_{\downarrow} \sim \left(\frac{T_C - T}{T_C} \right)^{\beta}$$

$$\text{Experiment}: \quad \beta = 0.322 \pm 0.005$$

$$\text{Theory}: \quad \beta = 0.325 \pm 0.002$$

"Universality" at continuous phase transitions (Wilson, Fisher, Kadanoff, ...)

Types of order

In 1980, the first ordered phase beyond symmetry breaking was discovered.

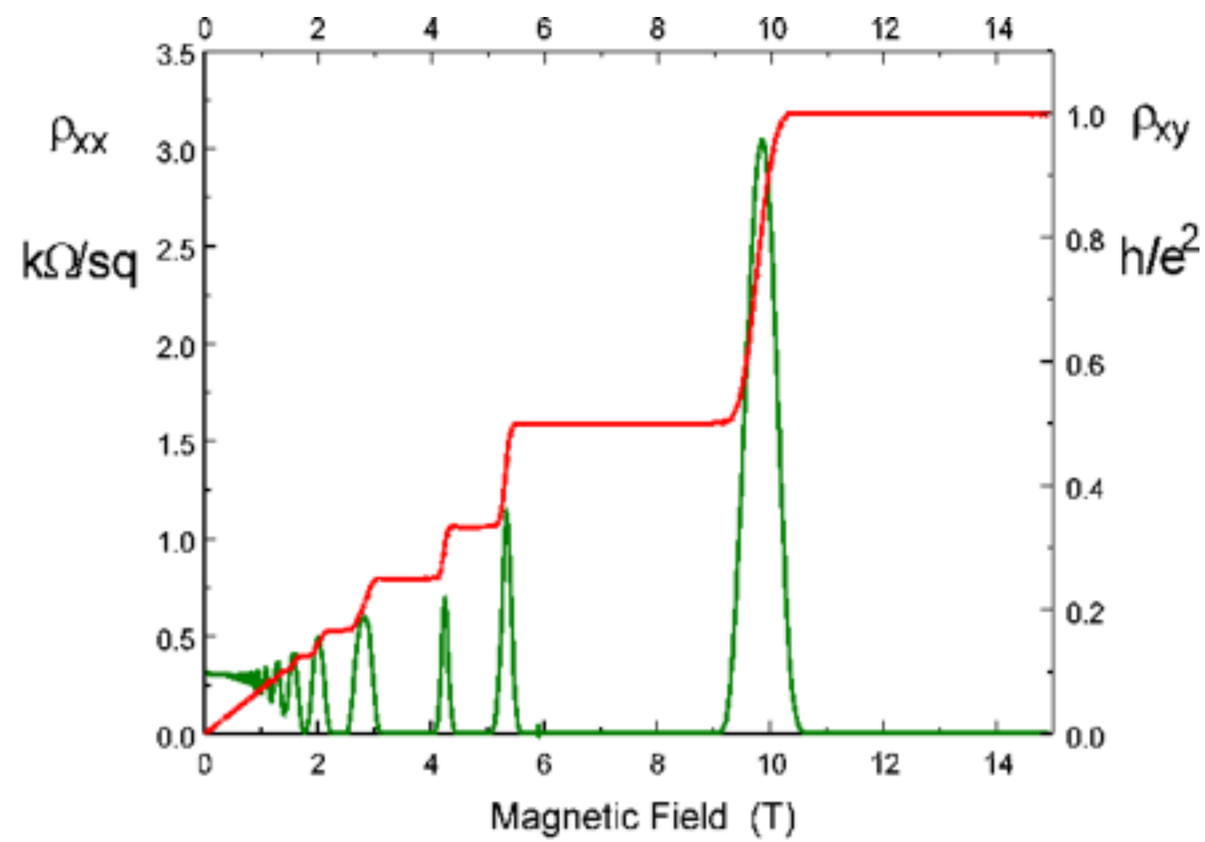
Electrons confined to a plane and in a strong magnetic field show, at low enough temperature, plateaus in the "Hall conductance":

force *I* along *x* and measure *V* along *y*

on a plateau, get

$$\sigma_{xy} = n\frac{e^2}{h}$$

at least within 1 in 10^9 or so.



What type of order causes this precise quantization?

Note I: the AC Josephson effect between superconductors similarly allows determination of *e*/*h*.

Note II: there are also *fractional* plateaus, about which more later.

Topological order

What type of order causes the precise quantization in the Integer Quantum Hall Effect (IQHE)?

Definition I:

In a topologically ordered phase, some physical response function is given by a "topological invariant".

What is a topological invariant? How does this explain the observation?

Definition II:

A topological phase is insulating but always has metallic edges/surfaces when put next to vacuum or an ordinary phase.

What does this have to do with Definition I?

"Topological invariant" = quantity that does not change under continuous deformation

(A third definition: phase is described by a "topological field theory")

Traditional picture: Landau levels

Normally the Hall ratio is (here *n* is a density)

$$R_H = \frac{I_x}{V_y B} = \frac{1}{nec} \Rightarrow \sigma_{xy} = \frac{nec}{B}$$

Then the value (now *n* is an integer)

$$\sigma_{xy} = n\frac{e^2}{h}$$

corresponds to an areal density $$\frac{n}{2\pi\ell^2} = neB/hc.$$

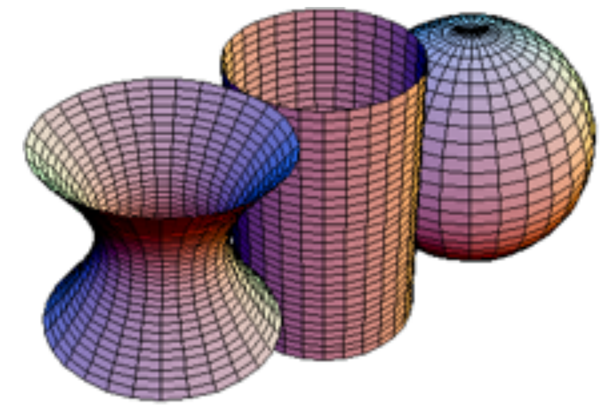
This is exactly the density of "Landau levels", the discrete spectrum of eigenstates of a 2D particle in an orbital magnetic field, spaced by the cyclotron energy. The only "surprise" is how precise the quantization is.

Topological invariants

Most *topological* invariants in physics arise as integrals of some *geometric* quantity.

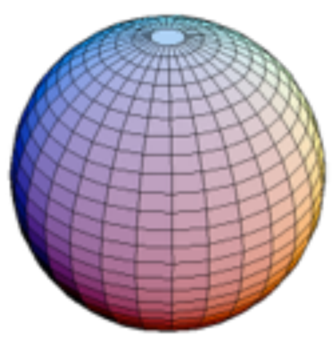
Consider a two-dimensional surface.

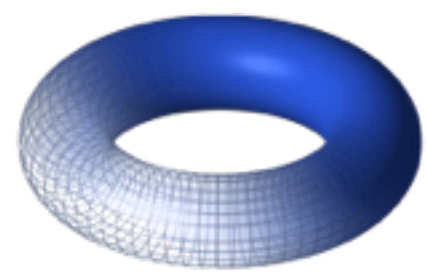
At any point on the surface, there are two radii of curvature.
We define the signed "Gaussian curvature" $\kappa = (r_1 r_2)^{-1}$



from left to right, equators have negative, 0, positive Gaussian curvature

Now consider *closed* surfaces.





The area integral of the curvature over the whole surface is "quantized", and is a topological invariant (Gauss-Bonnet theorem).

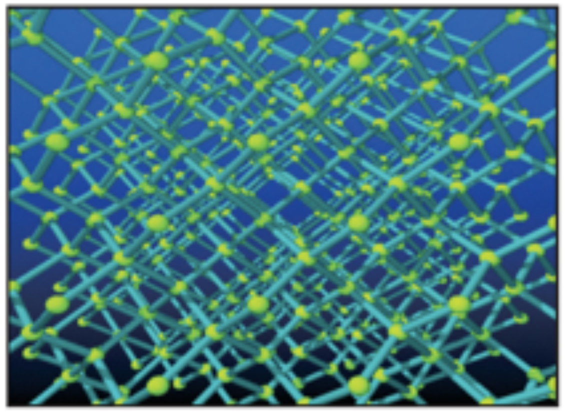
$$\int_M \kappa \, dA = 2\pi\chi = 2\pi(2-2g)$$

where the "genus" g = 0 for sphere, 1 for torus, n for "n-holed torus".

Topological invariants

Good news:
for the invariants in the IQHE and topological insulators, we need one fact about solids

Bloch's theorem:
One-electron wavefunctions in a crystal
(i.e., periodic potential) can be written



$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r})$$

where *k* is "crystal momentum" and *u* is periodic (the same in every unit cell).

Crystal momentum *k* can be restricted to the Brillouin zone, a region of *k*-space with periodic boundaries.
As *k* changes, we map out an "energy band". Set of all bands = "band structure".

The Brillouin zone will play the role of the "surface" as in the previous example,

and one property of quantum mechanics, the Berry phase

which will give us the "curvature".

Berry phase

What kind of "curvature" can exist for electrons in a solid?

Consider a quantum-mechanical system in its (nondegenerate) ground state.

The adiabatic theorem in quantum mechanics implies that, if the Hamiltonian is now changed slowly, the system remains in its time-dependent ground state.

But this is actually very incomplete (Berry).

When the Hamiltonian goes around a *closed loop k(t)* in parameter space, there can be an irreducible *phase*

$$\phi = \oint \mathcal{A} \cdot d\mathbf{k}, \quad \mathcal{A} = \langle \psi_k | - i\nabla_k | \psi_k \rangle$$

relative to the initial state.

Why do we write the phase in this form?
Does it depend on the choice of reference wavefunctions?

Michael Berry

Berry phase

Why do we write the phase in this form?
Does it depend on the choice of reference wavefunctions?

$$\phi = \oint \mathcal{A} \cdot d\mathbf{k}, \quad \mathcal{A} = \langle \psi_k | -i\nabla_k | \psi_k \rangle$$

If the ground state is non-degenerate, then the only freedom in the choice of reference functions is a local phase:

$$\psi_k \rightarrow e^{i\chi(k)} \psi_k$$

Under this change, the "Berry connection" *A* changes by a gradient,

$$\mathcal{A} \rightarrow \mathcal{A} + \nabla_k \chi$$

Michael Berry

just like the vector potential in electrodynamics.

So loop integrals of *A* will be gauge-invariant,
as will the *curl* of *A*, which we call the "Berry curvature".

$$\mathcal{F} = \nabla \times \mathcal{A}$$

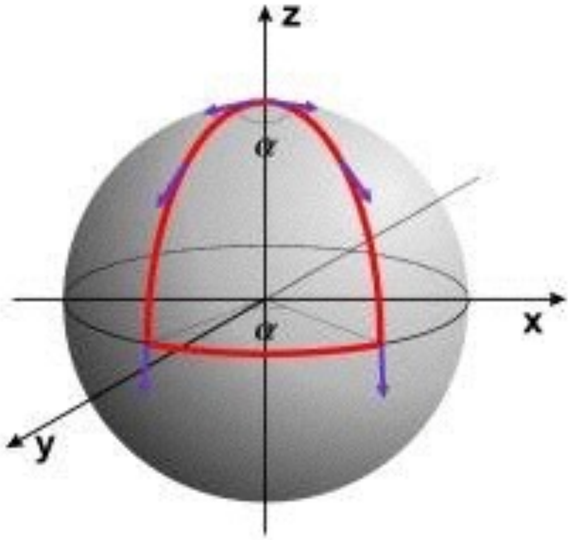
Berry phase: an example

Consider the Zeeman Hamiltonian for a spin-half moving in a magnetic field whose direction varies in time,

$$H = -\frac{g_s \mu B_0}{\hbar} \hat{\mathbf{n}}(t) \cdot \mathbf{S}$$

The resulting Berry phase around a closed path on the Bloch sphere is proportional to the (signed) area enclosed.

One can view this as the Aharonov-Bohm phase from the flux of a magnetic monopole located at the center of the Bloch sphere.



A sign of topology: when such a magnetic monopole has nonzero flux, there is no globally well-defined gauge for A (the gauge singular at the north pole has a "Dirac string" coming in that pole, e.g.).
Consequently, there is no globally well-defined smooth choice of wave functions for the Bloch sphere, as having such a smooth choice would lead to a smooth A.

Berry phase in solids

In a solid, the natural parameter space is electron momentum.

The change in the electron wavefunction *within the unit cell* leads to a Berry connection and Berry curvature:

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r})$$

$$\mathcal{A} = \langle u_{\mathbf{k}} | -i\nabla_k | u_{\mathbf{k}} \rangle \qquad \mathcal{F} = \nabla \times \mathcal{A}$$

S. S. Chern

We keep finding more physical properties that are determined by these quantum geometric quantities.

The first was that the integer quantum Hall effect in a 2D crystal follows from the integral of *F* (like Gauss-Bonnet!). Explicitly,

$$n = \sum_{bands} \frac{i}{2\pi} \int d^2k \left(\left\langle \frac{\partial u}{\partial k_1} \middle| \frac{\partial u}{\partial k_2} \right\rangle - \left\langle \frac{\partial u}{\partial k_2} \middle| \frac{\partial u}{\partial k_1} \right\rangle \right) \qquad \mathcal{F} = \nabla \times \mathcal{A}$$

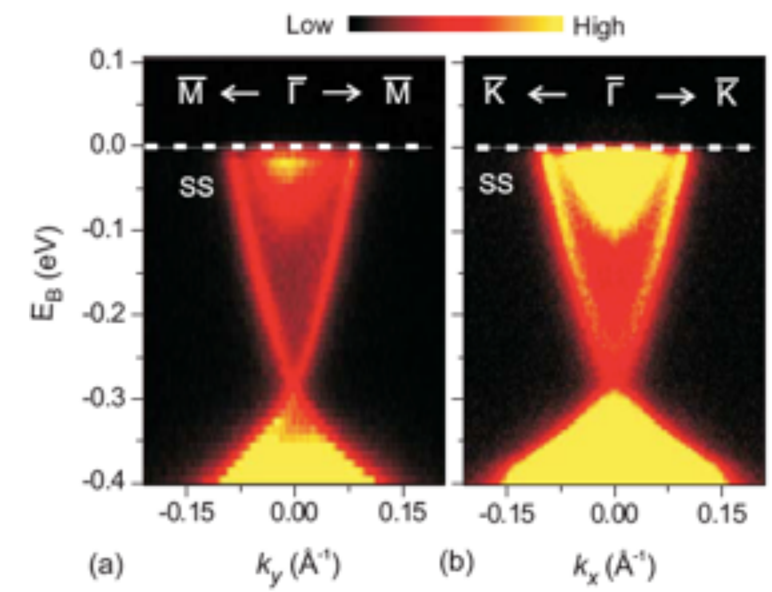
$$\sigma_{xy} = n\frac{e^2}{h}$$

TKNN, 1982

"first Chern number"

Berry phase in solids

Every simple gauge-invariant object made from A and F seems to mean something physically. We can identify several types of Berry-phase phenomena of nearly free electrons:

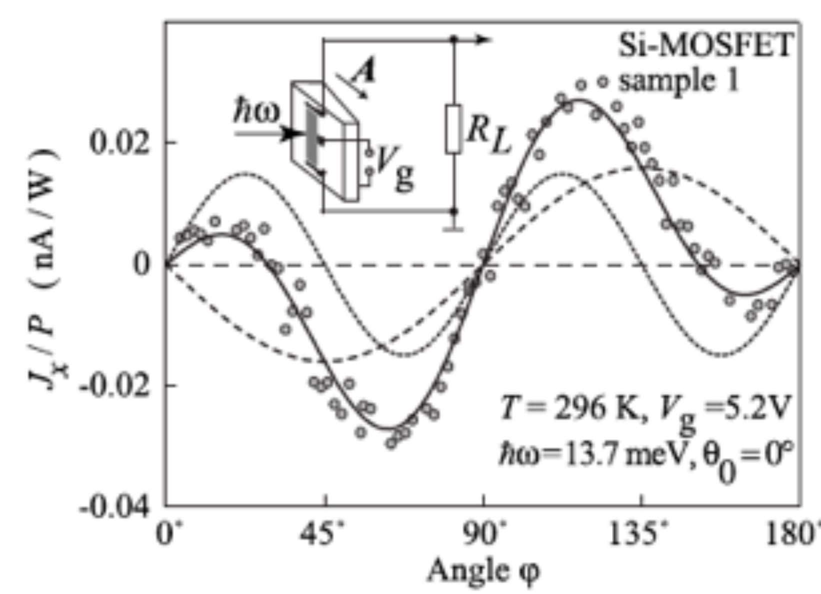


Insulators:

Topological phases independent of symmetry:
Examples: 2D and 4D QHE (1982,1988)

Topological phases dependent on symmetry
Examples: 2D and 3D Z2 topological insulators (2005,2007)

The Berry-phase approach to understanding these leads to expressions that are physically meaningful without symmetries:



Examples: electrical polarization (1987-1990);
magnetoelectric effect (2009-2010)

Metals: Several long-observed phenomena in metals are now believed to be Berry-phase effects. If time permits, will give a quick description of 3 (1999,2010,2012).

The importance of the edge

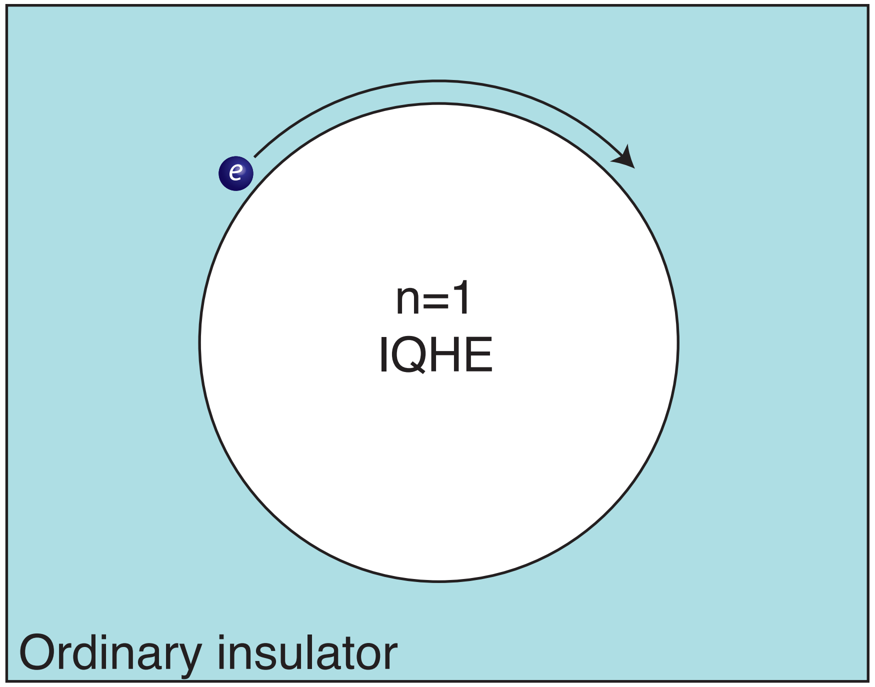
But wait a moment...

This invariant exists if we have energy bands that are either full or empty, i.e., a "band insulator".

How does an *insulator* conduct charge?

Answer: (Laughlin; Halperin)

There are *metallic edges* at the boundaries of our 2D electronic system, where the conduction occurs.



These metallic edges are "chiral" quantum wires (one-way streets). Each wire gives one conductance quantum (e^2/h).

$$\sigma_{xy} = n\frac{e^2}{h}$$

The topological invariant of the *bulk* 2D material just tells how many wires there *have* to be at the boundaries of the system.

How does the bulk topological invariant "force" an edge mode?

The importance of the edge

The topological invariant of the *bulk* 2D material just tells how many wires there *have* to be at the boundaries of the system.

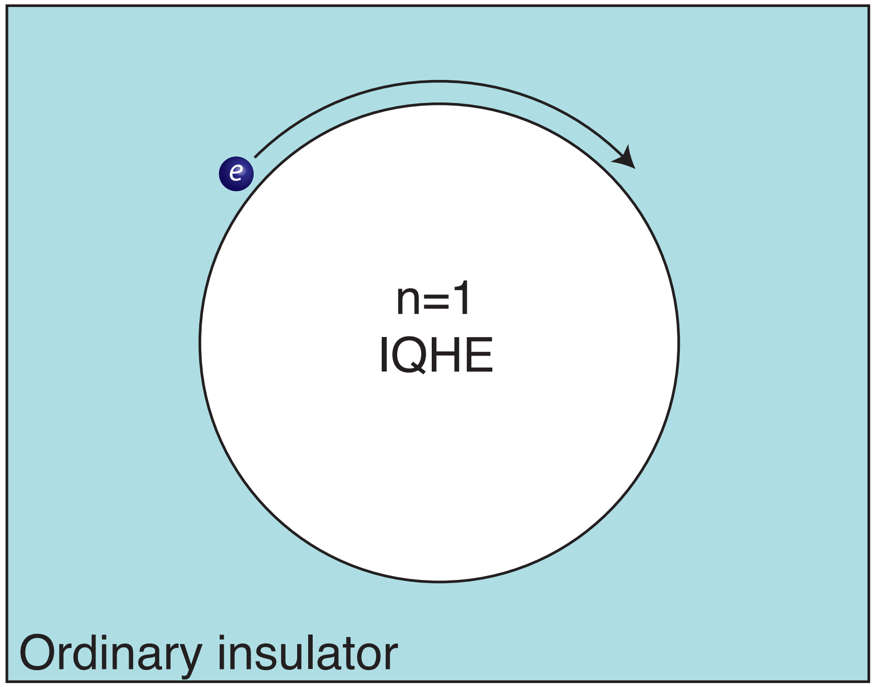
How does the bulk topological invariant "force" an edge mode?

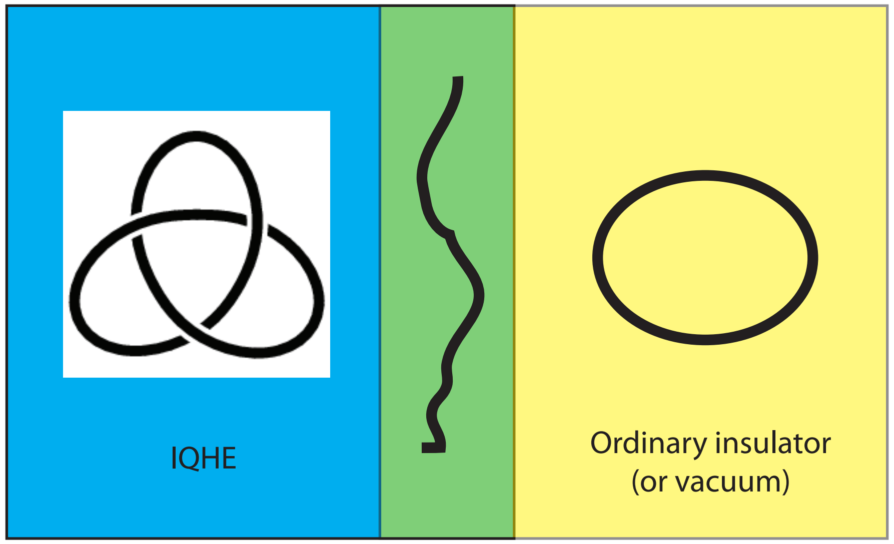
Answer:

Imagine a "smooth" edge where the system gradually evolves from IQHE to ordinary insulator. The topological invariant must change.

But the definition of our "topological invariant" means that, *if the system remains insulating* so that every band is either full or empty, the invariant cannot change.

∴ the system must not remain insulating.





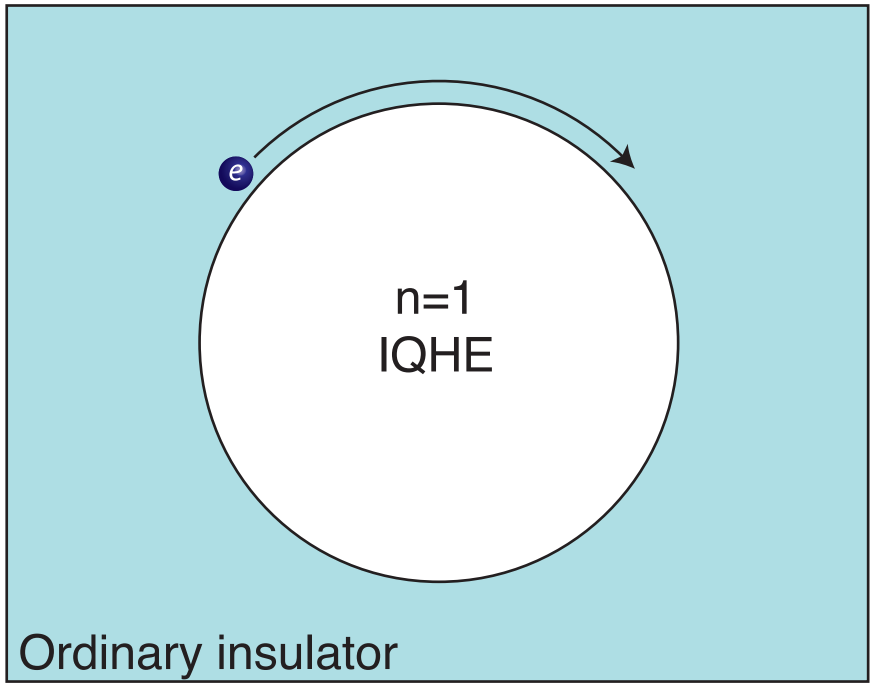
(What is "knotted" are the electron wavefunctions)

2005-present and “topological insulators”

The same idea will apply in the new topological phases discovered recently:

a “topological invariant”, based on the Berry phase, leads to a nontrivial edge or surface state at any boundary to an ordinary insulator or vacuum.

However, the physical origin, dimensionality, and experiments are all different.



We discussed the IQHE so far in an unusual way. The magnetic field entered only through its effect on the Bloch wavefunctions (no Landau levels!).

This is not very natural for a magnetic field.
It is ideal for spin-orbit coupling in a crystal.

The "quantum spin Hall effect"

Spin-orbit coupling appears in nearly every atom and solid. Consider the standard atomic expression

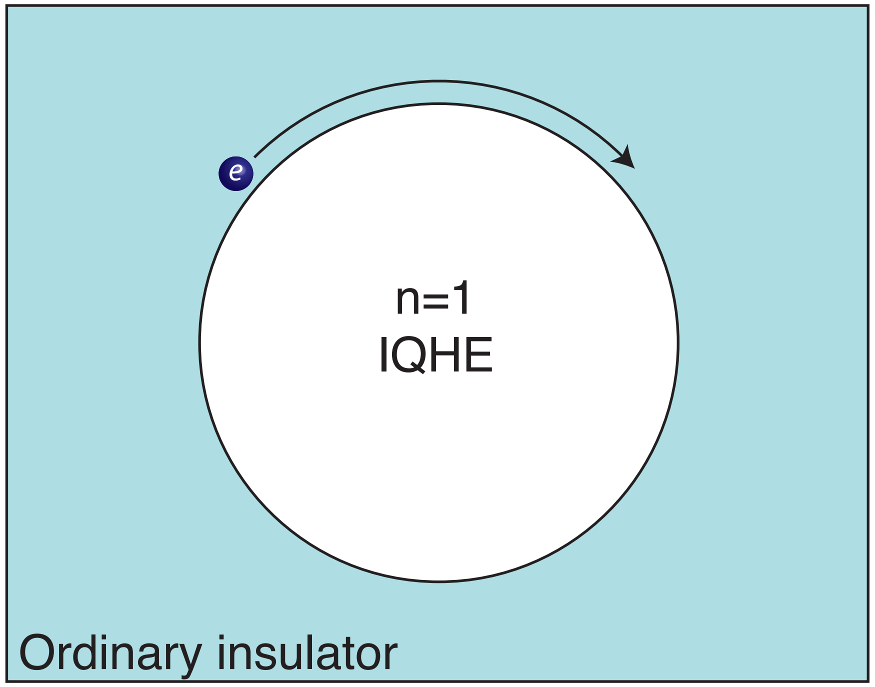
$$H_{SO} = \lambda \mathbf{L} \cdot \mathbf{S}$$

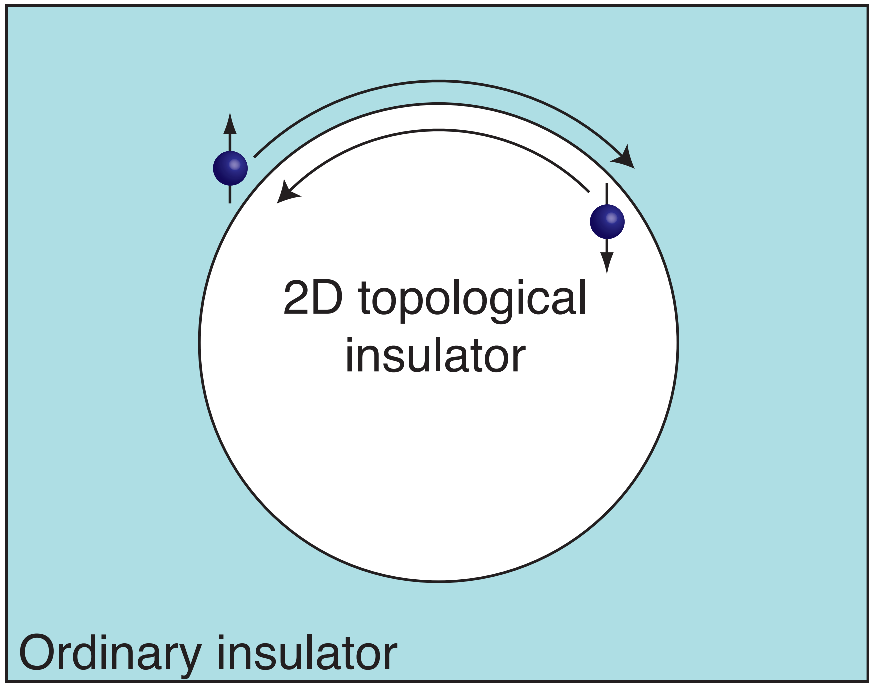
For a given spin, this term leads to a momentum-dependent force on the electron, somewhat like a magnetic field.

The spin-dependence means that the *time-reversal symmetry* of SO coupling (even) is different from a real magnetic field (odd).

It is possible to design lattice models where spin-orbit coupling has a remarkable effect: (Murakami, Nagaosa, Zhang 04; Kane, Mele 05)

spin-up and spin-down electrons are in IQHE states, with opposite "effective magnetic fields".



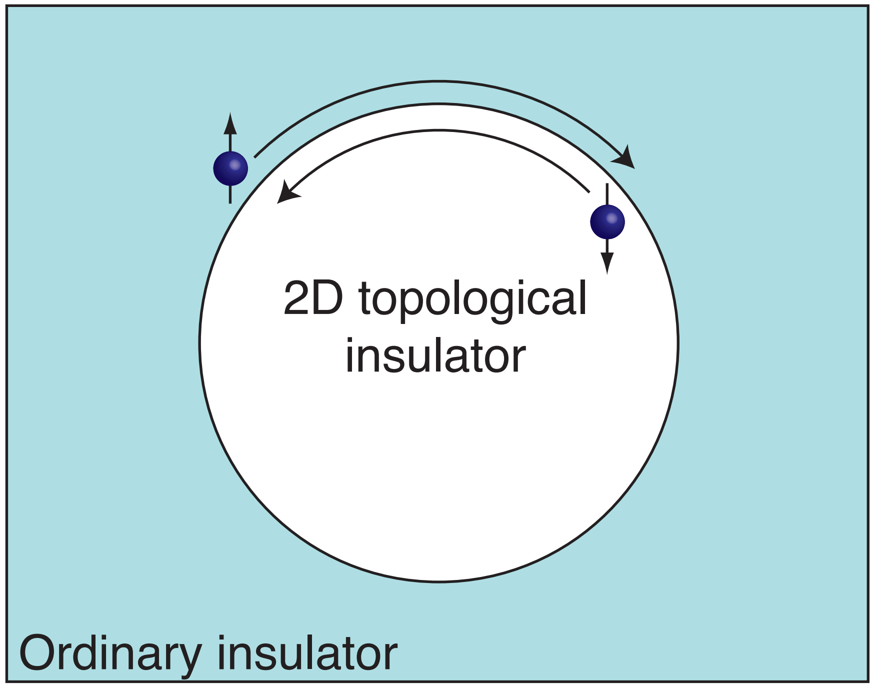


The "quantum spin Hall effect"

In this type of model, electron spin is conserved, and there can be a "spin current".

An applied electrical field causes oppositely directed Hall currents of up and down spins.

The charge current is zero, but the "spin current" is nonzero, and even quantized!



$$\mathcal{J}_j^i = \sigma_H^s \epsilon_{ijk} E_k$$

However...

1. In real solids there is no conserved direction of spin.

2. So in real solids, it was expected that "up" and "down" would always mix and the edge to disappear.

3. The theory of the above model state is just two copies of the IQHE.

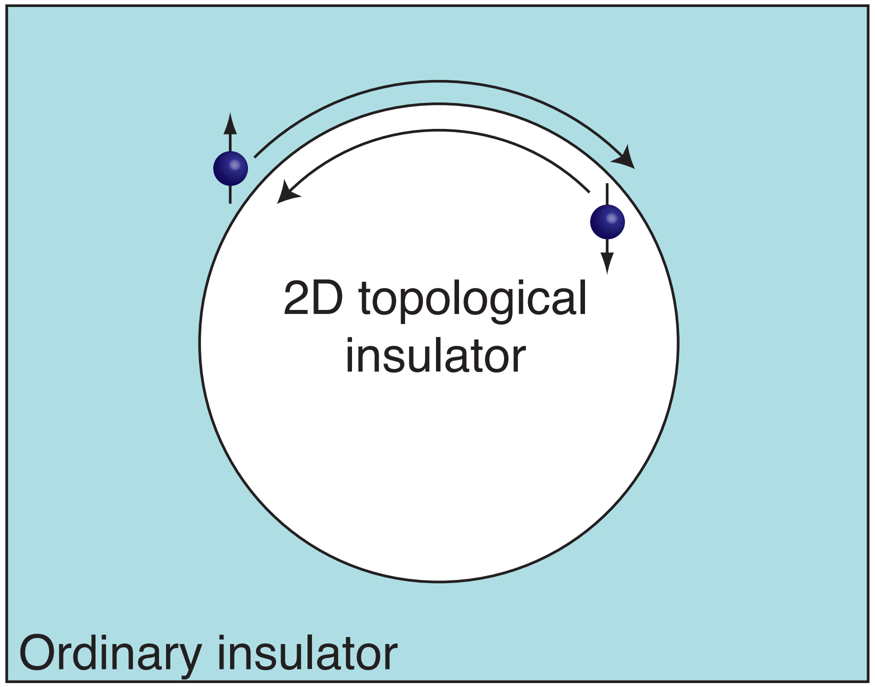
The 2D topological insulator

It was shown in 2005 (Kane and Mele) that, in real solids with all spins mixed and no "spin current", something of this physics does survive.

In a material with only spin-orbit, the "Chern number" mentioned before always vanishes.

Kane and Mele found a new topological invariant in time-reversal-invariant systems of fermions.

But it isn't an integer! It is a Chern *parity* ("odd" or "even"), or a "Z2 invariant".



Systems in the "odd" class are "2D topological insulators"

1. Where does this "odd-even" effect come from?
2. What is the Berry phase expression of the invariant?
3. How can this edge be seen?

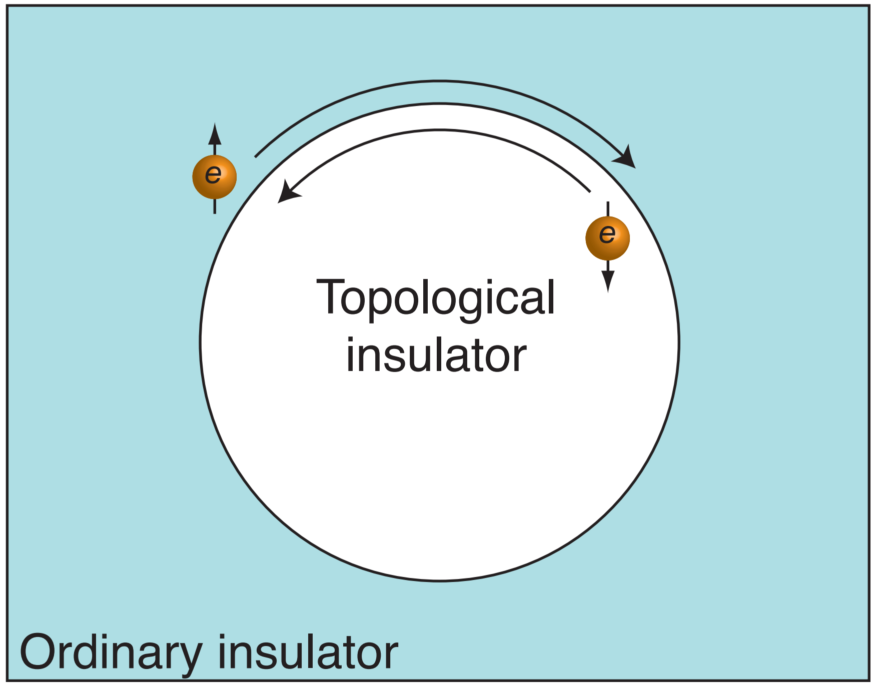
The "Chern insulator" and QSHE

Haldane showed that although *broken time-reversal* is necessary for the QHE, it is not necessary to have a net magnetic flux.

Imagine constructing a system ("model graphene") for which spin-up electrons feel a pseudofield along z, and spin-down electrons feel a pseudofield along -z.

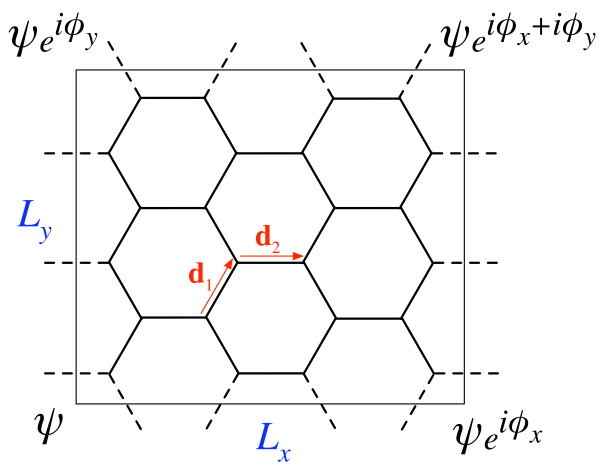
Then *SU(2)* (spin rotation symmetry) is broken, but time-reversal symmetry is not:

an edge will have (in the simplest case)
a clockwise-moving spin-up mode
and a counterclockwise-moving
spin-down mode
(Murakami, Nagaosa, Zhang, '04)



Example: Kane-Mele-Haldane model for graphene

The spin-independent part consists of a tight-binding term on the honeycomb lattice, plus possibly a sublattice staggering



$$H_0 = -t\sum_{\langle ij\rangle} c_{i\sigma}^\dagger c_{j\sigma} + \lambda_v \sum_i \xi_i c_{i\sigma}^\dagger c_{i\sigma}$$

$$\xi_i = \begin{cases} 1 & \text{if i in } A \text{ sublattice} \\ -1 & \text{if i in } B \text{ sublattice} \end{cases}$$

The first term gives a semimetal with Dirac nodes (as in graphene).

The second term, which appears if the sublattices are inequivalent (e.g., BN), opens up a (spin-independent) gap.

When the Fermi level is in this gap, we have an ordinary band insulator.

Example: Kane-Mele-Haldane model for graphene

The spin-independent part consists of a tight-binding term on the honeycomb lattice, plus possibly a sublattice staggering

$$H_0 = -t \sum_{\langle ij \rangle} c^\dagger_{i\sigma} c_{j\sigma} + \lambda_v \sum_i \xi_i c^\dagger_{i\sigma} c_{i\sigma}$$

The spin-dependent part contains two SO couplings

$$H' = i\lambda_{SO} \sum_{\langle\langle ij \rangle\rangle} v_{ij} c^\dagger_i s^z c_j + i\lambda_R \sum_{\langle ij \rangle} c^\dagger_i (\mathbf{s} \times \hat{\mathbf{d}}_{ij})_z c_j$$

The first spin-orbit term is the key: it involves second-neighbor hopping (v_{ij} is ±1 depending on the sites) and Sz. It opens a gap in the bulk and acts as the desired "pseudofield" if large enough. $v_{ij} \propto (\mathbf{d_1} \times \mathbf{d_2})_z$

Claim: the system with an SO-induced gap is fundamentally different from the system with a sublattice gap: it is in a different phase.
It has gapless edge states for *any* edge (not just zigzag).

Example: Kane-Mele-Haldane model for graphene

$$H_0 = -t \sum_{\langle ij \rangle} c^\dagger_{i\sigma} c_{j\sigma} + \lambda_v \sum_i \xi_i c^\dagger_{i\sigma} c_{i\sigma}$$

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Without Rashba term (second SO coupling), have two copies of Haldane's IQHE model. All physics is the same as IQHE physics.

The Rashba term violates conservation of Sz--how does this change the phase? Why should it be stable once up and down spins mix?

Invariants in T-invariant systems?

If a quantum number (e.g., Sz) can be used to divide bands into "up" and "down", then with T invariance, one can define a "spin Chern integer" that counts the number of Kramers pairs of edge modes:

$$n_\uparrow + n_\downarrow = 0, n_\uparrow - n_\downarrow = 2n_s$$

What about T-invariant systems?

If a quantum number (e.g., Sz) can be used to divide bands into "up" and "down", then with T invariance, one can define a "spin Chern number" that counts the number of Kramers pairs of edge modes:

$$n_\uparrow + n_\downarrow = 0, n_\uparrow - n_\downarrow = 2n_s$$

For general spin-orbit coupling, there is no conserved quantity that can be used to classify bands in this way, and no integer topological invariant.

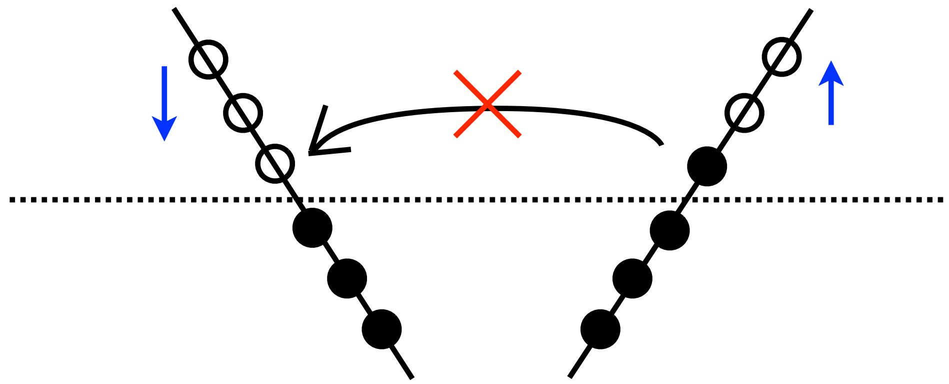
Instead, a fairly technical analysis shows

1. each pair of spin-orbit-coupled bands in 2D has a Z2 invariant (is either "even" or "odd"), essentially as an integral over half the Brillouin zone;

2. the state is given by the overall Z2 sum of occupied bands:
if the sum is odd, then the system is in the "topological insulator" phase

The 2D topological insulator

1. Where does this “odd-even” effect come from?



In a time-reversal-invariant system of electrons, all energy eigenstates come in degenerate pairs.

The two states in a pair cannot be mixed by any T-invariant perturbation. (disorder)

So an edge with a single Kramers pair of modes is perturbatively stable (C. Xu-JEM, C. Wu et al., 2006).

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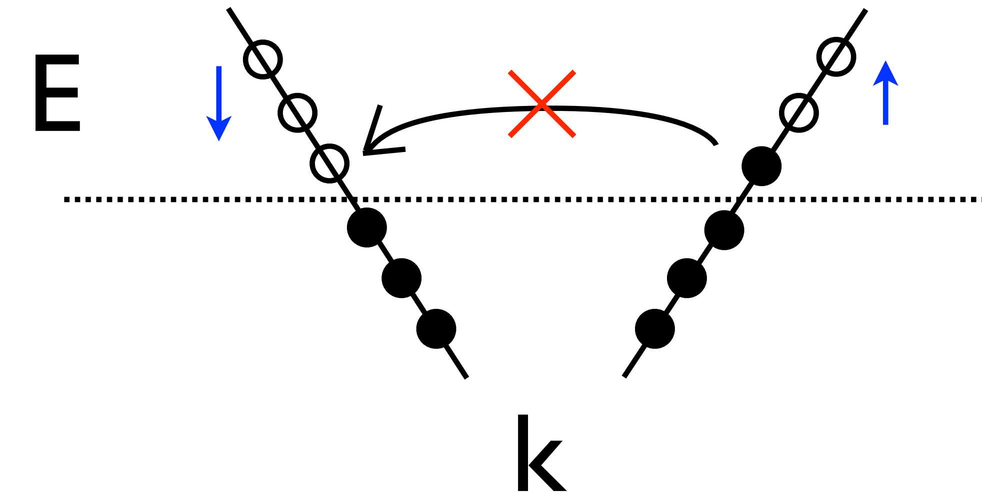
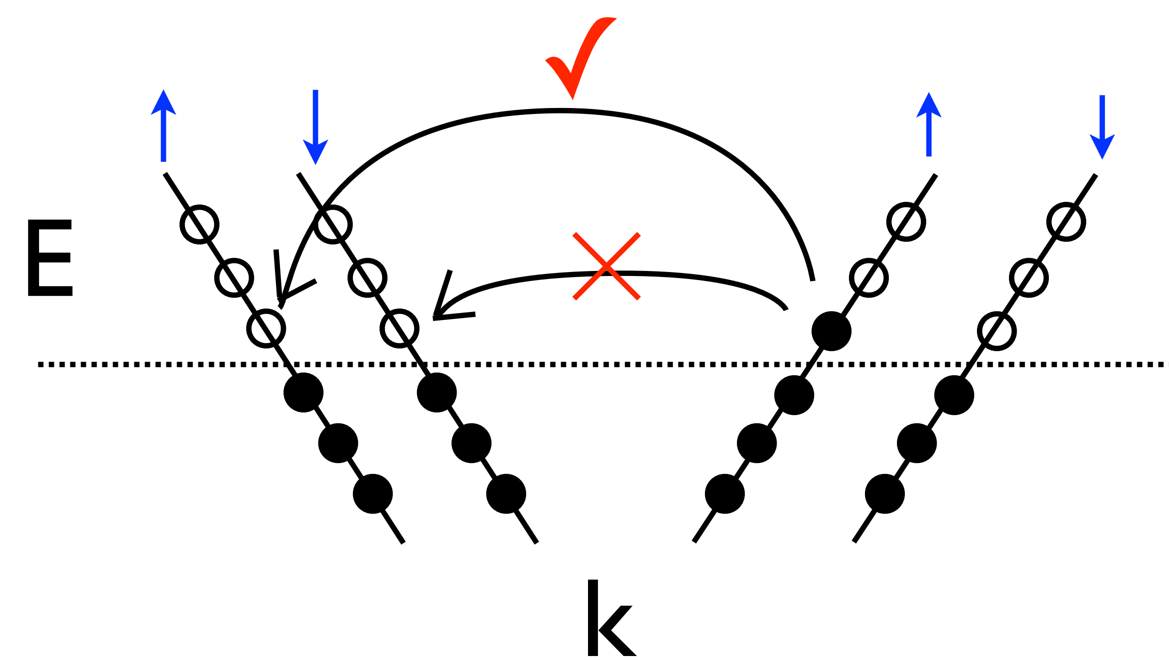


The two states in a pair cannot be mixed by any T-invariant perturbation. (disorder)

So an edge with a single Kramers pair of modes is perturbatively stable (C. Xu-JEM, C. Wu et al., 2006).

But this rule does not protect an ordinary quantum wire with 2 Kramers pairs:



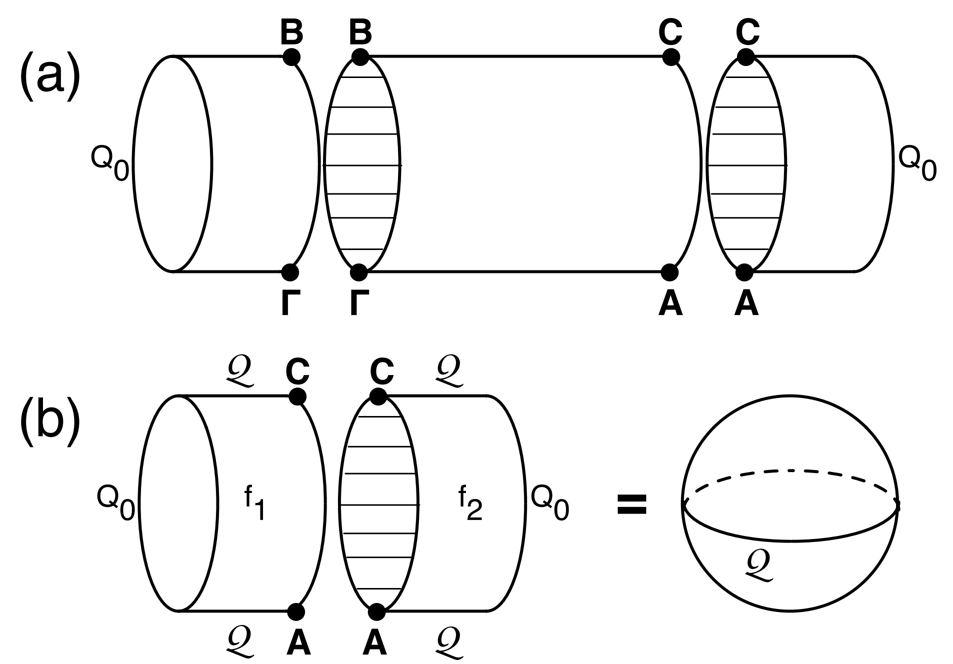
The topological vs. ordinary distinction depends on time-reversal symmetry.

The 2D topological insulator

2. What is the Berry phase expression of the invariant?

It is an integral over *half* the Brillouin zone,

$$D = \frac{1}{2\pi}\left[\oint_{\partial(EBZ)} d\mathbf{k}\cdot\mathcal{A} - \int_{EBZ} d^2\mathbf{k}\,\mathcal{F}\right] \bmod 2$$



3. How can this edge be seen?

Analogy for approach

But the EBZ is an open manifold. How can we "make it into a closed manifold"?

Our approach is similar to a standard construction in 1+1-dimensional QFT: the Wess-Zumino term in the nonlinear sigma model into a nonabelian Lie group *G*:

$$S_0 = -\frac{k}{8\pi} \int_{S^2} \mathcal{K}(g^{-1}\partial^\mu g, g^{-1}\partial_\mu g)$$

$$S_{WZ} = -\frac{2\pi k}{48\pi^2} \int_{B^3} \epsilon_{\mu\nu\lambda} \mathcal{K}(g^{-1}\partial_\mu g, [g^{-1}\partial_\nu g, g^{-1}\partial_\lambda g])$$

Here the Wess-Zumino term is computed using an arbitrary contraction of g in the unit ball B3 that agrees with the specified configuration on the boundary S2.

Such contractions always exist because $\pi_2(G) = 0$

Different contractions can be topologically inequivalent: combining two balls B3 gives a sphere S3, and the difference between two contractions is classified by $\pi_3(G) = \mathbb{Z}$
The path integral is contraction-independent if k is an integer.

Experimental signatures

Key physics of the edges: robust to disorder and hence good *charge* conductors .

The topological insulator is therefore detectable by measuring the two-terminal conductance of a finite sample: should see maximal 1D conductance.

$$G = \frac{2e^2}{h}$$

In other words, *spin transport does not have to be measured* to observe the phase.

Materials recently proposed: Bi, InSb, strained Sn (3d), HgTe (2d) (Bernevig, Hughes, and Zhang, *Science* (2006); *experiments by Molenkamp et al. (2007) see an edge, but $G \sim 0.3\ G_0$)*

The 2D topological insulator

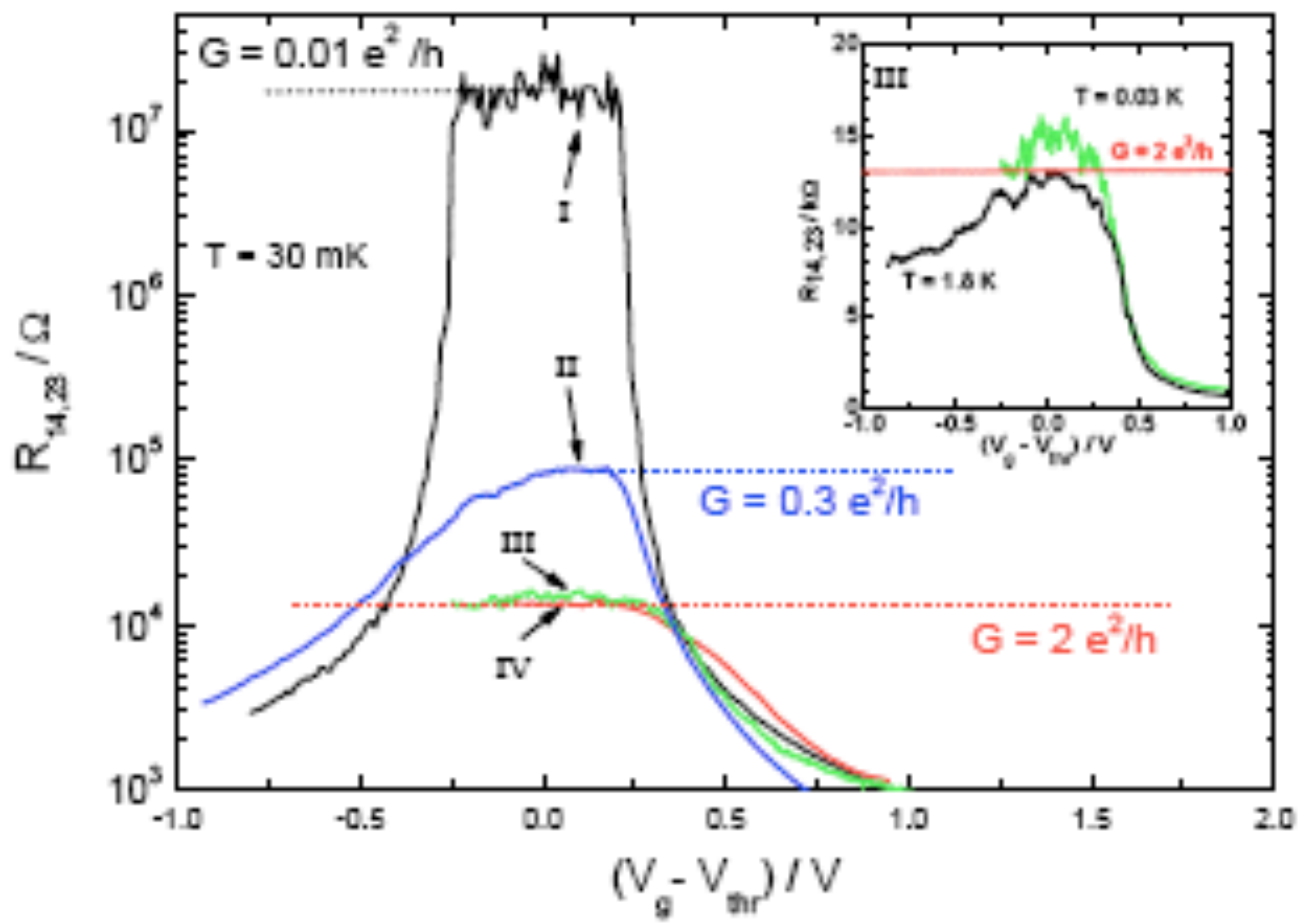
Key: the topological invariant predicts the "number of quantum wires".

While the wires are not one-way, so the Hall conductance is zero, they still contribute to the *ordinary* (two-terminal) conductance.

There should be a low-temperature *edge* conductance from one spin channel at each edge:

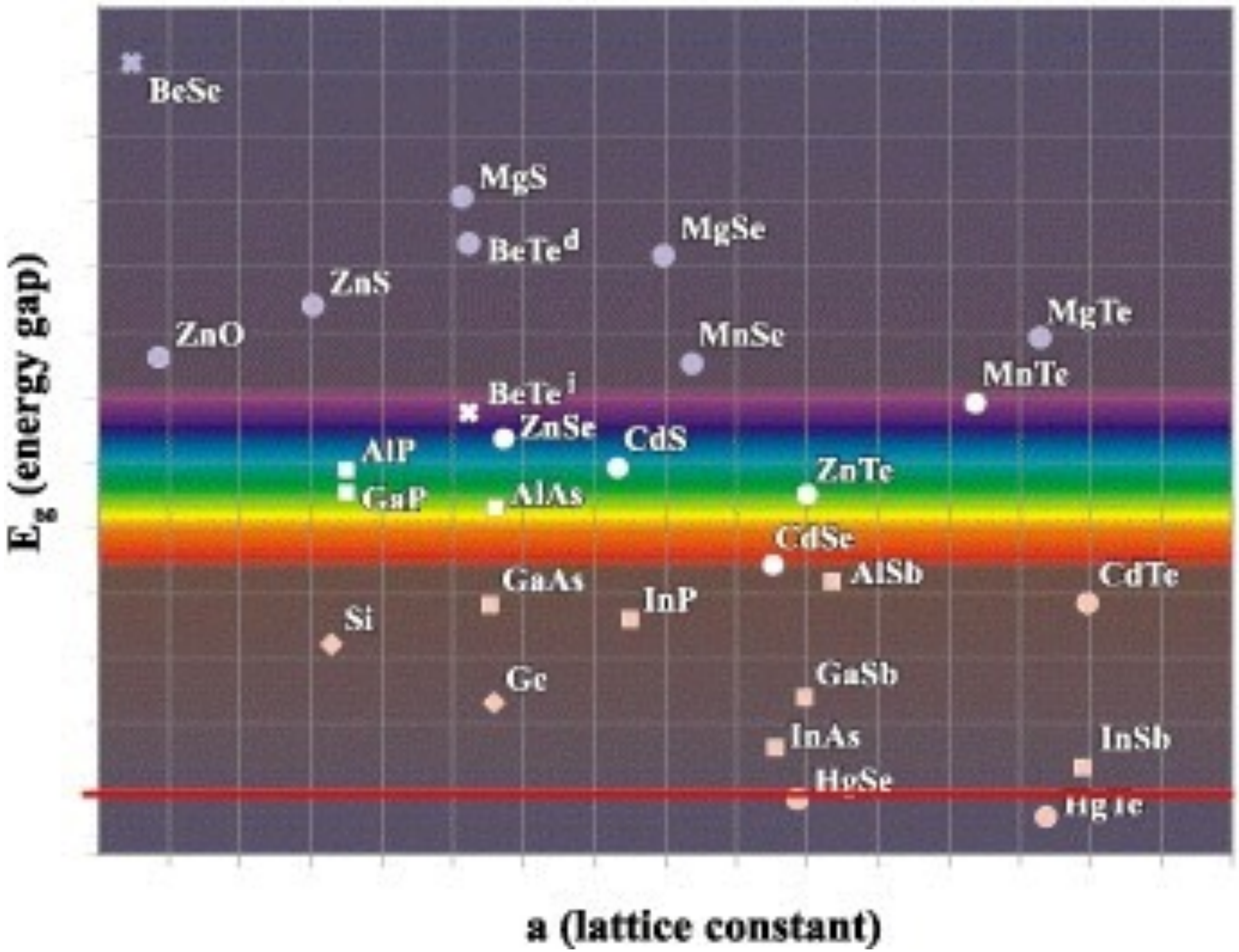
$$G = \frac{2e^2}{h}$$

König et al., *Science* (2007)



Laurens Molenkamp

This appears in (Hg,Cd)Te quantum wells as a quantum Hall-like plateau *in zero magnetic field.*



"Negative" bandgap of
HgTe = inverted band

Review of 3D facts

The 2D conclusion is that band insulators come in two classes:
ordinary insulators (with an even number of edge modes, generally 0)
"topological insulators" (with an odd number of Kramers pairs of edge modes, generally 1).

What about 3D? The only 3D IQHE states are essentially layered versions of 2D states:
Mathematically, there are three Chern integers:

Cxy (for *xy* planes in the 3D Brillouin torus), *Cyz*, *Cxz*

There are similar layered versions of the topological insulator, but these are not very stable; intuitively, adding parities from different layers is not as stable as adding integers.

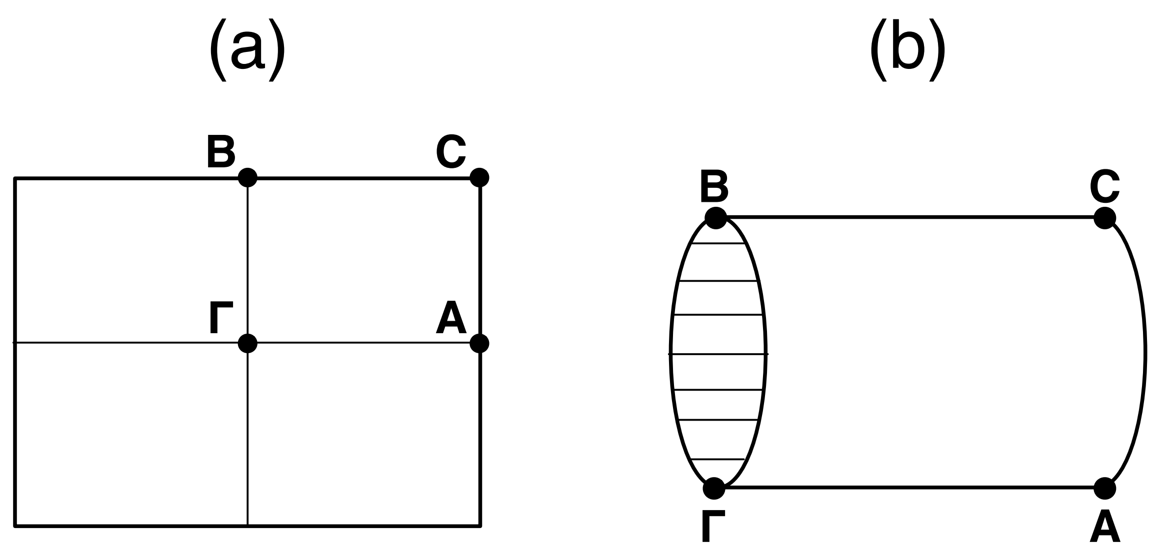
However, there is an unexpected 3D topological insulator state that does not have any simple quantum Hall analogue. For example, it cannot be realized in any model where up and down spins do not mix!

General description of invariant from JEM and L. Balents, PRB RC 2007.
The connection to physical consequences in inversion-symmetric case (proposal of BiSb, Dirac surface state): Fu, Kane, Mele, PRL 2007. See also R. Roy, PRB 2009.

Build 3D from 2D

Note that only at special momenta like k=0 is the "Bloch Hamiltonian" time-reversal invariant: rather, k and -k have T-conjugate Hamiltonians. Imagine a square BZ:

$$H(-k) = TH(k)T^{-1}$$

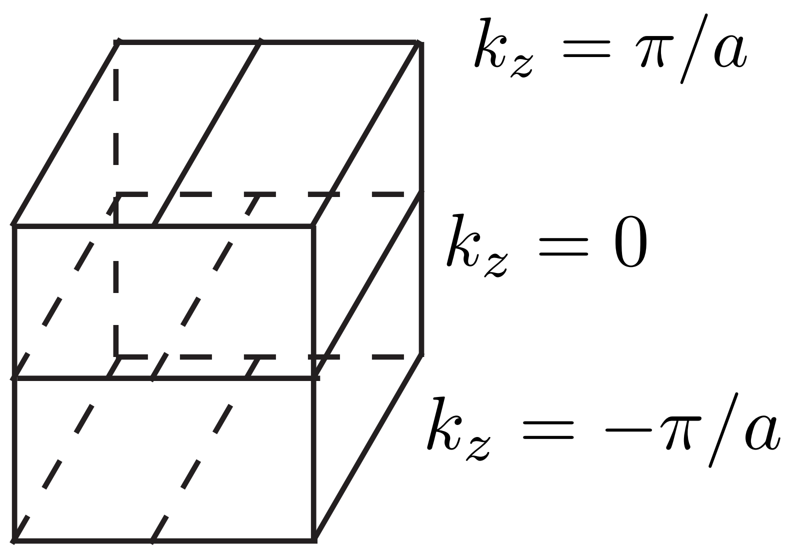


"effective BZ"

In 3D, we can take the BZ to be a cube (with periodic boundary conditions):

think about xy planes

2 inequivalent planes look like 2D problem

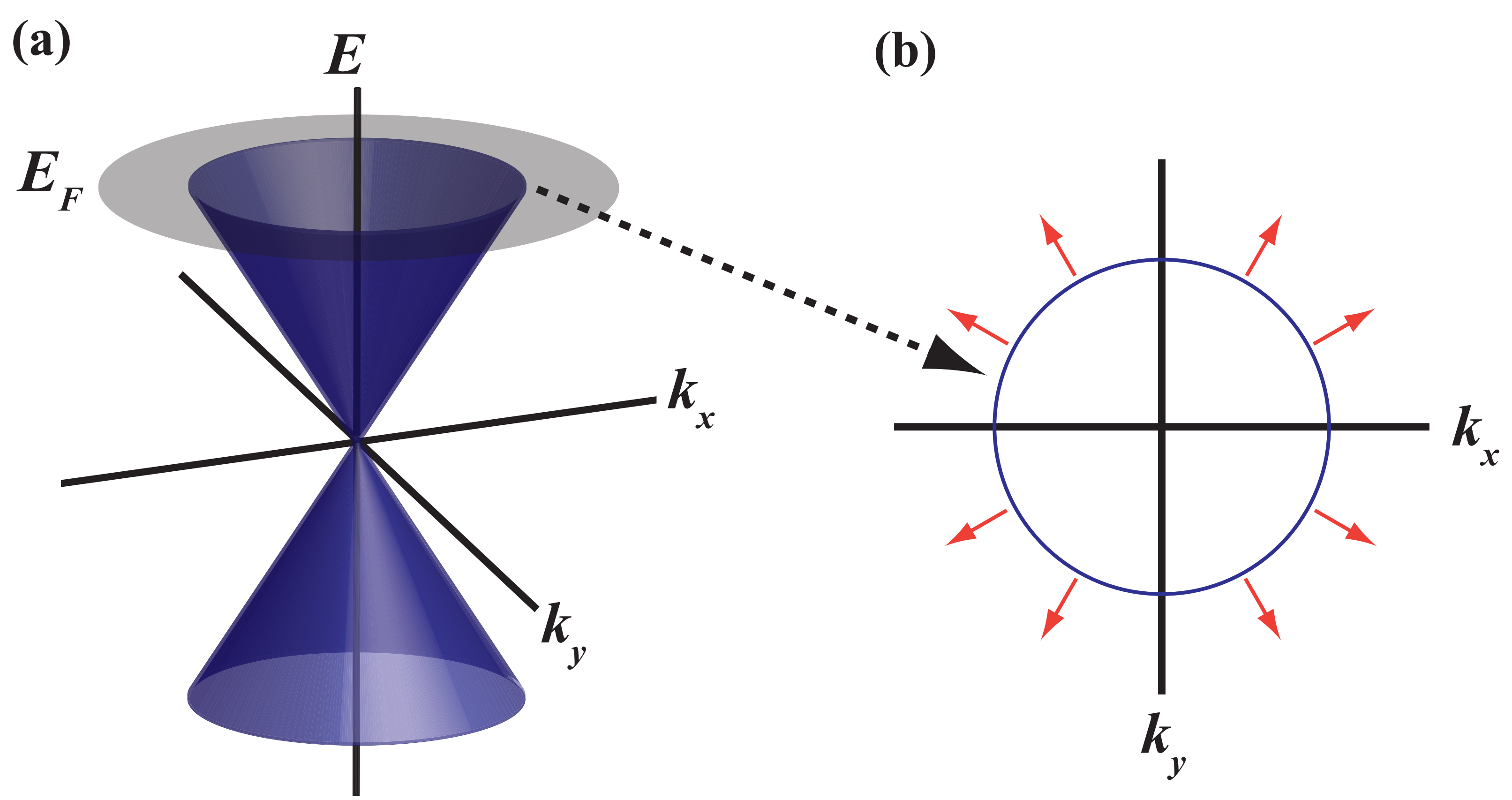


3D "strong topological insulators" go from an 2D *ordinary* insulator to a 2D *topological* insulator (or vice versa) in going from k_z=0 to k_z=±π/a.

This is allowed because intermediate planes have no time-reversal constraint.

Topological insulators in 3D

1. This fourth invariant gives a robust 3D "strong topological insulator" whose metallic surface state in the simplest case is a single "Dirac fermion" (Fu-Kane-Mele, 2007)



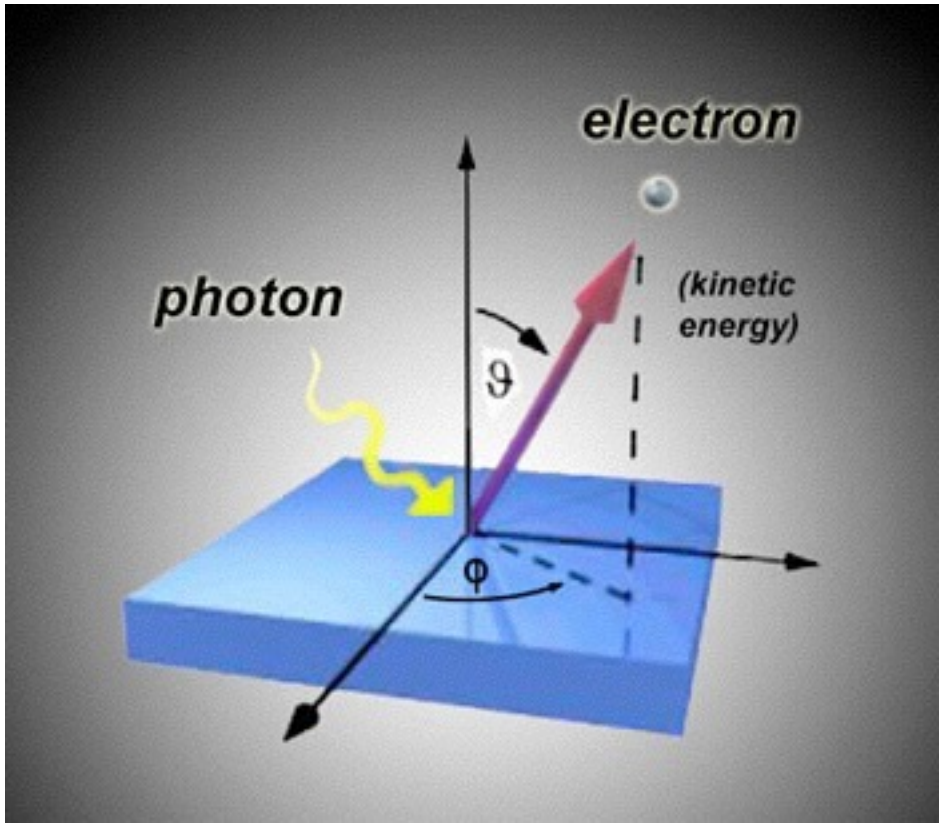
2. Some fairly common 3D materials might be topological insulators! (Fu-Kane, 2007)

Claim:
Certain insulators will *always* have metallic surfaces with strongly spin-dependent structure

How can we look at the metallic surface state of a 3D material to test this prediction?

ARPES of topological insulators

Imagine carrying out a "photoelectric effect" experiment very carefully.



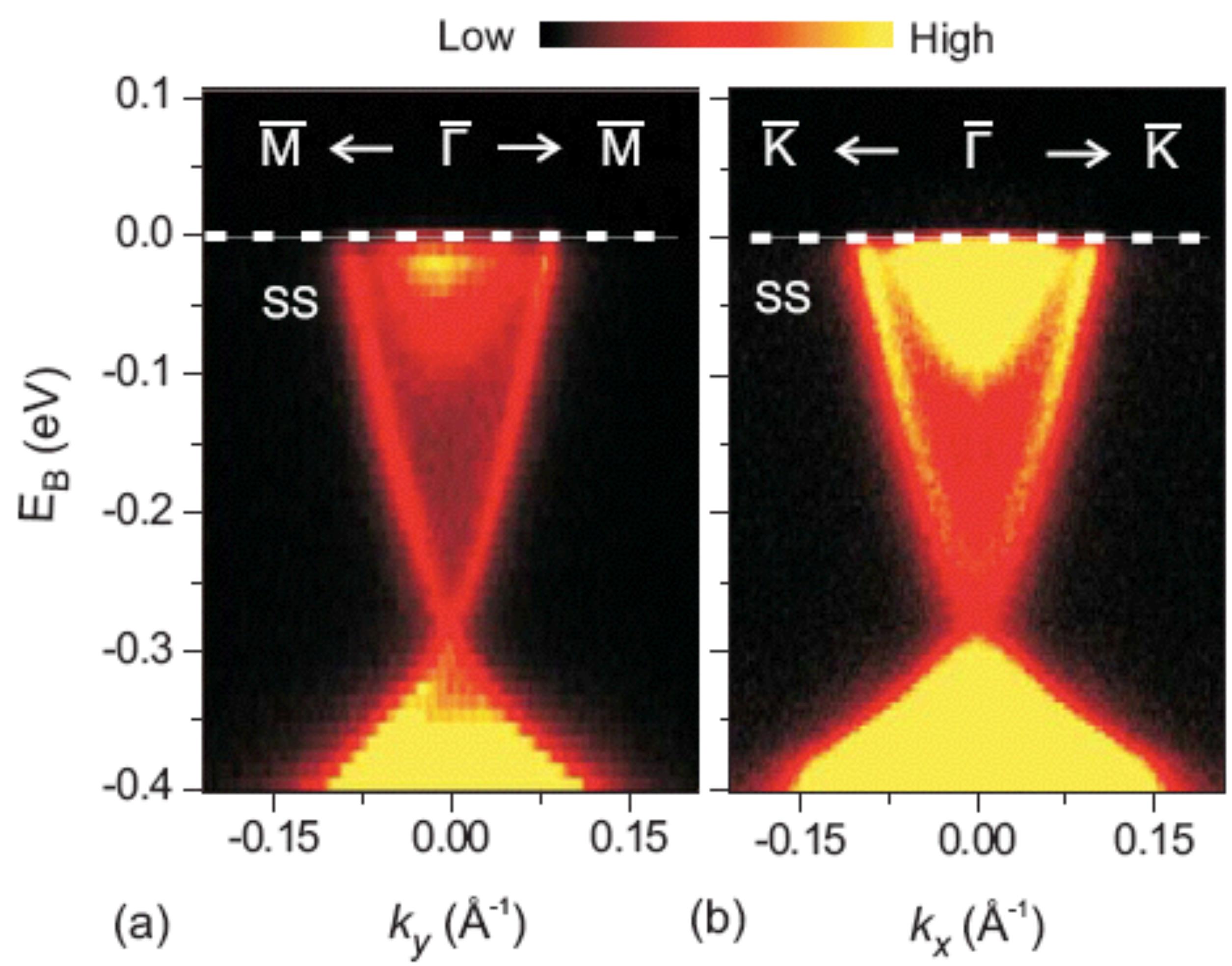
Measure as many properties as possible of the outgoing electron
to deduce the momentum, energy, and spin it had while still in the solid.

This is "angle-resolved photoemission spectroscopy", or ARPES.

ARPES of topological insulators

First observation by D. Hsieh et al. (Z. Hasan group), Princeton/LBL, 2008.

This is later data on Bi_2Se_3 from the same group in 2009:

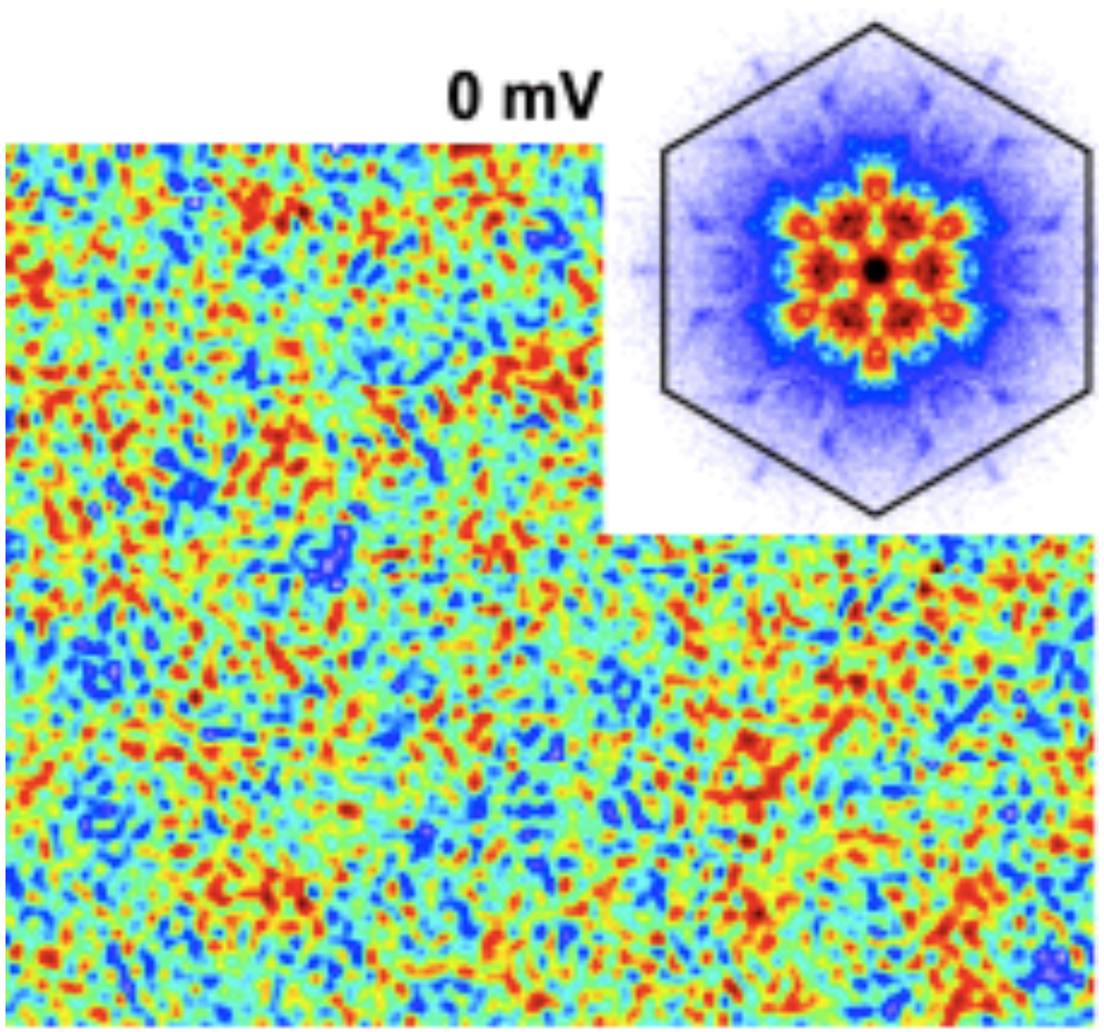


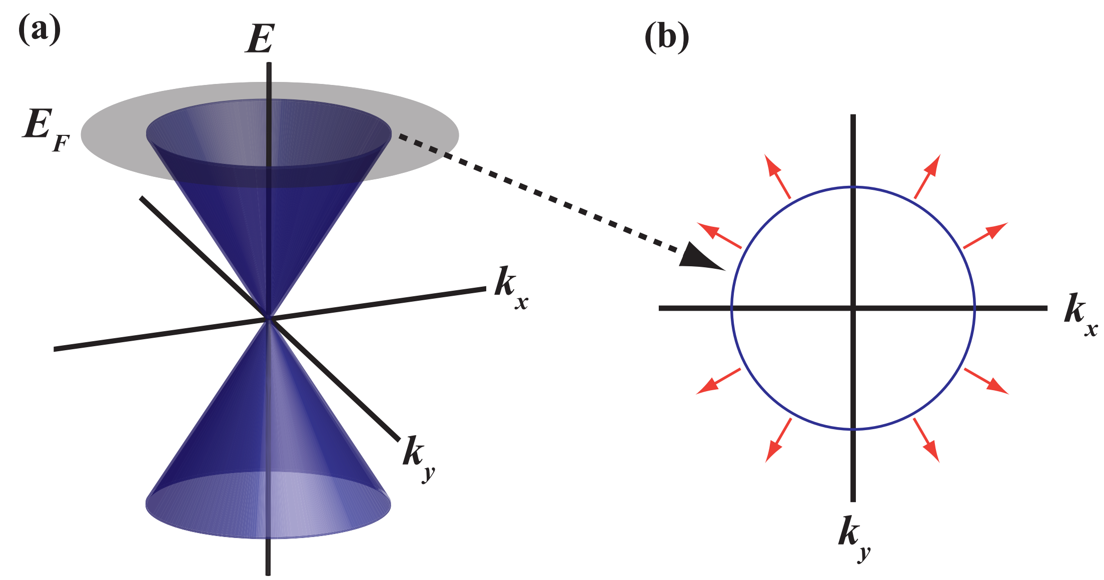
The states shown are in the "energy gap" of the bulk material--in general no states would be expected, and especially not the Dirac-conical shape.

STM of topological insulators

The surface of a simple topological insulator like Bi_2Se_3 is "1/4 of graphene": it has the Dirac cone but no valley or spin degeneracies.

Scanning tunneling microscopy image (Roushan et al., Yazdani group, 2009)





STM can see the absence of scattering within a Kramers pair (cf. analysis of superconductors using quasiparticle interference, D.-H. Lee and S. Davis).

Periodic table of one-fermion TIs

In every dimension, of the 10 Altland-Zirnbauer symmetry classes, there are 3 with integer invariants and 2 with Z2 invariants. But different symmetry classes are topological in different dimensions.

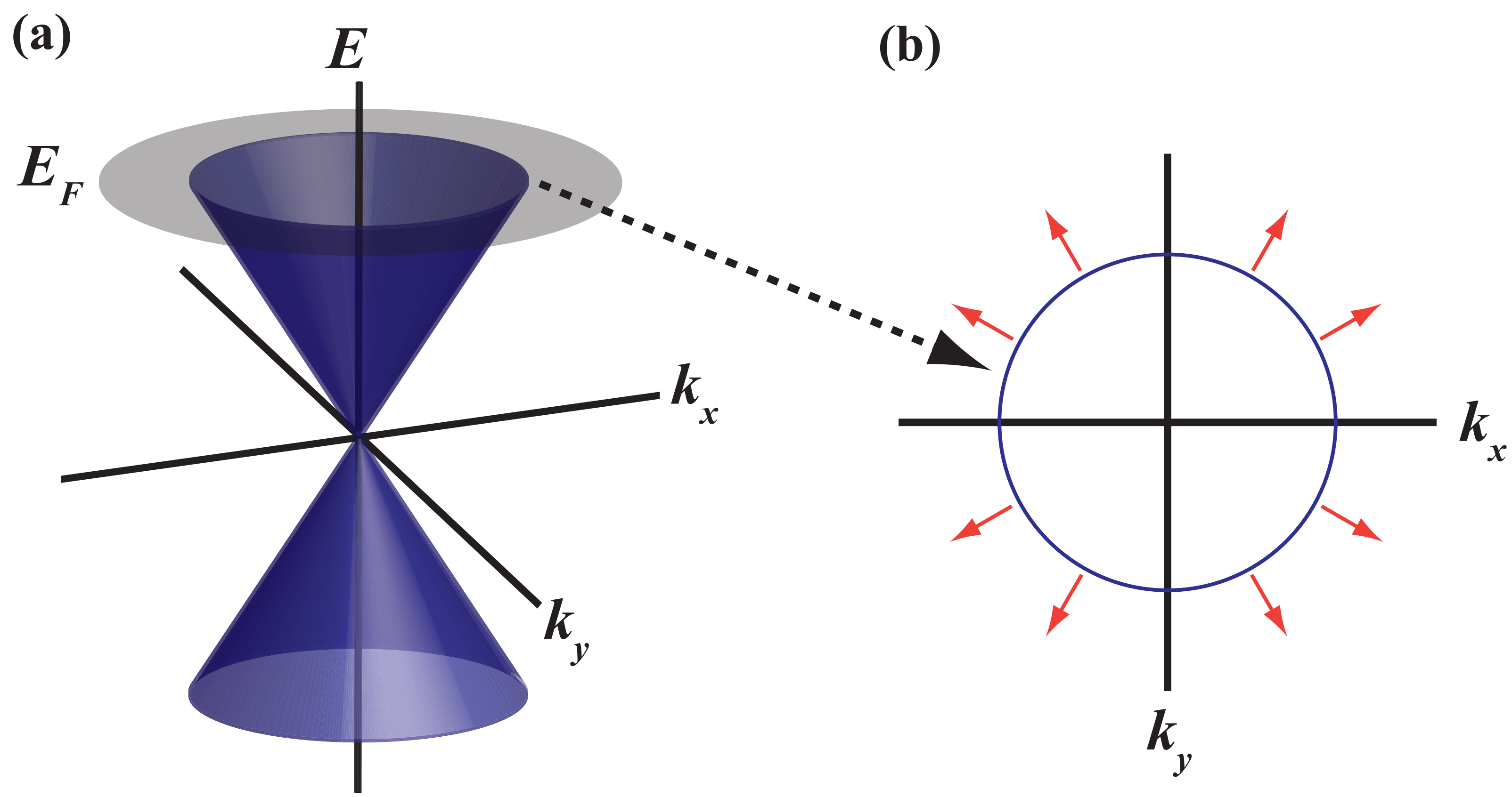
In the table below, A = unitary class (no symmetry).
AII = symplectic class (time-reversal symmetry that squares to -1)

TABLE II. Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (a) in the complex symmetry classes, predicted from the chiral U(1) anomaly, and (b) in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the chiral anomaly in the presence of both background gravity and U(1) gauge field (green).

Cartan\\d	0	1	2	3	4	5	6	7	8	9	10	11	···
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	···
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	···
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	···
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	···
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	···
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	···
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	···
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	···
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	···
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	···

Spintronic applications of 3D TIs

This is a very active area on the archive, but most of what is discussed is very simple:



a charge current at one TI surface has a nonzero average spin. The same is true for a Rashba quantum well, where the two electron sheets almost cancel; in a TI there is only one sheet and the effect is much stronger.

Stability, or
Phases versus points

True quantum phases in condensed matter systems should be robust to *disorder* and *interactions*.

Examples:
The Fermi gas is robust to repulsive interactions in 2D and 3D (the "Fermi liquid") but *not* in 1D. In 1D, conventional metallic behavior is only seen at one fine-tuned point in the space of interactions.

The Fermi gas is robust to disorder in 3D but not in 1D or 2D (*Anderson localization*): the clean system is only a point in phase space in 1D or 2D.

The IQHE is a phase robust to both disorder and interactions.

What about the SQHE? Is it a new phase of condensed matter?

Remark on simple generalization of IQHE topology

TKNN, 1982: the Hall conductance is related to an integral over the magnetic Brillouin zone: $\sigma_{xy} = n\frac{e^2}{h}$

$$n = \sum_{bands} \frac{i}{2\pi} \int d^2k \left(\left\langle \frac{\partial u}{\partial k_1} \middle| \frac{\partial u}{\partial k_2} \right\rangle - \left\langle \frac{\partial u}{\partial k_2} \middle| \frac{\partial u}{\partial k_1} \right\rangle \right)$$

Niu, Thouless, Wu, 1985: many-body generalization
more generally, introducing "twist angles" around the two circles of a torus and considering the (assumed unique) ground state as a function of these angles,

$$n = \int_0^{2\pi} \int_0^{2\pi} d\theta \, d\varphi \, \frac{1}{2\pi i} \left| \left\langle \frac{\partial \phi_0}{\partial \varphi} \middle| \frac{\partial \phi_0}{\partial \theta} \right\rangle - \left\langle \frac{\partial \phi_0}{\partial \theta} \middle| \frac{\partial \phi_0}{\partial \varphi} \right\rangle \right|$$

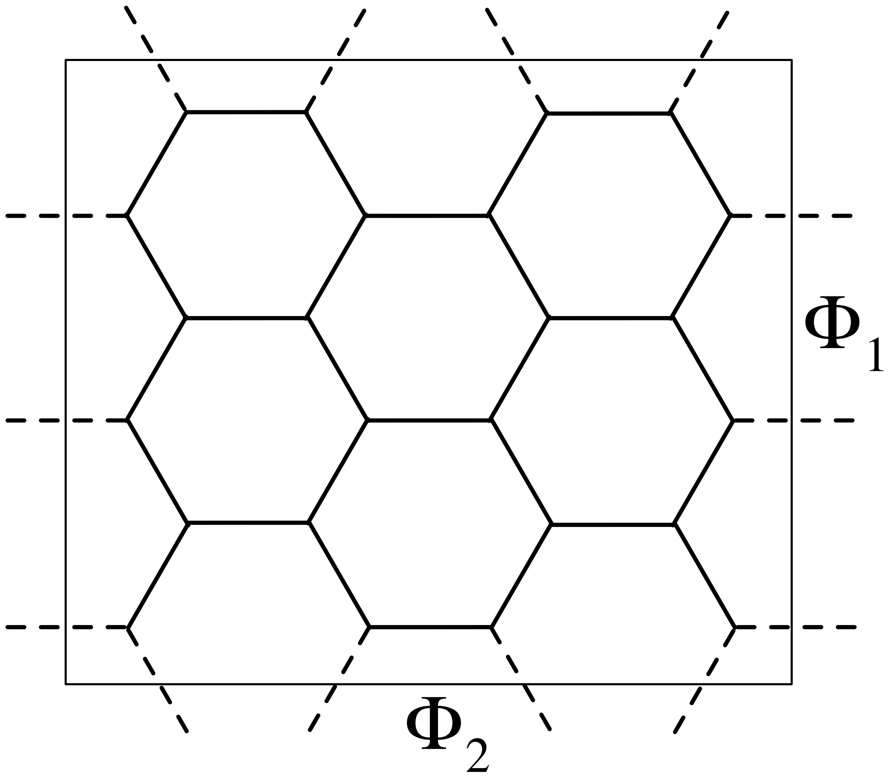
This quantity is an integer.
For T-invariant systems, all ordinary Chern numbers are zero.

Redefining the Berry phase with disorder

Suppose that the parameters in *H* do not have exact lattice periodicity.

Imagine adding boundary phases to a finite system, or alternately considering a "supercell". Limit of large supercells -> disordered system.

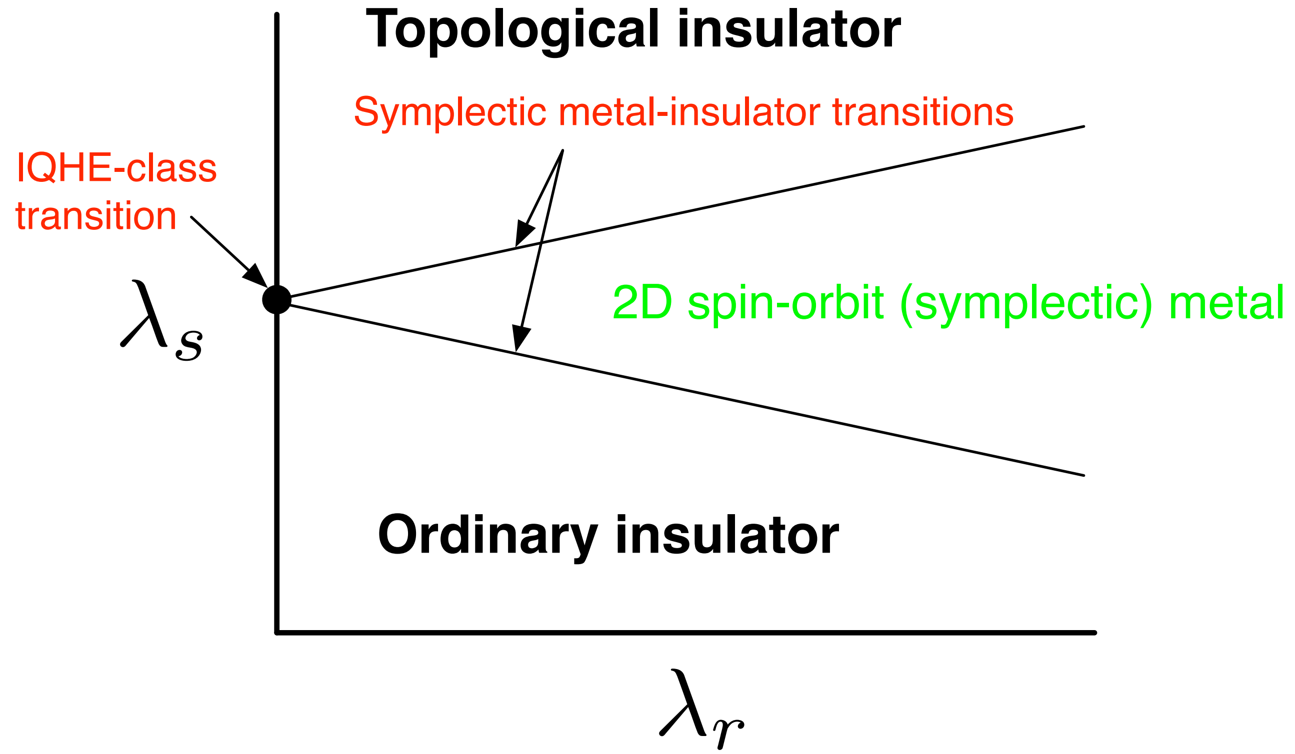
Effect of boundary phase is to shift *k*: alternate picture of topological invariant is in terms of half the (Φ_1,Φ_2) torus.

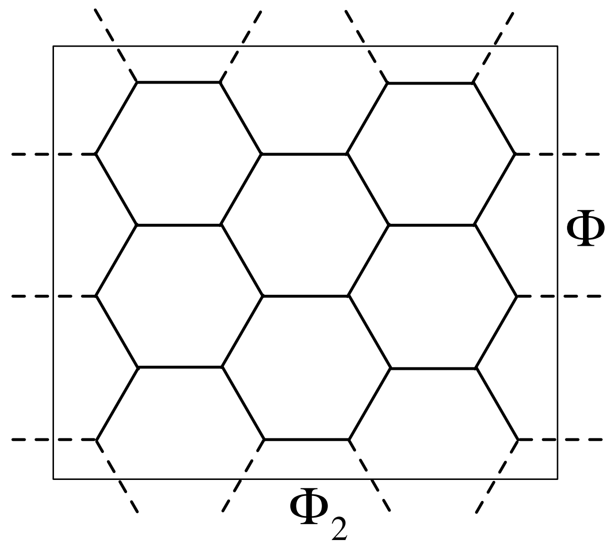


Can define Chern parities by pumping, analogous to Chern numbers, and study phase diagram w/disorder

The 2D topological insulator with disorder

Spin-orbit *T=0* phase diagram (fix spin-independent part): instead of a point transition between ordinary and topological insulators, have a symplectic metal in between.





We compute this numerically using Fukui-Hatsugai algorithm (PRB 2007) to compute invariants in terms of *boundary phases* (A. Essin and JEM, PRB 2007). See also Obuse et al., Onoda et al. for other approaches with higher accuracy->scaling exponents for transitions; Ryu et al. for theory.

Summary of recent experiments

1. There are now at least 3 strong topological insulators that have been seen experimentally (Bi_xSb_{1-x}, Bi_2Se_3, Bi_2Te_3).

2. Their metallic surfaces exist in zero field and have the predicted form.

3. These are fairly common bulk 3D materials (and also ^{3}He B).

4. The temperature over which topological behavior is observed can extend up to room temperature or so.

What's left

What is the physical effect or response that defines a topological insulator beyond single electrons?

(What are they good for?)

Are there more profound consequences of geometry and topologiy?
Lecture 2: Many basic phenomena in matter
Lecture 3: New types of particles, with new types of statistics
Lecture 4:The future

But first we need a few basic notions from topology.

Berry phase review

Why do we write the phase in this form?
Does it depend on the choice of reference wavefunctions?

$$\phi = \oint \mathcal{A} \cdot d\mathbf{k}, \quad \mathcal{A} = \langle \psi_k | -i\nabla_k | \psi_k \rangle$$

If the ground state is non-degenerate, then the only freedom in the choice of reference functions is a local phase:

$$\psi_k \to e^{i\chi(k)} \psi_k$$

Under this change, the "Berry connection" *A* changes by a gradient,

$$\mathcal{A} \to \mathcal{A} + \nabla_k \chi$$

just like the vector potential in electrodynamics.

So loop integrals of *A* will be gauge-invariant,
as will the *curl* of *A*, which we call the "Berry curvature".

$$\mathcal{F} = \nabla \times \mathcal{A}$$

How can we picture *A*?

$$\phi = \oint \mathcal{A} \cdot d\mathbf{k}, \quad \mathcal{A} = \langle \psi_k | -i\nabla_k | \psi_k \rangle$$

To get a physical interpretation of what *A* means, note that
if we consider a plane wave exp(*i k r*), then the vector potential
just gives the position **r**.

Now in a periodic crystal, the position can't be uniquely defined,
but we nevertheless expect that *A* might reflect something to do
with the position of the wavefunction *within the unit cell.*

$$\mathcal{F} = \nabla \times \mathcal{A}$$

What about non-magnetic insulators?

Electrical polarization: another simple Berry phase in solids
(Will eventually give another picture of topological insulators)

Sum the integral of *A* over bands: in one spatial dimension,

$$P = \sum_v e \int \frac{dq}{2\pi} \langle u_v(q)| - i\partial_q |u_v(q)\rangle$$

Intuitive idea: think about the momentum-position commutation relation,

$$A = \langle u_k| - i\nabla_k |u_k\rangle \approx \langle r \rangle$$

There is an ambiguity of *e* per transverse unit cell, the "polarization quantum."

Note: just as dA=F is a "closed form" and very useful to define Chern number, in 4 dimensions there is a "second Chern form"

Fact from cohomology:
Odd dimensions have Chern-Simons forms that have a "quantum" ambiguity;
Even dimensions have Chern forms that are quantized.

But what does *F* do?

It is useful to get some intuition about what the Berry F means in simpler physical systems first.

Its simplest consequence is that it modifies the semiclassical equations of motion of a Bloch wavepacket:

$$\frac{dx^a}{dt} = \frac{1}{\hbar}\frac{\partial \epsilon_n(\mathbf{k})}{\partial k_a} + \mathcal{F}_n^{ab}(\mathbf{k})\frac{dk_b}{dt}.$$

a "magnetic field" in momentum space.

The anomalous velocity results from changes in the electron distribution *within the unit cell*: the Berry phase is connected to *the electron spatial location.*

Example I: the intrinsic anomalous Hall effect in itinerant magnets
still no universal agreement on its existence

Example II: helicity-dependent photocurrents in optically active materials
(Berry phases in nonlinear transport)

But what does *F* do?

Example I: the anomalous Hall effect in itinerant magnets

An electrical field **E** induces a transverse current through the anomalous velocity if *F* is nonzero averaged over the ground state.

$$\frac{dx^a}{dt} = \frac{1}{\hbar}\frac{\partial \epsilon_n(\mathbf{k})}{\partial k_a} + \mathcal{F}_n^{ab}(\mathbf{k})\frac{dk_b}{dt}.$$

A nonzero Hall current requires T breaking; microscopically this follows since time-reversal symmetry implies

$$\mathcal{F}^{ab}(\mathbf{k}) = -\mathcal{F}^{ab}(-\mathbf{k}).$$

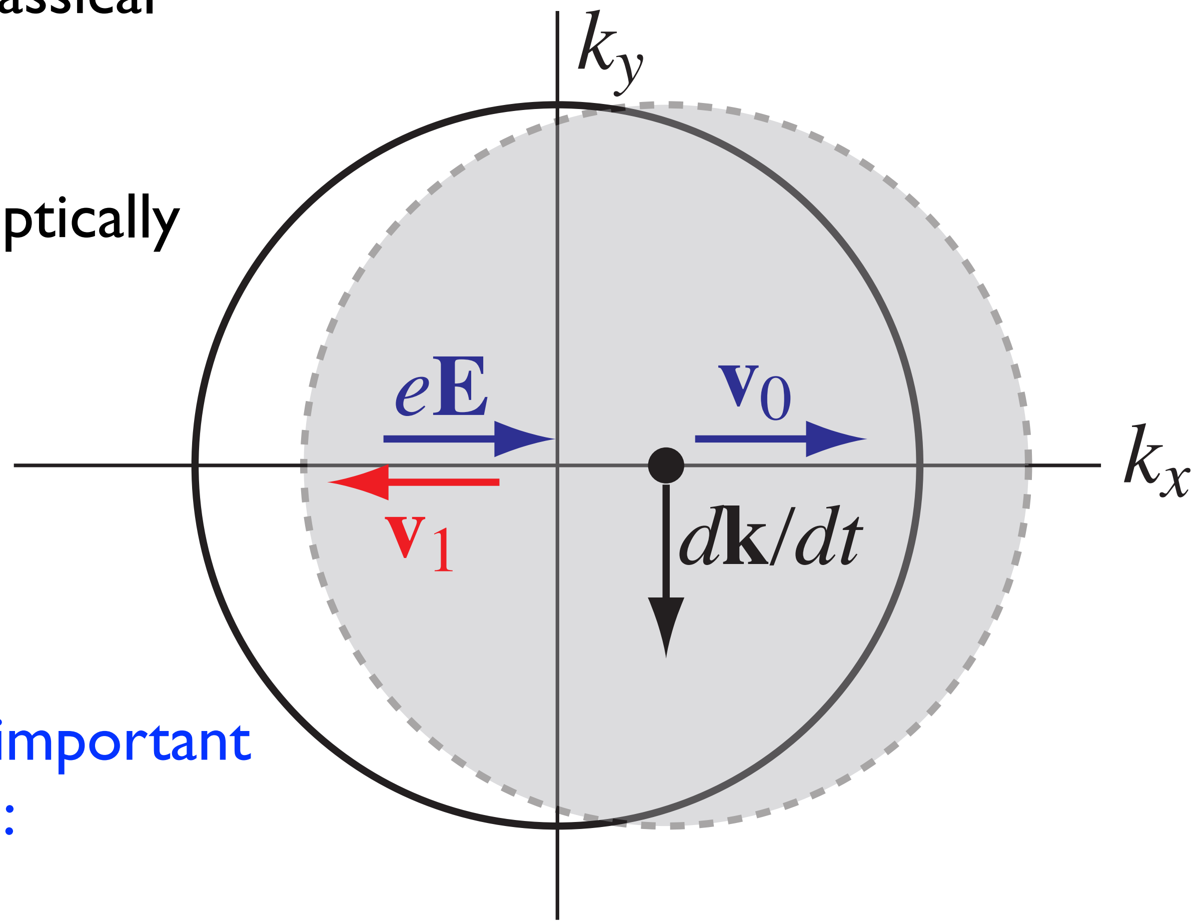
Smit's objection: in steady state the electron distribution is stationary; why should the anomalous velocity contribute at all?

(In a quantum treatment, the answer is as if *dk/dt* resulted only from the macroscopic applied field, which is mostly consistent with experiment)

But what does *F* do?

To try to resolve the question of what the semiclassical equation means:

Example II: helicity-dependent photocurrents in optically active materials
(Berry phases in nonlinear transport)



In a T-symmetric material, the Berry phase is still important at finite frequency. Consider circular polarization:

The small deviation in the electron distribution generated by the electrical field gives an anomalous velocity contribution that need not average to zero over the wave.

Smit vs. Luttinger

The resulting formula has 3 terms, of which one is "Smit-type" (i.e., nonzero even with the full **E**) and two are "Luttinger-type".

$$\beta = \frac{\partial F}{\partial k_x}$$

$$\mathbf{j}_{dc} = \frac{\beta n e^3}{2\hbar^2} \frac{1}{1/\tau^2 + \omega^2} \Big[i\omega (E_x E_y^* - E_y E_x^*) \hat{\mathbf{x}} + 1/\tau (E_x E_y^* + E_y E_x^*) \hat{\mathbf{x}} + |E_x|^2 \hat{\mathbf{y}} \Big].$$

(JEM and J. Orenstein, 2009). The full semiclassical transport theory of this effect was given by Deyo, Golub, Ivchenko, and Spivak (arXiv, 2009).

We believe that the circularly switched term actually explains a decade of experiments on helicity-dependent photocurrents in GaAs quantum wells.

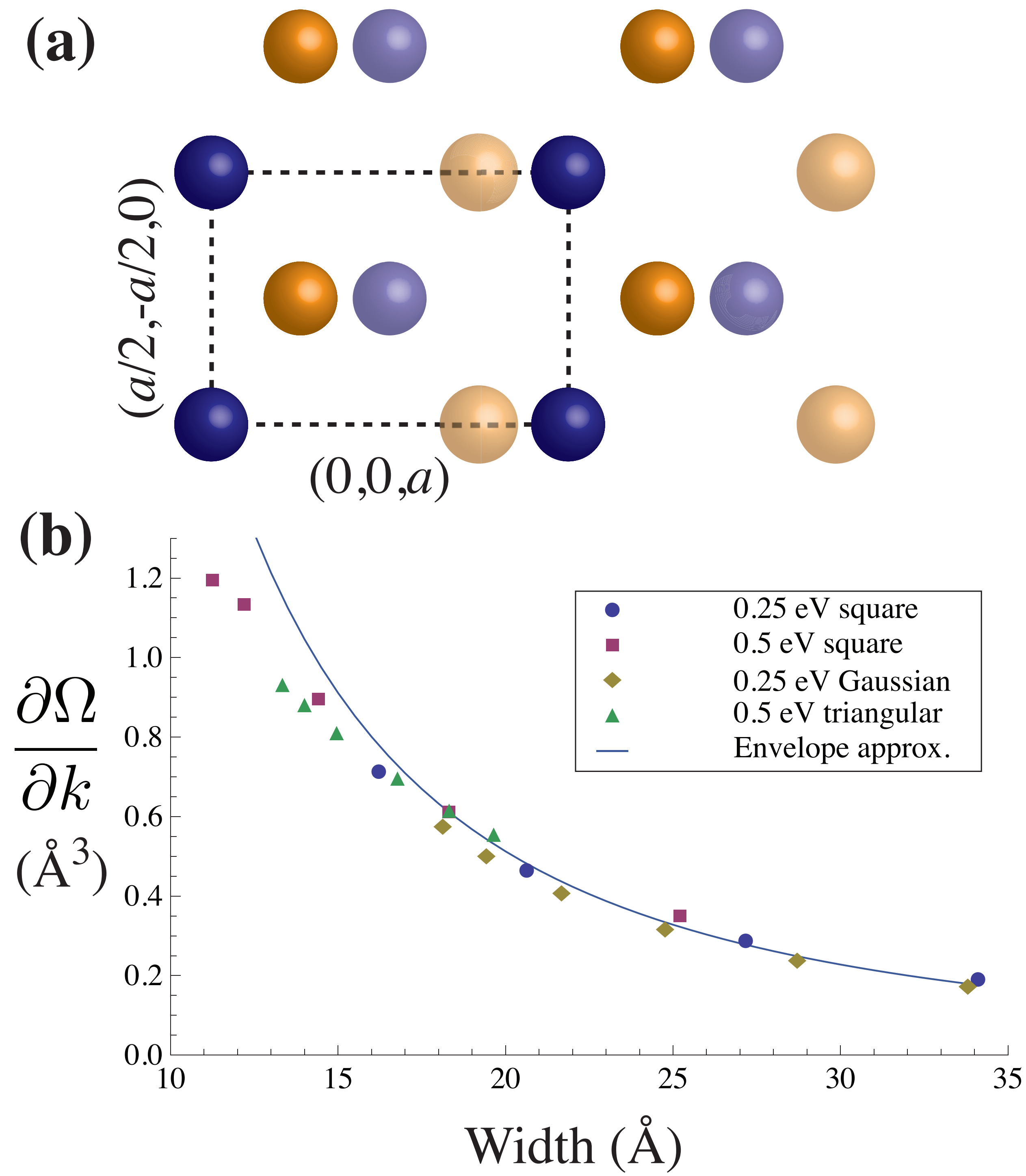
Bulk GaAs has too much symmetry to allow the effect; these quantum wells show the effect because the well confinement breaks the symmetry ("confinement-induced Berry phase").

Confinement-induced Berry phases

Bulk GaAs has too much symmetry to allow the effect; these quantum wells show the effect because the well confinement breaks the symmetry ("confinement-induced Berry phase").

Our numerics and envelope approximation suggest
a magnitude of 1 nA for incident power 1W in a (110) well, which is consistent with experiments by S. D. Ganichev et al. (Regensburg).

Only one parameter of GaAs is needed to describe F at the Brillouin zone origin: symmetries force



$$\mathcal{F} = \lambda \left(k_x(k_y^2 - k_z^2), k_y(k_z^2 - k_x^2), k_z(k_x^2 - k_y^2) \right), \lambda \approx 410 \mathring{A}^3$$

Electrodynamics in insulators

We know that the constants ε and μ in Maxwell's equations can be modified inside an ordinary insulator.

Particle physicists in the 1980s considered what happens if a 3D insulator creates a new term ("axion electrodynamics", Wilczek 1987)

$$\Delta\mathcal{L}_{EM} = \frac{\theta e^2}{2\pi h}\mathbf{E}\cdot\mathbf{B} = \frac{\theta e^2}{16\pi h}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}.$$

This term is a total derivative, unlike other magnetoelectric couplings.
It is also "topological" by power-counting.

The angle θ is periodic and odd under T.

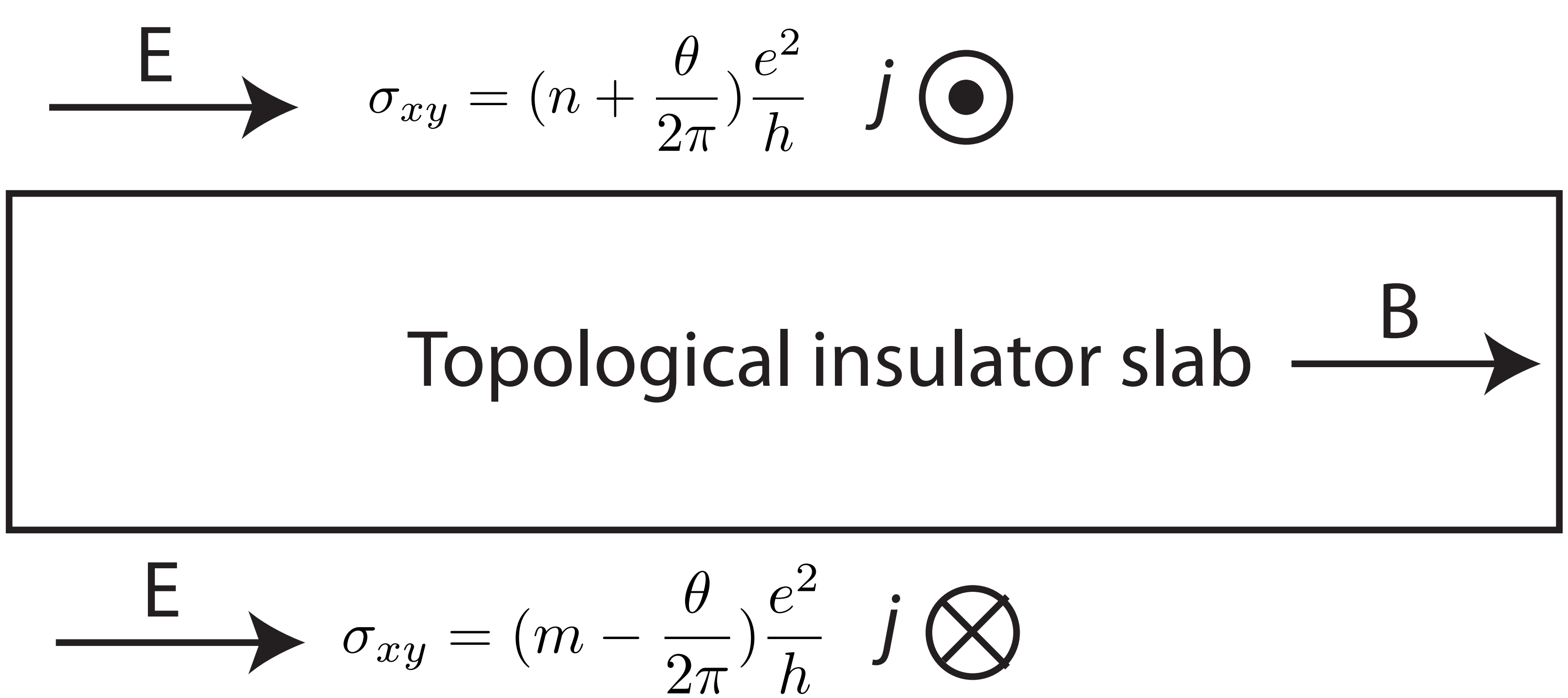
A T-invariant insulator can have *two* possible values: 0 or π.

Axion E&M

$$\Delta\mathcal{L}_{EM} = \frac{\theta e^2}{2\pi h}\mathbf{E}\cdot\mathbf{B} = \frac{\theta e^2}{16\pi h}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}.$$

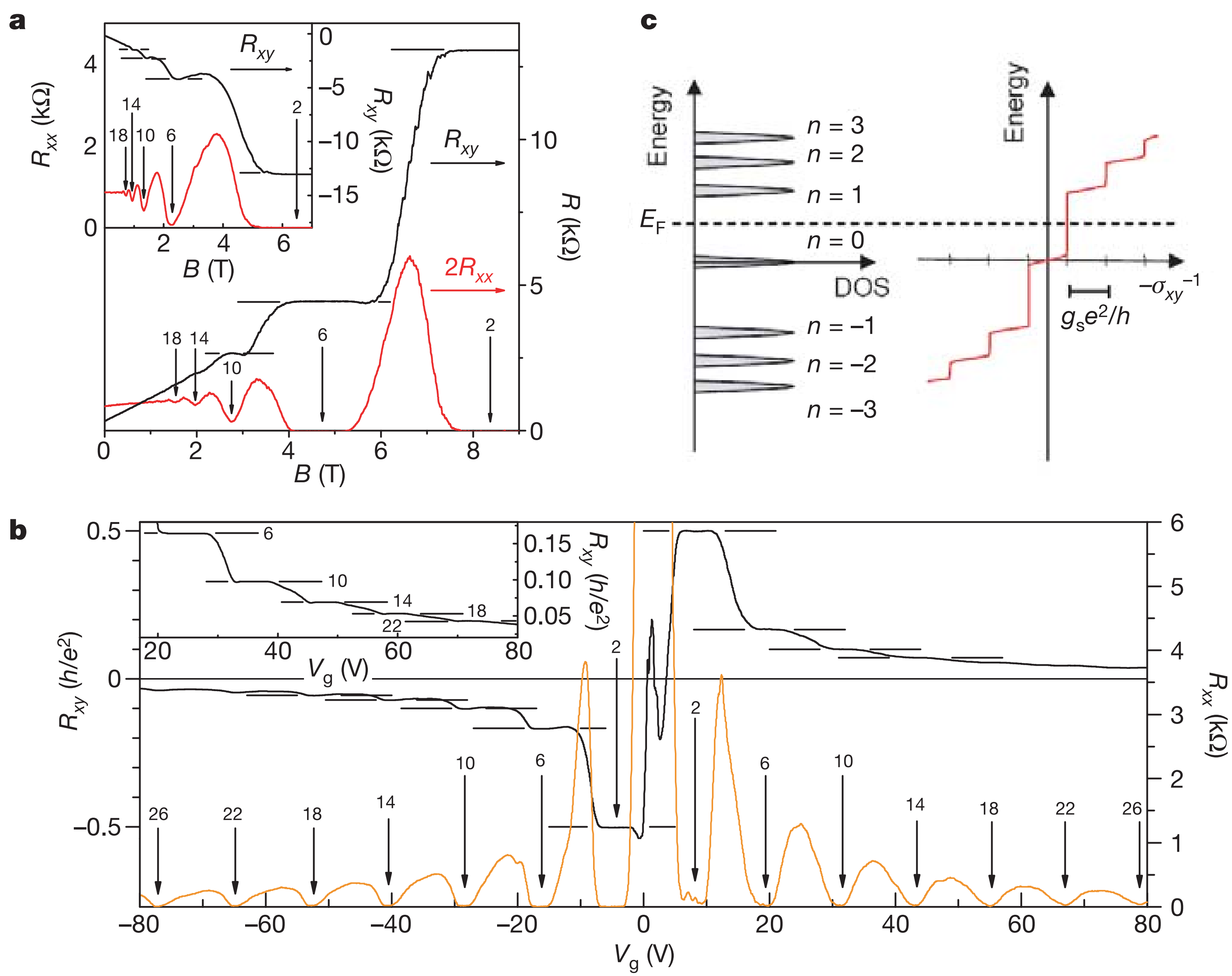
This explains a number of properties of the 3D topological insulator when its surfaces become gapped by breaking T-invariance:

Magnetoelectric effect:
applying B generates polarization P, applying E generates magnetization M)



Graphene QHE

The connection is that a single Dirac fermion contributes a *half-integer QHE*: this is seen directly in graphene if we recall the extra fourfold degeneracy. (Columbia data shown below)



Topological response

Idea of "axion electrodynamics in insulators"

there is a "topological" part of the magnetoelectric term

$$\Delta \mathcal{L}_{EM} = \frac{\theta e^2}{2\pi h} \mathbf{E} \cdot \mathbf{B} = \frac{\theta e^2}{16\pi h} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}.$$

that is measured by the orbital magnetoelectric polarizability

$$\theta \frac{e^2}{2\pi h} = \frac{\partial M}{\partial E} = \frac{\partial}{\partial E} \frac{\partial}{\partial B} H = \frac{\partial P}{\partial B}$$

and computed by integrating the "Chern-Simons form" of the Berry phase

$$\theta = -\frac{1}{4\pi} \int_{\mathrm{BZ}} d^3k \, \epsilon_{ijk} \, \mathrm{Tr}[\mathcal{A}_i \partial_j \mathcal{A}_k - i\frac{2}{3} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k] \qquad (2)$$

(Qi, Hughes, Zhang, 2008; Essin, JEM, Vanderbilt 2009)

This integral is quantized only in T-invariant insulators, but contributes in all insulators.

Topological response

Many-body definition: the Chern-Simons or second Chern formula does not directly generalize. However, the quantity dP/dB *does* generalize:
a clue is that the "polarization quantum" combines nicely with the flux quantum.

$$\frac{\Delta P}{B_0} = \frac{e/\Omega}{h/e\Omega} = e^2/h.$$

So dP/dB gives a *bulk*, *many-body* test for a topological insulator.

(Essin, JEM, Vanderbilt 2009)

$\frac{e^2}{h}$ = contact resistance in 0D or 1D
= Hall conductance quantum in 2D
= magnetoelectric polarizability in 3D

Orbital magnetoelectric polarizability

One mysterious fact about the previous result:

We indeed found the "Chern-Simons term" from the semiclassical approach.

But in that approach, it is not at all clear why this should be the only magnetoelectric term from orbital motion of electrons.

More precisely: on general symmetry grounds, it is natural to decompose the tensor into *trace* and *traceless* parts

$$\frac{\partial P^i}{\partial B^j} = \frac{\partial M_j}{\partial E_i} = \alpha^i_j = \tilde{\alpha}^i_j + \alpha_\theta \delta^i_j.$$

The traceless part can be further decomposed into symmetric and antisymmetric parts.
(The antisymmetric part is related to the "toroidal moment" in multiferroics;
cf. M. Fiebig and N. Spaldin)

But consideration of simple "molecular" models shows that even the trace part is not always equal to the Chern-Simons formula...

Orbital magnetoelectric polarizability

Computing orbital dP/dB in a fully quantum treatment reveals that there are additional terms in general. (Essin et al., 1002.0290)
For dM/dE approach and numerical tests, see Malashevich, Souza, Coh, Vanderbilt, 1002.0300.

$$\begin{aligned}
\alpha^i_j &= (\alpha_I)^i_j + \alpha_{CS}\delta^i_j \\
(\alpha_I)^i_j &= \sum_{\substack{n\,\mathrm{occ}\\ m\,\mathrm{unocc}}} \int_{\mathrm{BZ}} \frac{d^3k}{(2\pi)^3} \mathrm{Re}\left\{ \frac{\langle u_{n\mathbf{k}}|e\,\not{r}^i_{\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|e(\mathbf{v}_{\mathbf{k}}\times \not{r}_{\mathbf{k}})_j - e(\not{r}_{\mathbf{k}}\times\mathbf{v}_{\mathbf{k}})_j - 2i\partial H'_{\mathbf{k}}/\partial B^j|u_{n\mathbf{k}}\rangle}{E_{n\mathbf{k}}-E_{m\mathbf{k}}} \right\} \\
\alpha_{CS} &= -\frac{e^2}{2\hbar}\epsilon_{abc}\int_{\mathrm{BZ}}\frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\left[\mathcal{A}^a\partial^b\mathcal{A}^c - \frac{2i}{3}\mathcal{A}^a\mathcal{A}^b\mathcal{A}^c\right].
\end{aligned}$$

The "ordinary part" indeed looks like a Kubo formula of electric and magnetic dipoles.

Not inconsistent with previous results:
in topological insulators, time-reversal means that only the Berry phase term survives.

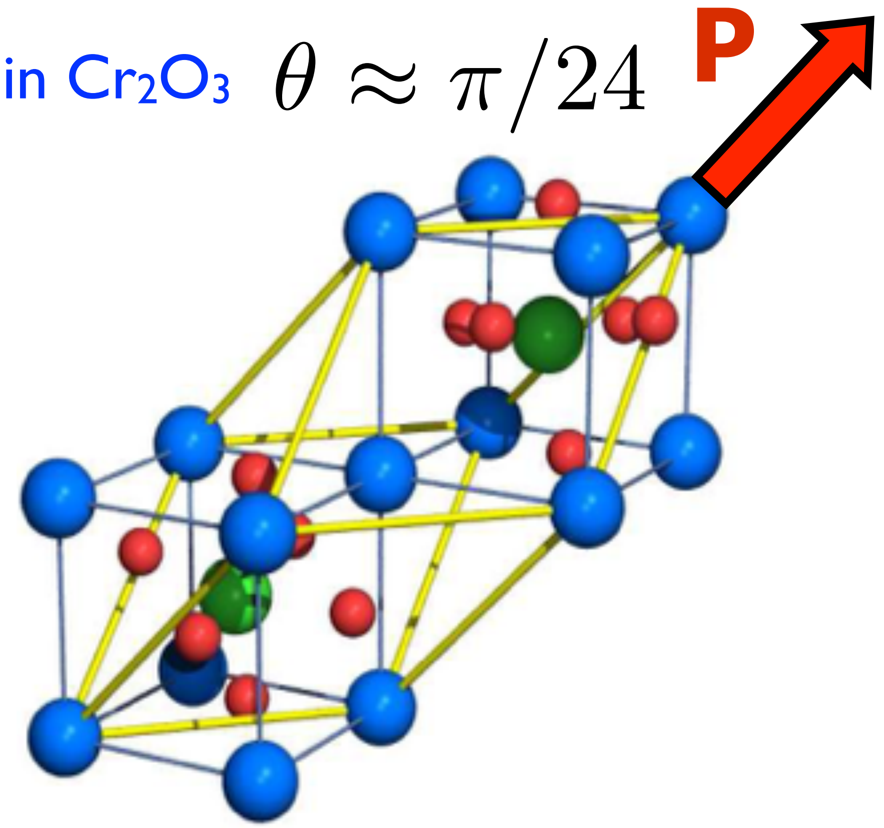
There is an "ordinary part" and a "topological part", which is scalar but is the only nonzero part in TIs. But the two are not physically separable in general.
Both parts are nonzero in multiferroic materials.

Magnetoelectric theory: a spinoff of TIs

This leads to a general theory for the orbital magnetoelectric response tensor in a crystal, including contributions of all symmetries (Essin, Turner, Vanderbilt, JEM, 2010).

It is not a pure Berry phase in general, but *it is in topological insulators*.

Such magnetoelectric responses have been measured, e.g., in Cr_2O_3 (Obukhov, Hehl, et al.). $\theta \approx \pi/24$



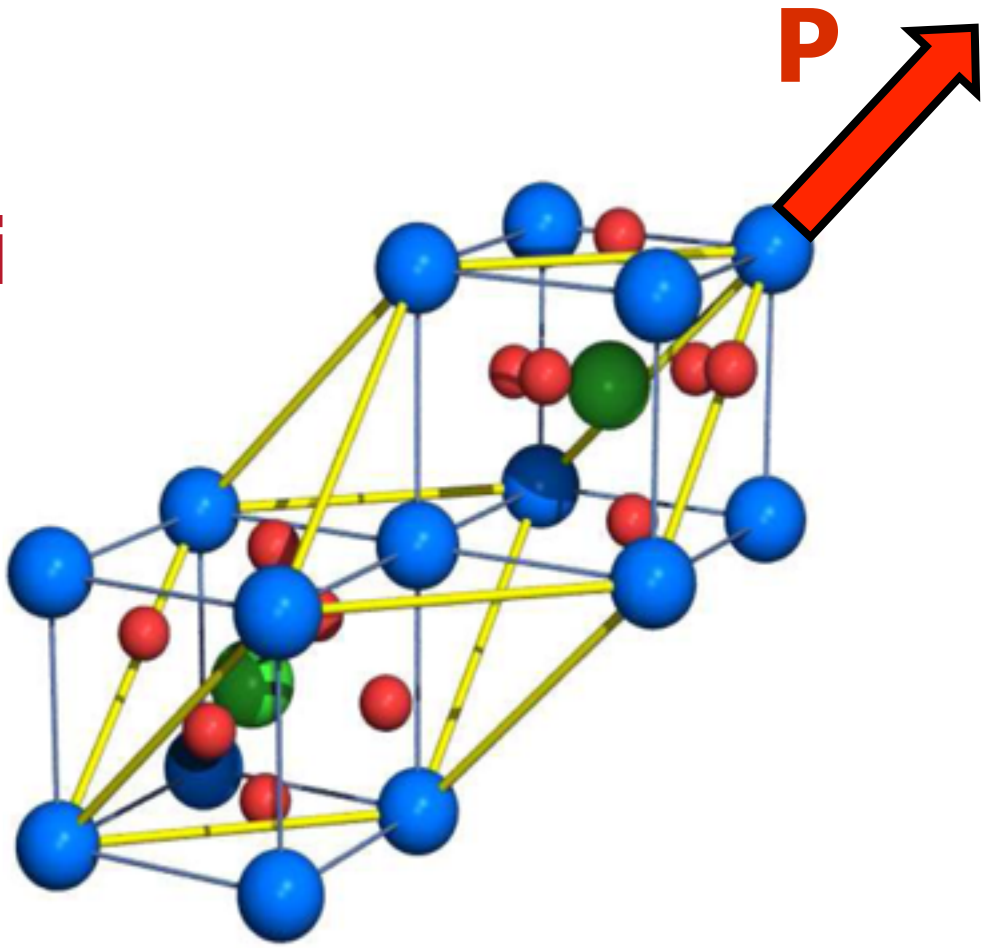
Example of the ionic "competition": $BiFeO_3$

Can make a 2x2 table of "magnetoelectric mechanisms":
(ignore nuclear magnetism)

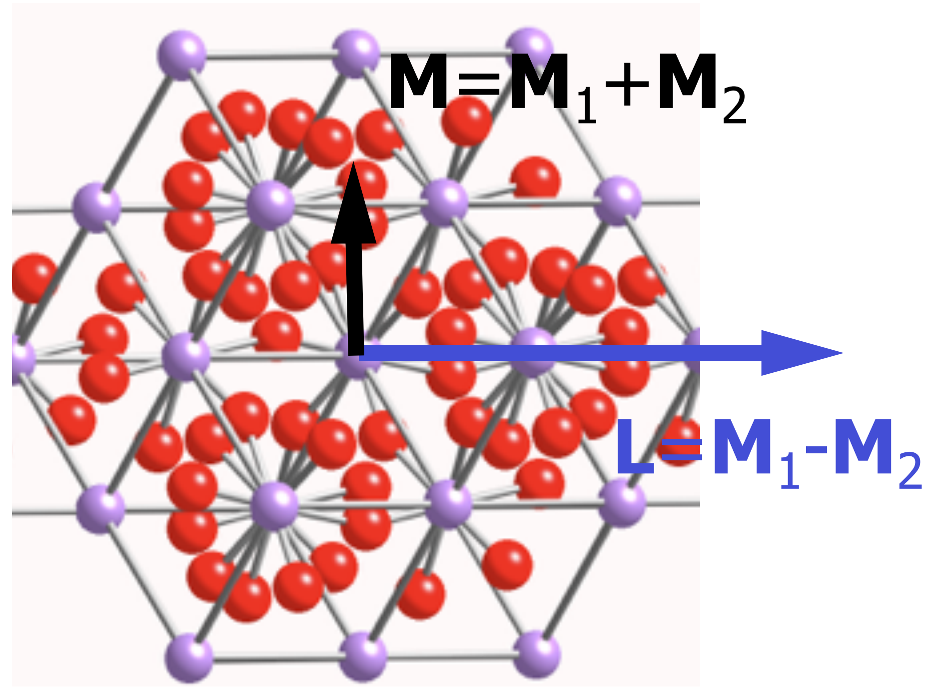
electronic P, orbital M	ionic P orbital M
electronic P, spin M	ionic P spin M

electronic P effects (left column) should be faster and less fatiguing than magnetoelectric effects requiring ionic motion.

The competition: $BiFeO_3$, a high-T multiferroic Coupled polar (P), antiferromagneti (L), and ferromagnetic (M) orders



Figs. courtesy R. Ramesh



$BiFeO_3$ BULK

- Rhombohedral R3c: a=3.96Å, α=89.46º
- No inversion symmetry, but "close"
- T_N~ 650K; T_C ~ 1120K
- Spiral, canted AFM order
- P ~ 6 $\mu C/cm^2$

$BiFeO_3$ FILM on (100) $SrTiO_3$

- Tetragonal distortion a=3.91Å, c=4.06Å
- Homogeneous, canted AFM order
- Giant ME effect: P ~ 90 $\mu C/cm^2$

Lecture III

1. How is the TI surface metal different from an ordinary metal?

2. What are other geometric/topological effects in simple metals?

3. What is the effective theory (similar in spirit to Ginzburg-Landau) for Abelian quantum Hall states?

Theme: When is the factor of 2 between an ordinary metal and the 2D and 3D edge/ surface states more than just a factor of 2?

How can we tell in normal-state transport that the 3D TI surface is different from both graphene and conventional 2DEGs?

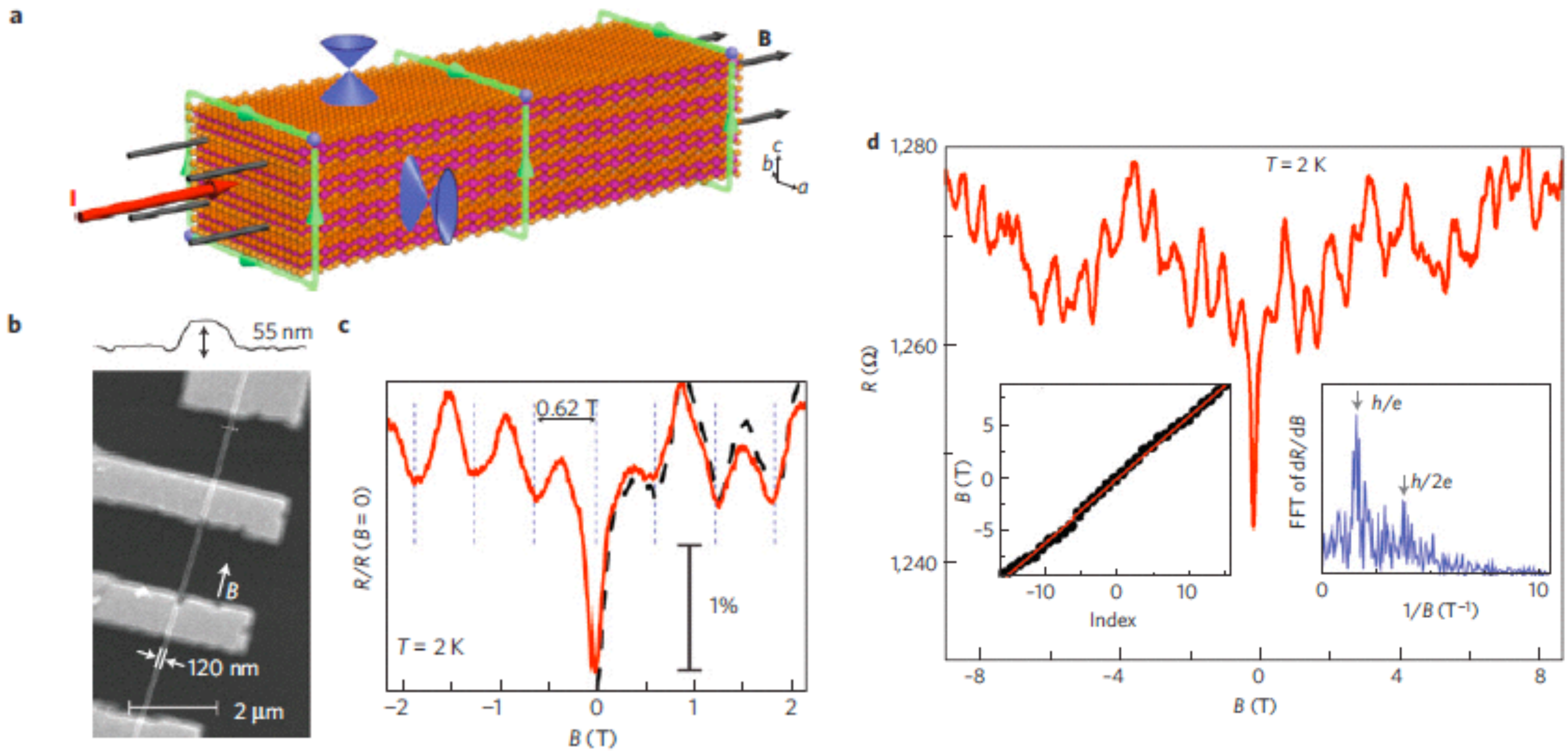
$$H = v(\sigma^x \pi_y - \sigma^y \pi_x) + \frac{g\mu_B}{2}\mathbf{B}\cdot\boldsymbol{\sigma}$$

$$(\boldsymbol{\pi} = \mathbf{p} + e\mathbf{A})$$

Beyond just having Dirac fermions, we would like a way to count them that does not require achieving quantized Hall conditions.

Conductance along a Bi2Se3 nanoribbon pierced by magnetic flux

Use geometry to isolate surface even when bulk is conducting.



H. Peng et al. (Y. Cui group), Nature Materials 9, 225 (2010)

Berry phases in transport

Puzzle: Stanford nanowire experiment (Yi Cui et al., Nature Materials)

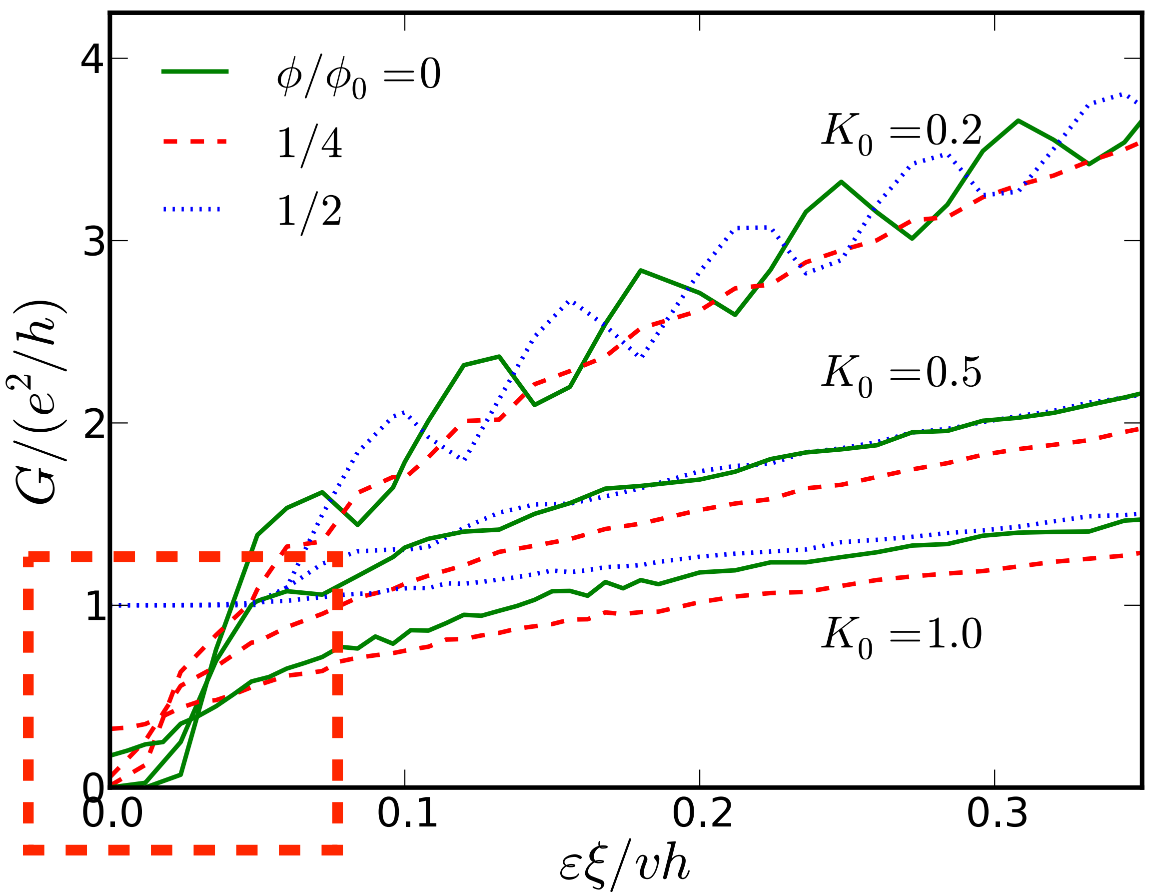
sees Aharonov-Bohm (h/e) oscillations, as expected for a clean system, rather than Sharvin & Sharvin (h/2e), as expected for a diffusive metallic cylinder.

The *sign* is also not what is expected in the strong-disorder limit: the Berry phase protects a mode at pi flux, rather than 0 flux as in a nanotube.

Intuition: spin-momentum locking means tha spin direction rotates through 2π as electro circles the cylinder. This gives a - sign that is compensated by the π flux.

(Bardarson, Brouwer, JEM, PRL 2010; Zhang and Vishwanath, PRL 2010)

K_0 = disorder strength

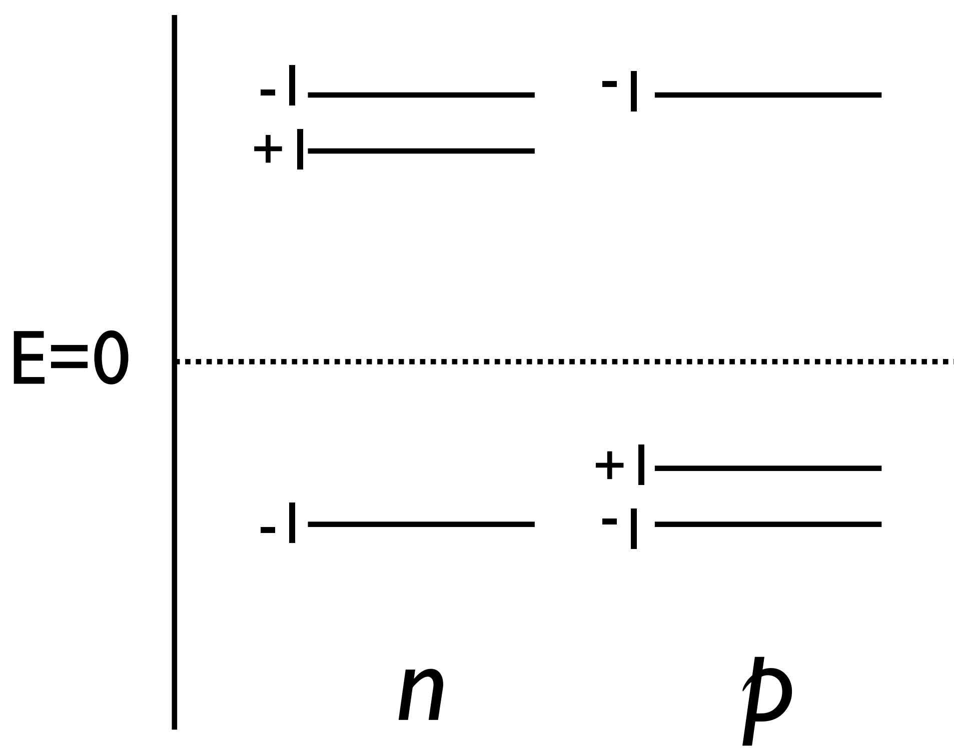


Scaled chemical potential relative to Dirac point

TSC for Majoranas and quantum computing?

Note that most of the special physics of the Moore-Read state can be understood from BdG theory of a p+ip superconductor (Read and Green, 1999).

In general, Majorana fermions appear as zero-energy solutions of Bogoliubov-de Gennes equation for quasiparticles in a superconductor



$$H = \frac{1}{2}\mathbf{v}^\dagger \tilde{G}\mathbf{v}$$

doubled (BdG) spectrum bands can become degenerate away from E=0

Majorana fermion = “half” an ordinary fermion; appears in superconductors that are “half” an ordinary superconductor

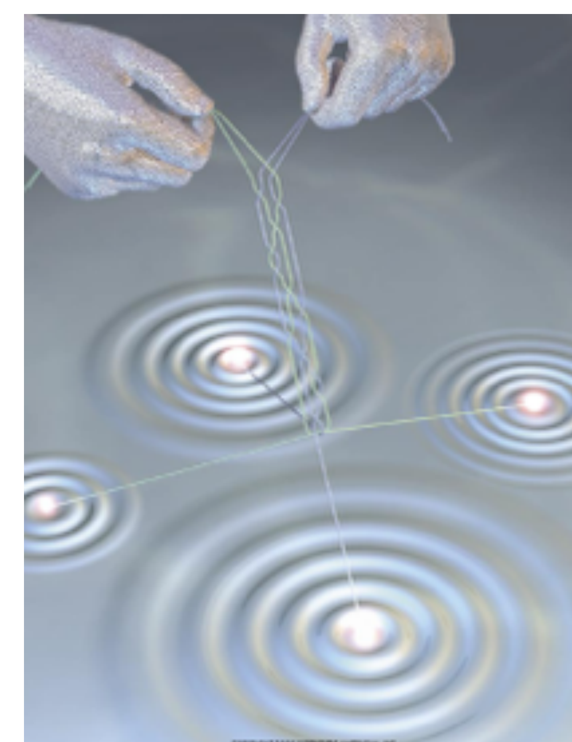
Quantum computing and memory

Majoranas for memory: 1 spinless Dirac fermion = one “qubit”:
there are two states, occupied and empty

$$\gamma_1 = (c^\dagger + c), \quad \gamma_2 = i(c^\dagger - c)$$

Majoranas alone might not be quite good enough for a universal “quantum computer”--not enough operations in the braid group?

Can either try to fix this or try to find more complex states with a “universal” representation of the braid group (12/5, 4/7)



New particles from interactions using topological insulators

1. Correlation and new particles:
There are two ways to make "Majorana fermions" from topological insulators:

Method I: start from a different "universality class", a topological *superconductor* driven by interactions (3He is an example)

Method II: build the Majorana fermions using "ordinary" topological insulators and "normal" superconductors

2. Can make a new type of vortex just by biasing a thin film of topological insulator.

Proximity effect and quantum computing

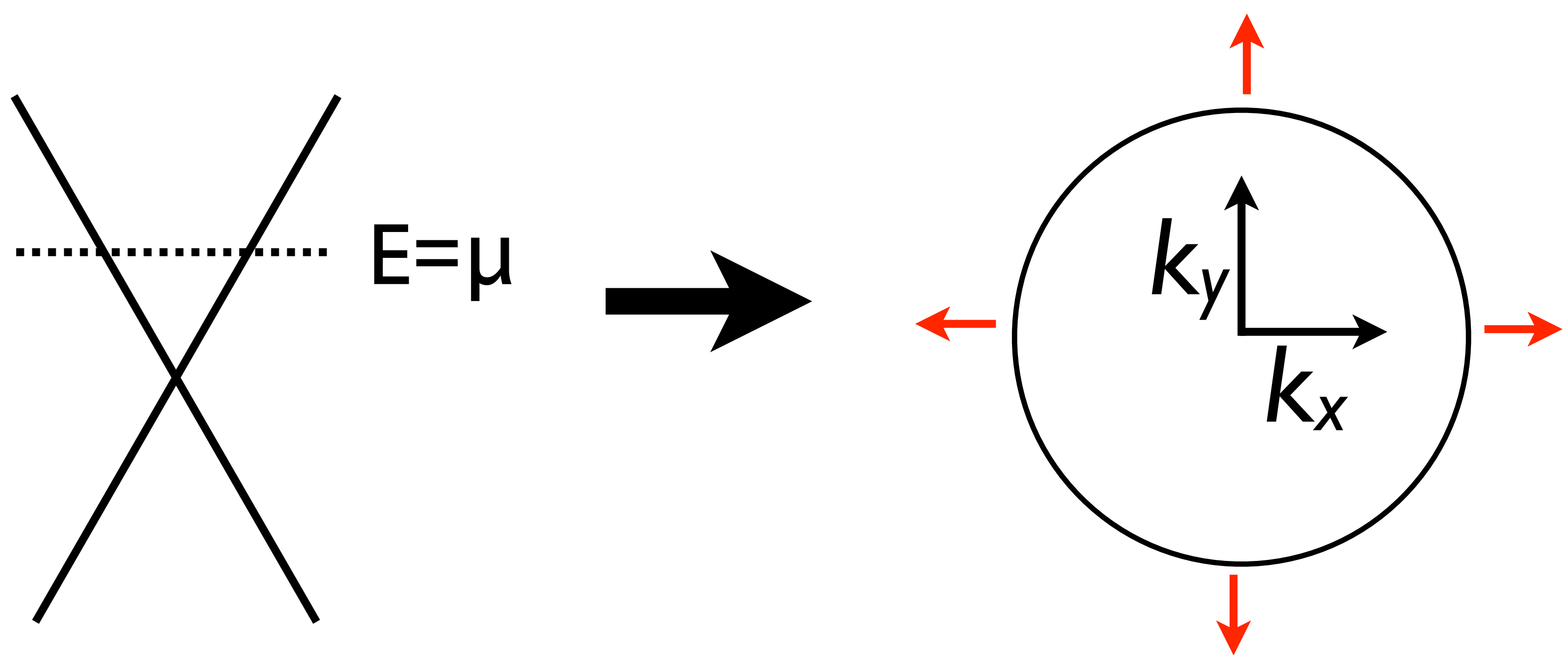
A natural question is whether the surface of a Z2 topological insulator is stable beyond single-particle models.

Time-reversal-breaking perturbations (coupling to a magnetic material or magnetic field) certainly can gap the surface modes.

What about coupling to a superconductor?

Idea: an s-wave proximity effect term $$H = \sum_{\mathbf{k}} (\Delta c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} + h.c.)$$

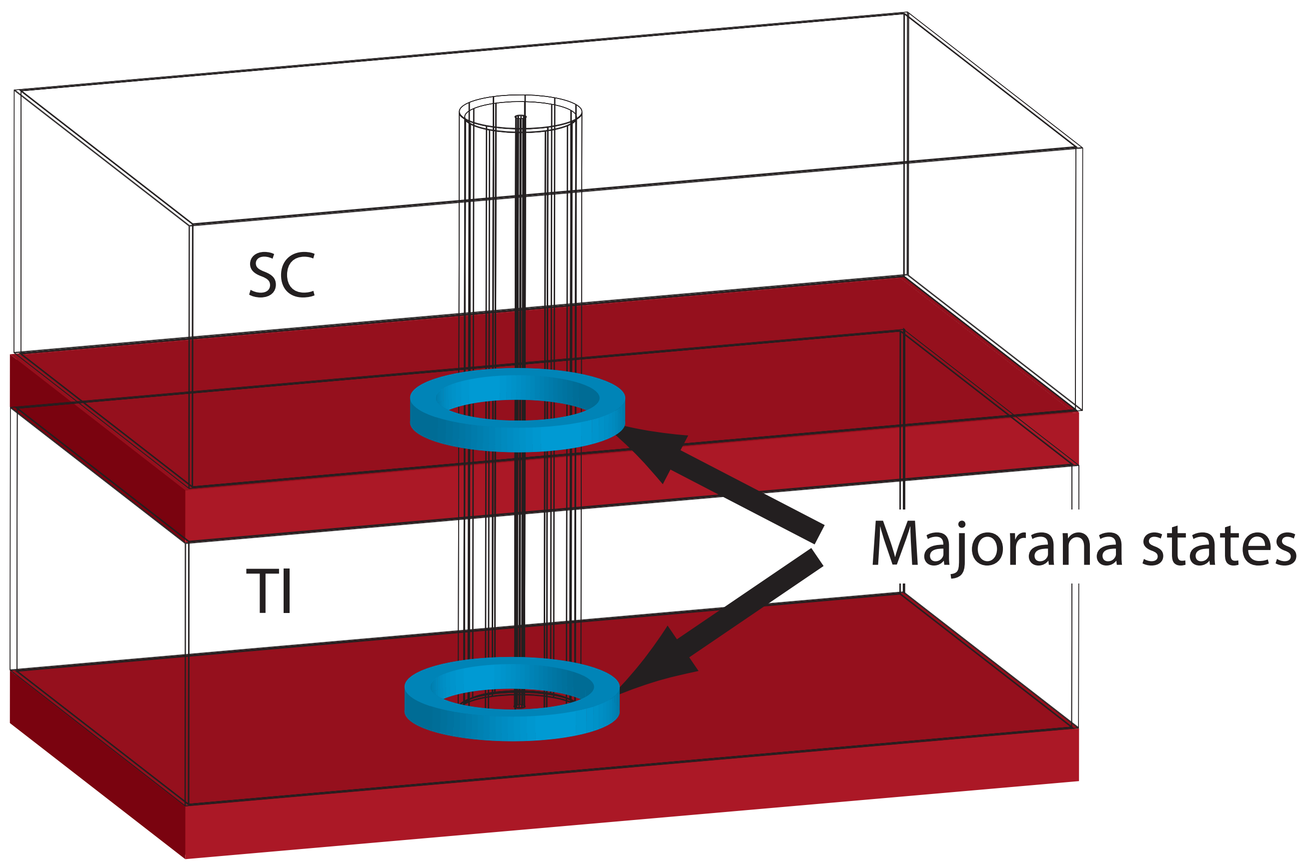
couples within the low-energy chiral fermion, and hence gives a "spinless" p-wave superconductor (Fu and Kane, PRL 2007).



Topological quantum computing

It turns out that the core of a magnetic vortex in a two-dimensional "*p*+*ip*" superconductor can have a Majorana fermion. (But we haven't found one yet.)

However, a superconducting layer with this property exists at the boundary between a 3D topological insulator and an ordinary 3D superconductor (Fu and Kane, 2007).



(Recent theoretical work by Sau et al. (Das Sarma) suggests that one doesn't even need a topological insulator. Another piece of breaking news: FQHE observed in graphene.)

Warmup: polarization in insulators

Electrical polarization: "simple" Berry phase effect in solids (took about 50 years to understand how to calculate polarization of a solid from its unit cell)

Sum the integral of *A* over bands: in one spatial dimension,

$$P = \sum_v e \int \frac{dq}{2\pi} \langle u_v(q)| - i\partial_q |u_v(q)\rangle$$

Intuitive idea: think about the momentum-position commutation relation

$$A = \langle u_k| - i\nabla_k |u_k\rangle \approx \langle r\rangle$$

More seriously: relate changes in *P* to currents moving through the unit cell.

Polarization isn't quantized in general; it is just a simple physical observable determined by the Berry phase. Note that there is an ambiguity *ne*.

Broader reason, in hindsight: E(k), the band structure, is k-symmetric with time-reversal, even with broken inversion. Anything related to inversion-breaking has to come from the wavefunction, and at low energy, usually from the Berry phase.

What about metals?

Claim: the biggest omission in Ashcroft and Mermin (standard solids text) is a term in the semiclassical equations of motion, the (Karplus-Luttinger) ***anomalous velocity***.

$$\frac{dx^a}{dt} = \frac{1}{\hbar}\frac{\partial \epsilon_n(\mathbf{k})}{\partial k_a} + \mathcal{F}_n^{ab}(\mathbf{k})\frac{dk_b}{dt}.$$

a "magnetic field" in momentum space.

The anomalous velocity results from changes in the electron distribution *within the unit cell*: the Berry phase is connected to *the electron spatial location.*

Example I: the intrinsic anomalous Hall effect in itinerant magnets (Fe, e.g.)

Example II: helicity-dependent photocurrents in optically active materials

Example III: optical rotation in gyrotropic/chiral materials with T symmetry

Can we get anything quantized/interesting in a metal?

Anomalous Hall effect (100+ years)

From Nagaosa et al., RMP 2011

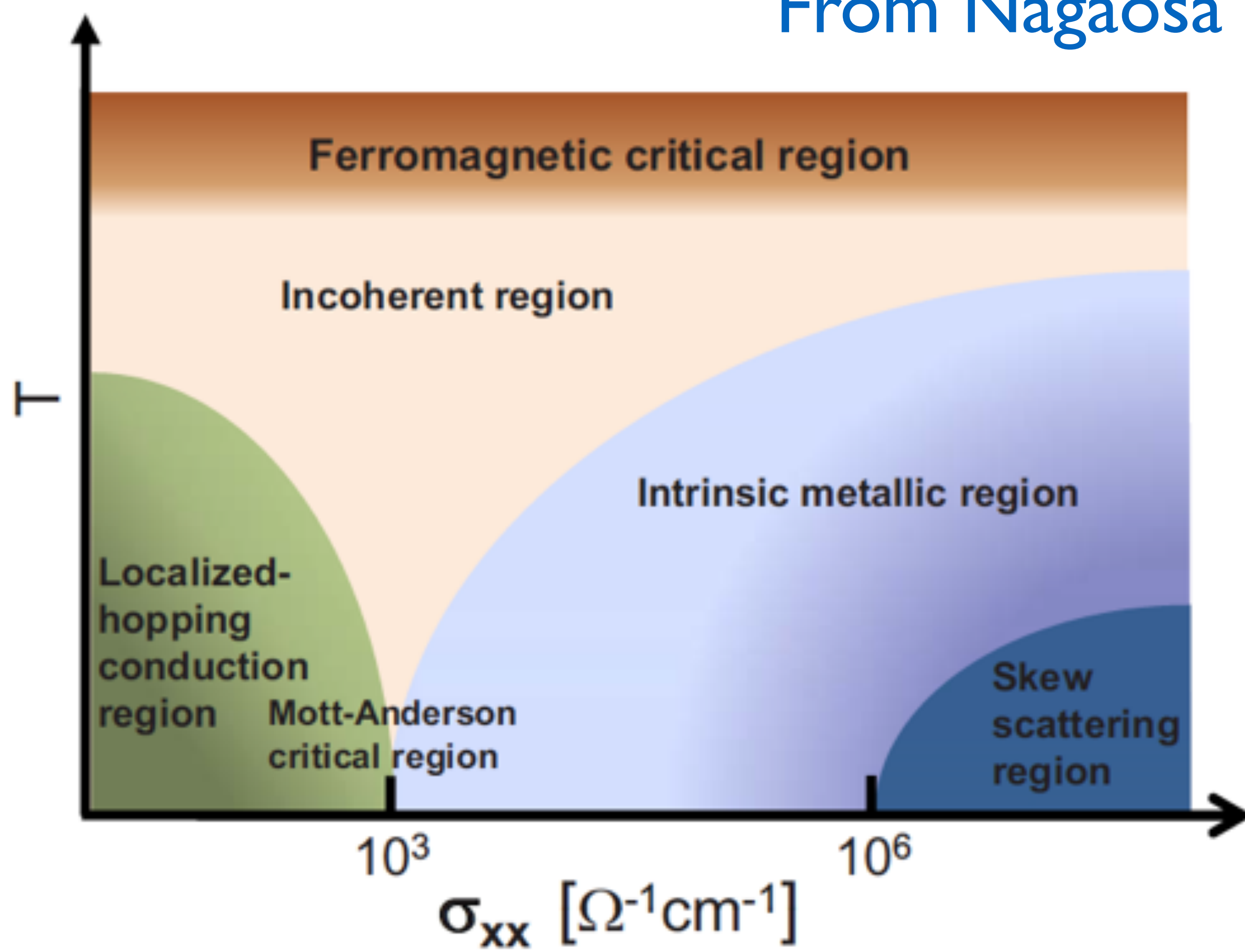


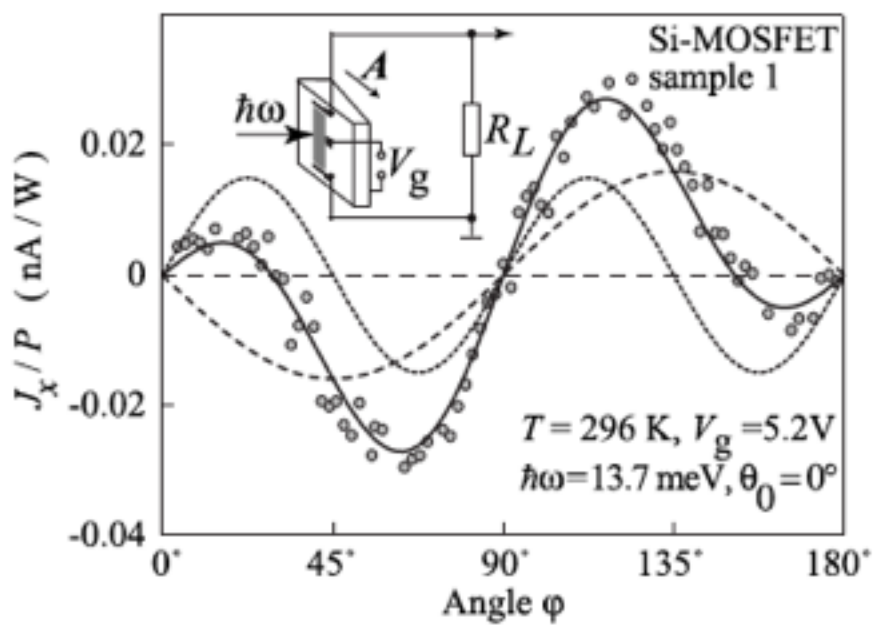
FIG. 47. (Color online) A speculative and schematic phase diagram for the anomalous Hall effect in the plane of the diagonal conductivity σ_{xx} and the temperature T.

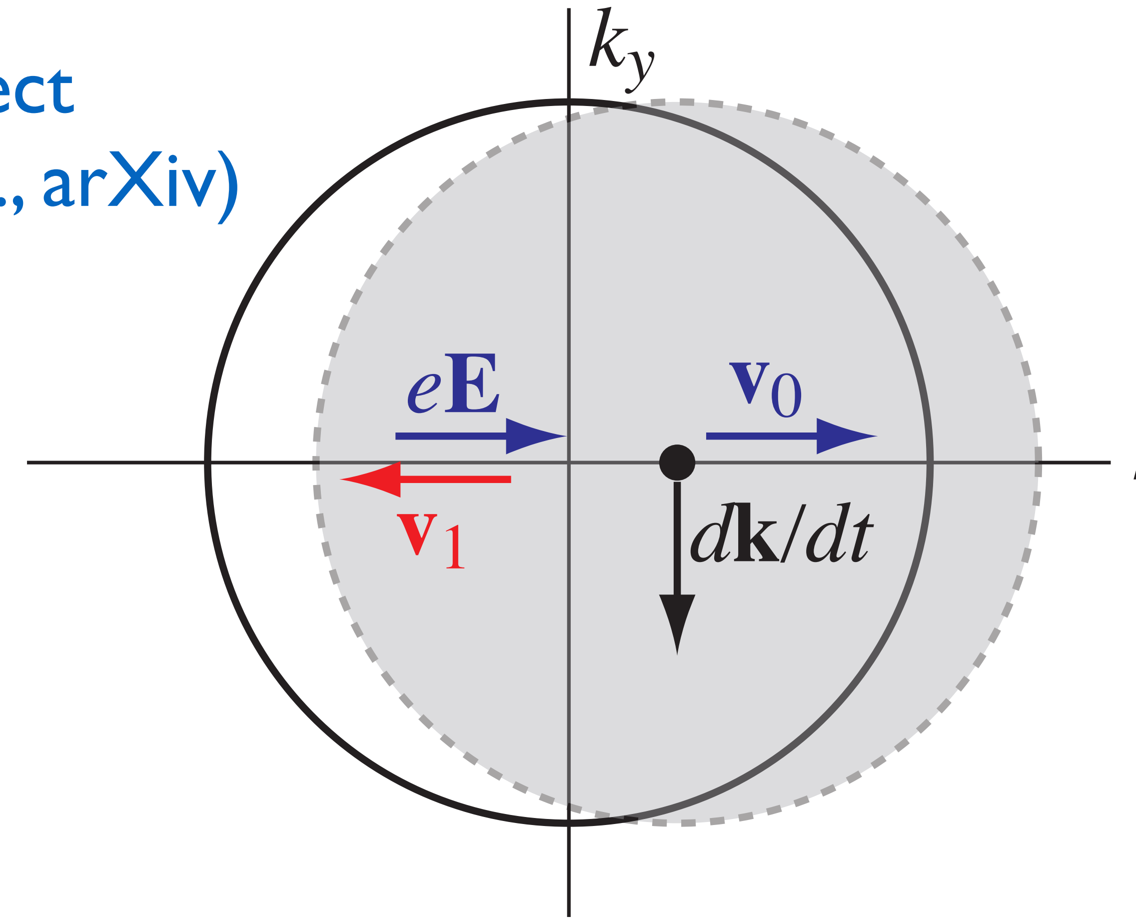
Sundaram and Niu, 1999
A topological approach:
J. Dahlhaus et al., PRB to appear

$$\sigma_{xy} = \frac{e^2}{h} \int_{\mathrm{FS}} d^2k \, \frac{F}{2\pi} + \mathrm{extrinsic}$$

Two other "mystery" effects:

Nonlinear optics: circular photogalvanic effect
(JEM and J. Orenstein, PRL 2010; Deyo et al., arXiv)



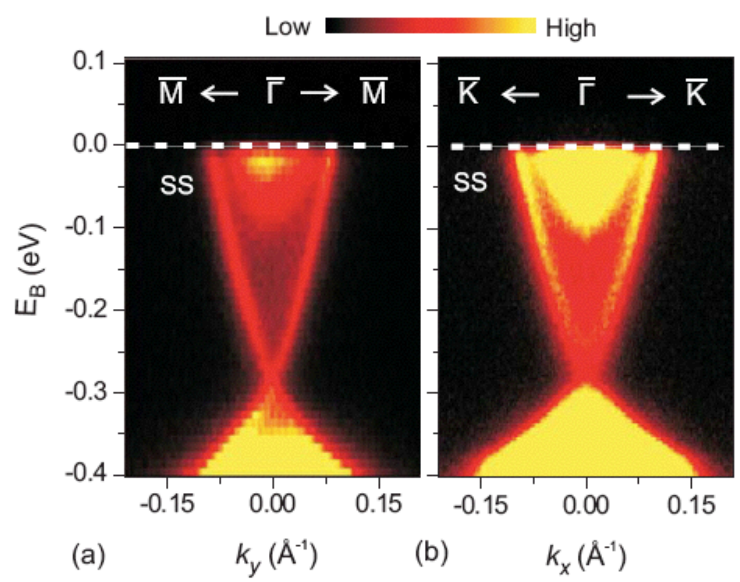


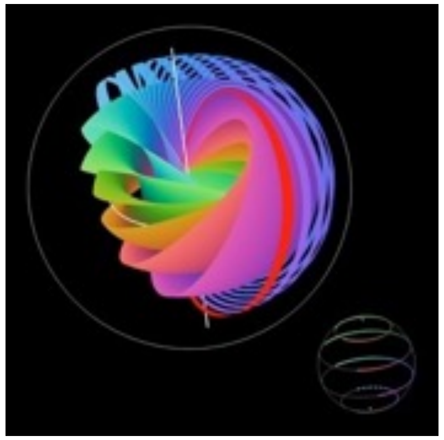
Currents are switched by the sense of circular polarization, as previously observed in a series of experiments by S.D. Ganichev et al. We believe this is entirely or almost entirely a Berry-phase effect.

Linear optics: Chiral materials (and sugar water!) can show optical rotation in transmission, the Faraday effect, even without time-reversal breaking. (J. Orenstein and JEM, PRB 2012, motivated by cuprates)

The future

1. "Topological insulators" exist in two and three dimensions in zero magnetic field.





In the 2D case, they have surface Dirac fermions with an unusual spin structure.

Are there correlated topological insulators and superconductors?
Are there "fractional" topological insulators?

Can we use these materials to create new particles?

Topological field theory of QHE

How can we describe the topological order in the quantum Hall effect?

Standard answer: Chern-Simons Landau-Ginzburg theory
(Girvin & MacDonald; Zhang, Hansson, and Kivelson; Read; ...)

$$L_{CS} = -\frac{k}{4\pi}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + j^\mu a_\mu, \qquad j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

There is an "internal gauge field" *a* that couples to electromagnetic *A*.

Integrating out the internal gauge field *a* gives a Chern-Simons term for *A*, which just describes a quantum Hall effect:

$$L_{QHE} = -\frac{1}{4k\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

There is a difference in principle between the topological field theory and the topological term generated for electromagnetism; they are both Chern-Simons terms.

Topological field theory of QHE

What good is the Chern-Simons theory? (Wen)

$$L_{CS} = -\frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + j^\mu a_\mu, \qquad j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

The bulk Chern-Simons term is not gauge-invariant on a manifold with boundary.

It predicts that a quantum Hall droplet must have a chiral boson theory at the edge:

$$S = \frac{k}{4\pi}\int \partial_x\phi(\partial_t\phi - v\partial_x\phi)\,dx\,dt$$

For fractional quantum Hall states, the chiral boson is a "Luttinger liquid" with strongly non-Ohmic tunneling behavior.

Experimentally this is seen qualitatively--perhaps not quantitatively.

Topological field theory of TI

For the topological insulator, we know many properties.

Two standard defining properties in the 3D case:

1. When T is unbroken, there are gapless surfaces with an odd number of Dirac fermions.

2. When T is broken weakly, there is a half-integer quantum Hall effect at the surface, which is equivalent to a bulk EM term

$$\Delta \mathcal{L}_{EM} = \frac{\theta e^2}{2\pi h} \mathbf{E} \cdot \mathbf{B} = \frac{\theta e^2}{16\pi h} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}.$$

Can we find an internal topological field theory that can capture the gapless surface and, when gapped, capture the "axion electrodynamics" term for electromagnetism?

In the 2D case, a useful defining property is that a pi flux insertion in the bulk captures an odd number of Kramers singlets (Fu-Kane, Essin-Moore, Ran-Vishwanath-Lee, Qi-Zhang)

Topological field theory of TI

For the two-dimensional topological insulator, we know that an example of the state is provided by a pair of integer quantum Hall states for "spin-up" and "spin-down".

We can write the resulting combination of two Chern-Simons theories in a basis of two fields *a* and *b* with different time-reversal properties:

$$L_{BF} = \frac{1}{\pi}\varepsilon^{\mu\nu\lambda}(b_\mu \partial_\nu a_\lambda + A_\mu \partial_\nu b_\lambda)$$

This is known as 2D "BF theory", since the topological part couples the field *b* and the field strength *F* of *a*. It is time-reversal even, unlike CS theory.

Its edge has two oppositely propagating boson modes. In the above we have written the coupling to electromagnetism, and indeed we obtain the localized states around a pi flux.

The sources of a and b are charge density and spin density.

This theory was previously studied in CM in the context of superconductivity (Oganesyan, Hansson, Sondhi 2004).

What about 3D?

Unlike Chern-Simons theory, BF theory exists in 3D and still describes time-reversal-invariant systems.

$$L_{BF} = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda\rho}a_\mu\partial_\nu b_{\lambda\rho} + \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda\rho}A_\mu\partial_\nu b_{\lambda\rho} + C\varepsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu\partial_\lambda A_\rho$$

Now b is a two-form and there are two possible couplings to the EM field.

One is T-invariant and the other is not; we expect it to be generated by a T-breaking perturbation at a surface, and indeed it is a boundary term.

The electromagnetic current contains both contributions from a and b.

$$J^\mu_{EM} = J^\mu_b + J^\mu_a = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} + \frac{1}{8\pi^2}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu(\theta\partial_\lambda a_\rho)$$

The two-form b contains information about electric and magnetic polarizations, which can be viewed as a density of intrinsically line-like objects (think about field lines).

Facts about 3D BF

1. With the T-breaking perturbation, we obtain "axion electrodynamics".

2. Without it, we obtain a bosonized representation of a 2D Fermi surface.

Sketch:

As in the FQHE, the bulk topological field theory is not gauge invariant on a manifold with boundary.

It forces boundary degrees of freedom and a topological zero-energy kinetic term.

For BF theory in 3D, the boundary degrees of freedom are a scalar and vector boson, coupled in a first-order Lagrangian. (Hansson-Oganesyan-Sondhi)

These are exactly the degrees of freedom required to represent canonically a single Dirac fermion with time-reversal symmetry (Cho-Moore).

The velocity and filling of the Dirac fermion are set by nonuniversal surface physics, as in the FQHE case.

Facts about 3D BF

1. With the T-breaking perturbation, we obtain "axion electrodynamics".

2. Without it, we obtain a bosonized representation of a 2D Fermi surface.

3. We can reproduce the flow of charge through flux tubes ("wormhole effect", Rosenberg, Guo, Franz, PRB 2010).

Future: we can modify the bulk coefficient of BF theory and obtain fractional braiding statistics of point-like and line-like objects. This seems to be different from the existing "parton" constructions of 3D fractional topological insulators.

A challenge in connecting to experimental reality: at the 1D edge of the FQHE, needed not just the chiral boson but "vertex operators"

$$e^{i\alpha\phi}$$

To imitate Wen's discovery and understand whether the surface could have new excitations in the fractional case, obtained by changing the braiding statistics in the bulk, need to understand equivalent of vertex operators for a 2D fractional Fermi surface.

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