Supersymmetric Virial Expansion for Propagation Probabilities and Response Functions of Gaussian Ensembles of the Almost Diagonal RMTs

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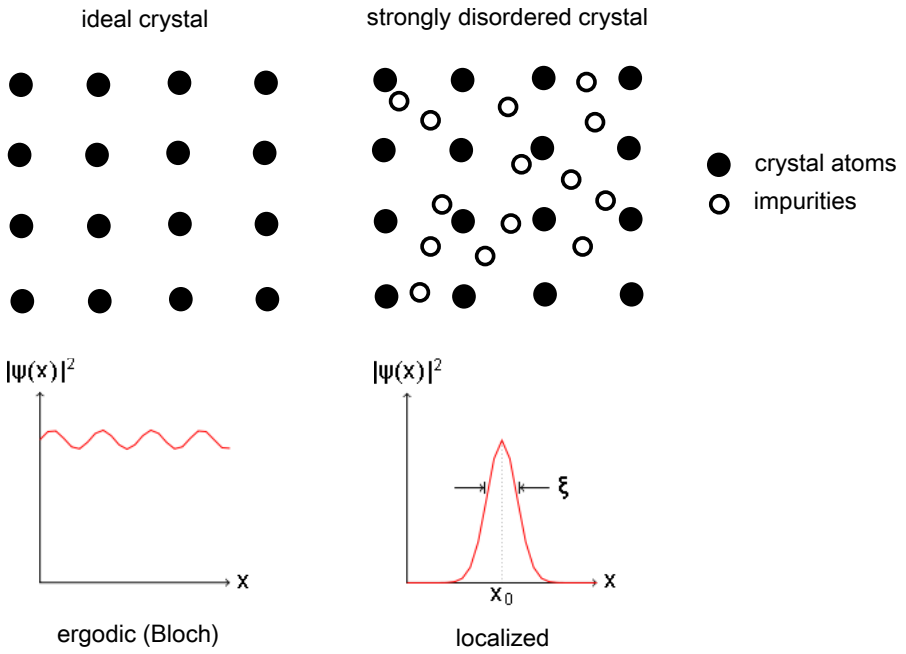
Abstract

In this diploma thesis, the propagation probability is calculated for disordered systems of different underlying symmetry classes (orthogonal and unitary) at the Anderson localization transition point in the strong multifractality regime or in the case of multifractal insulators. A good toy model to analyze such systems is the almost diagonal power law banded random matrix theory (PLBRMT). This is a model of a one-dimensional chain with long-range hopping which decay outside a band with bandwidth $\mathcal{B} \ll 1$. In this model, the system's Hamiltonian is viewed as a real, symmetric or Hermitian $N \times N$ random matrix. All matrix elements are independent, Gaussian distributed random variables where the off-diagonal elements are parametrically smaller $H_{i \neq j} \sim \mathcal{B}$ than the diagonal ones outside of a band with bandwidth $\mathcal{B} \ll 1$ centered at the main diagonal. The standard non-linear supermatrix σ-model fails to be applicable as a solution method. That is why a supersymmetric version of a virial expansion which is analogous to the one known from the theory of imperfect, dilute gases is used. It is shown that the results of this thesis are in agreement on scaling predictions made by J. T. Chalker. On the other hand, our results can be straightforwardly used to calculate response functions of the mentioned systems.

1. Introduction

The consideration of perfect, periodic crystals has brought about deep insights regarding the physics of such objects at the beginning of the last century. One of the major discoveries has been the forming of energy bands for electrons moving as Bloch waves in the periodic crystal potential. This has lead to a classification of electronic systems by their band structures [1, 2]. It has become possible to distinguish between good metals, semiconductors and insulators. Yet, physical reality is quite different. Unfortunately, a perfect crystal is rather an idealization than a common occurrence. Usually, one must deal with disorder caused by impurities in the system of interest. These impurities provide some randomness and irregularity of the crystal potential and can thus cause electrons to get localized. The picture below depicts the situations, on the one hand, of an ideal crystal with delocalized (ergodic) wave functions and, on the other hand, of a strongly disordered crystal with localized states. ξ is the "diameter" of the area around x_0 in which an electron is localized and which is called localization length.



The theory of localized wave functions started when P. W. Anderson in his seminal paper [3], examined a model with random on-site energies and predicted that

electrons get localized depending on the strength of disorder and the dimensionality of the system. This model of a transition from extended to localized states is today referred to as the Anderson metal-insulator transition or Anderson localization. In his original work, Anderson considered a particle on a lattice with nearest neighbor couplings and random on-site energies. Such a model is normally hard to solve analytically as one has in principle to solve the Schrödinger equation for all realizations of the random disorder potential. Indeed, so far there exist merely analytical solutions of the Anderson model for dimensions $d \leq 2 + \epsilon$ ($\epsilon \ll 1$). Since Anderson's discovery, there has been lots of techniques developed to tackle disorder problems.

About 20 years later, based on ideas of D. J. Thouless [4], Abrahams et al. [5] formulated a rather phenomenological one-parameter scaling theory of localization. They considered the scaling function $\beta(g(L))$ where g denotes the dimensionless conductance in one, two and three dimensions as a function of the length L of a sample, and investigated its asymptotic behavior $g \longrightarrow \infty$ and $g \longrightarrow 0$ corresponding to very weak and strong disorder respectively. In interpolating between the two asymptotes, they drew the conclusion that one always has localized states in one and two dimensions no matter how weak disorder is.

In the mean time, field-theoretical approaches for a bit more quantative analysis of disordered systems were developed. F. Wegner used a replica non-linear σ-model [6] where he considered n copies (replicas) of a system to average over disorder and put $n \longrightarrow 0$ at the end. In 1983, K. B. Efetov published his supersymmetric version of the non-linear σ-model which combines bosonic (commuting) and fermionic (anti-commuting) degrees of freedom [7, 8]. This model managed to fuse together the already existing field of random matrix theory (RMT) [9, 10] and the analysis of the energy level and wave function statistics of small disordered metallic particles. Thus, RMT has proven to be a good toy model for disordered condensed matter systems.

The field of RMT dates back to E. P. Wigner, who originally studied the energy spectra of complex nuclei. Within the framework of RMT, one considers the Hamiltonian of the system under consideration as a $N \times N$ matrix consisting of independent and randomly Gaussian distributed matrix elements which fluctuate around zero mean with constant variance. These random matrices can be classified by their underlying symmetries according to F. J. Dyson's "Threefold Way" [11] resulting in the three Gaussian ensembles: orthogonal (GOE), unitary (GUE) and symplectic (GSE). In this thesis, we are only going to refer to the first two. Since the development of the Wigner-Dyson random matrix theory which applies to disordered, yet good metals, a variety of different random matrix models have emerged [12]. RMT models where the variances depend on the indices of the matrix elements have become interesting, especially banded RMTs which describe

the insulator side of the Anderson metal-insulator transition. These RMTs are characterized by variances which decay outside a band centered around the main diagonal. Another representative, the so-called critical power law banded RMT (PLBRMT), attracted special interest. It enables one to investigate the critical point of the Anderson transition and examine all the features the wave functions exhibit there [13]. These features range from strong amplitude fluctuations (multifractality) to a specific scaling behavior of wave function correlations (critical correlations). Recently, the class of almost diagonal critical PLBRMT has drawn attention [14, 15, 16]. This type of RMT is characterized by off-diagonal matrix elements which are parametrically smaller than the diagonal ones.

The main task of this diploma thesis is the investigation of the propagation probability in disordered systems with (almost) localized wave functions. Such systems can be fractal insulators which are close to the localization transition or critical systems at the point of the localization transition in the strong fractality regime.

Structure of the thesis

The thesis is structured in the following way:

- The subsequent chapter, an introduction to the field of Anderson physics is provided. Some early attempts to analyze disordered systems undergoing a transition from localized to delocalized states are presented and the features of wave functions at the critical point of the Anderson transition are discussed.

- Chapter 3 deals with the development of the field of random matrix theory and its application to disorderd systems. Furthermore, the idea behind the supersymmetric virial expansion is reviewed.

- In chapter 4, the supersymmetric virial expansion is applied to the Gaussian unitary and orthogonal ensembles of the almost diagonal random matric theory (ADRMT), which is the main scientific work done for this thesis. The calculations and results are presented.

- The last chapter contains a scaling discussion of the results obtained in chapter 4. Applying the critical power law banded RMT, it is verified if the so-called dynamical scaling assumption holds true and if it is sensitive to the phases of wave functions.

2. Anderson transition and criticality

This chapter is supposed to provide a short review of the critical phenomenon known as Anderson localization: a quantum particle in a metal, say an electron, may become localized depending on the disorder of the system.

2.1. Anderson model and localization

Today, it is well-known that a quantum particle can be localized due to a random potential the particle is moving in. Localization of, say, electrons in a metallic sample is caused by multiple interferences of the electrons' wave functions which scatter at the random disorder potential. More than 50 years ago, P. W. Anderson [3] pointed out that the localization of initially extended electron states is mainly dependent on the strength of disorder and the dimenion d of the system under consideration. Depending on the dimensionality, there is a critical amount of disorder causing extended wave functions to get exponentially localized such that the envelope of the wave functions become

$$|\psi(\vec{x})|_{\text{ext}} \sim 1 \longrightarrow |\psi(\vec{x})|_{\text{loc}} \sim e^{-\frac{|\vec{x}-\vec{x}_0|}{\xi}} , \tag{2.1}$$

where ξ denotes the localization length.

In his considerations, Anderson used a tight-binding model with random on-site energies and constant nearest neighbor couplings. The Hamiltonian of his model reads

$$\hat{H} = t \sum_{(i,j)} \hat{c}_i^\dagger \hat{c}_j + \sum_i \varepsilon_i \hat{c}_i^\dagger \hat{c}_i \, , \tag{2.2}$$

where (i, j) under the first summation symbol is to imply that the summation runs over nearest neighbor sites and t is the coupling constant controlling the hopping between these sites. As mentioned above, the on-site energies ε_i are assumed to be independent (not correlated) and equally distributed random variables reflecting disorder in the system. This model predicts a transition from a metallic phase of the system to an insulating one. Eq. (2.2) is also a good starting point for numerical studies.

This disorder-induced metal-insulator transition is highly sensitive to the underlying symmetries of the system. There are three basic symmetry classes disordered

mesoscopic systems can be classified by. These are the three Wigner-Dyson symmetry classes which can be characterized by the natural numbers $\beta = 1, 2, 4$ [9, 10]:

- **orthogonal**, $\beta = 1$: This class corresponds to all systems with time-reversal invariance and spin-rotational symmetry. The Hamiltonians $\hat{H}$ of such systems can thus be represented by real and symmetric matrices (the bar denotes the complex conjugate in this thesis)

$$H = \overline{H}\,, \quad H = H^T\,. \tag{2.3}$$

- **unitary**, $\beta = 2$: Systems where time-reversal symmetry is broken, e.g. by an applied external magnetic field, but with spin-rotational symmetry are contained in this class. Here, the matrices of the Hamiltonians can be chosen to be Hermitian

$$H = H^\dagger\,. \tag{2.4}$$

- **symplectic**, $\beta = 4$: This symmetry class includes time-reversal invariant systems and broken rotation invariance. In this class, the Hamiltonians can be expanded in terms of Pauli matrices σ_i

$$H = h_0\sigma_0 - i\sum_{i=1}^{3} h_i\sigma_i\,, \tag{2.5}$$

  where σ_0 is the unity matrix and h_i $(i = 1, 2, 3)$ are real, antisymmetric matrices. h_0 is real and symmetric.

We are going to focus only on the orthogonal and unitary classes here. The symplectic case and higher symmetry classes of disordered systems (see [13]) are beyond the scope of this diploma thesis.

2.1.1. One-parameter scaling at the localization transition

A phenomenological approach to the Anderson localization transition was made by Abrahams et al. [5] where the basis of considerations is the scaling function $\beta(g)$ defined as the logarithmic derivative of the dimensionless conductance $g = \frac{\hbar}{e^2}G$ with respect to the length L of the system (see also [17])

$$\beta(g) = \frac{d\ln g}{d\ln L}\,. \tag{2.6}$$

The dimensionless conductance is assumed to serve as the scaling parameter indicating localization. The sign of the β-function then signals whether the system is in the localized or the metallic regime. The discussion of the β-function is

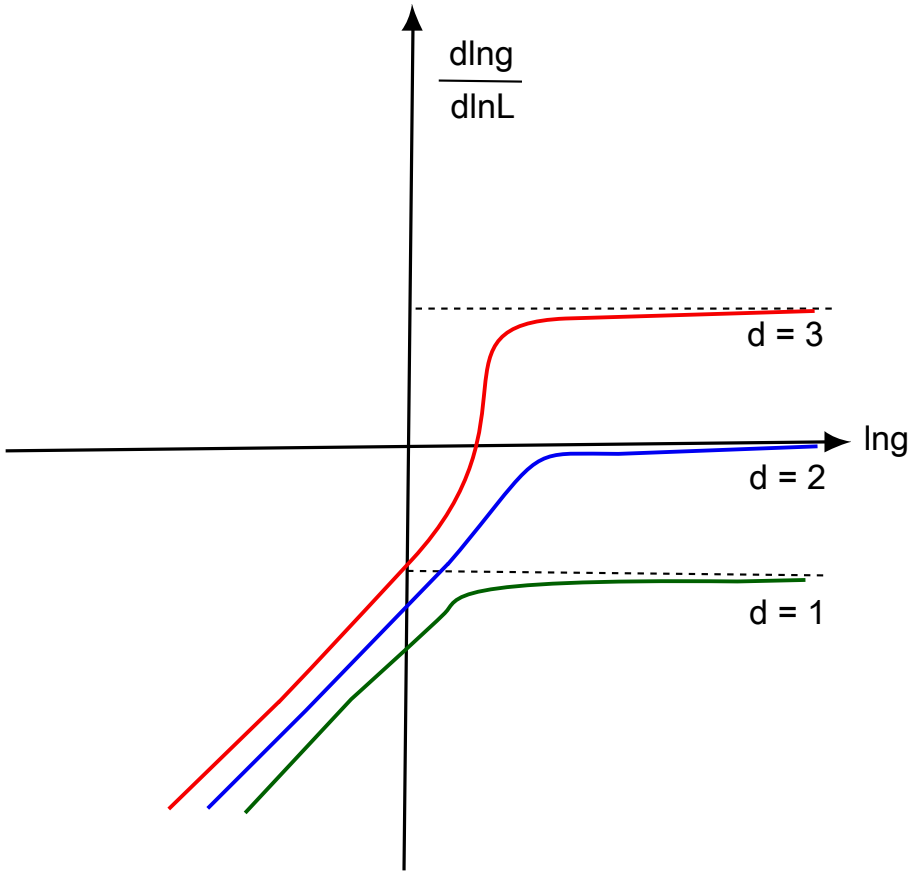


Figure 2.1.: Qualitative plot of $\beta(g)$ vs. $\ln g$ for dimensions $d = 1, 2, 3$.

just based on simple asymptotic arguments about the dimensionless conductance in arbitrary dimensions d of the system. Considering a "hypercube" of size L^d, where L is larger than the mean free path l, one can say that, in the case of large conductances g, the macroscopic Ohmic law certainly holds

$$G(L) = \sigma L^{d-2} \tag{2.7}$$

with the conductivity σ given by the Drude formula

$$\sigma = \frac{ne^2\tau}{m} = \frac{ne^2 l}{\hbar k_F}\,. \tag{2.8}$$

n denotes the electron density and τ is the relaxation time between two scattering events $\tau = \frac{l}{v_F}$ with the electron velocity v_F at the Fermi edge. Thus, $\beta(g)$ acquires the form

$$\lim_{g \longrightarrow \infty} \beta(g) = d - 2\,. \tag{2.9}$$

In the opposite case of very small conductance, one can assume that electron states are exponentially localized and therefore

$$g(L) = g_0 e^{-\alpha L}\,, \tag{2.10}$$

where g_0 is a constant of the order of one. We get

$$\lim_{g\longrightarrow 0} \beta(g) = \lim_{g\longrightarrow 0} L\frac{d\ln g}{dL} = -\alpha L = \ln\left(\frac{g}{g_0}\right)\,, \tag{2.11}$$

which is independent of dimension. By interpolating between Eqs. (2.9) and (2.11) and assuming smoothness, it is possible to sketch $\beta(g)$ for the dimensions one, two and three (see Fig. 2.1). Qualitatively speaking, it is obious that in one dimension, the β-function is negative for all values of g. All electron states are hence localized even if disorder is very weak. In contrast to that, $\beta(g)$ in three dimensions shows a change in sign when going from small to high conductances. There is a g_c marking a transition point from the localized to the delocalized regime. The dimension two can be regarded as marginal case. $\beta(g)$ is negative everywhere, but asymptotically tends to zero, i.e. to a critical point.

2.1.2. $2+\epsilon$ expansion

The peculiarity of two dimensions and the question whether there may be a transition point for sufficiently large g can be investigated by a perturbative expansion for $g^{-1} \ll 1$. This makes it possible to analyze $\beta(g)$ in $2+\epsilon$ dimensions ($\epsilon \ll 1$). It was F. Wegner [18], who found expressions for the scaling function in different symmetry classes. In the following the a_i's are just numerical factors depending on the underlying symmetry class:

- **orthogonal**:

$$\beta_{\text{orth}}(g) = \epsilon - \frac{a_1}{g} - \frac{a_4}{g^4} + O\left(\frac{1}{g^5}\right) \tag{2.12}$$

- **unitary**:

$$\beta_{\text{unit}}(g) = \epsilon - \frac{a_2}{g^2} - \frac{a_4}{g^4} + O\left(\frac{1}{g^6}\right) \tag{2.13}$$

In the limit $\epsilon \longrightarrow 0$ ($d = 2$), $\beta(g)$ is always negative, i.e. there is no real transition point but for the case $g \longrightarrow \infty$.

2.2. Anderson localization: a critical phenomenon

Anderson localization exhibits all features of a critical transition and bears some resemblance to continuous thermodynamic phase transitions [13]. One is able to introduce an order parameter. This order parameter however has to be chosen in a non-trivial way in order to reflect the spontaneous breaking of symmetry at the phase transition point. It is suggested in [19, 20] that the order parameter needs to be a function. This order parameter function is closely related to the probability distribution of the local density of states. The symmetry-breaking relies on the fact that, in terms of scattering theory, the scattering matrix loses the unitarity which holds for systems without dissipation. Formally, dissipation enters a system by applying boundary conditions. As an example, one can think of a system with merely localized states coupled to a lead. These states do not experience any dissipation because they are all localized. Therefore, the S-matrix is unitary. When we cross the Anderson transition point to the metallic phase, all states are now delocalized and are sensitive to dissipation which results in a breakdown of unitarity of the S-matrix. In the close vicinity of the critical point, some further significant characteristics arise:

- **power-law behavior of physical quantities**: An observable $O(x)$ depending on a physical parameter x behaves like

$$O(x) \propto x^{\delta} \tag{2.14}$$

  when approaching the critical point. The exponent δ is called critical exponent. It is assumed to be universal and have the same value on both sides of the critical point. For example, the dc-conductivity σ and the localization length ξ (or correlation length in the metallic phase) behave like

$$\sigma \propto |E - E_c|^{s}\,, \quad \xi \propto |E - E_c|^{-\nu}\,, \tag{2.15}$$

  where E_c denotes the mobility edge, the critical energy at which the localization transition takes place. The critical exponents s and ν are related to each other by

$$s = \nu(d-2) \tag{2.16}$$

  with d being the dimension of the system (see [21]). Critical exponents at the Anderson transition point can be calculated in $2+\epsilon$ dimensions (ϵ-expansion) [18].

- **long-range correlations**: As one can see above, the correlation length (metallic side) or localization length (insulator side) ξ starts to diverge when approaching the localization transition point.

- **scaling invariance**: This is rather a hypothesis than a rigorously proven statement. All critical observables are invariant under rescaling the length scale λ and obey a homogeneity law

$$O(c\lambda) = c^{\Delta_O} O(\lambda)\,. \tag{2.17}$$

Δ_O is called the scaling dimension and c is the rescaling parameter. This means a rescaling of λ is supposed to be compensated by a rescaling of $O(\lambda)$. Due to this behavior under rescaling, critical exponents and scaling dimensions are related to each other. The diffusion constant D at the mobility edge, for instance, scales in the following way [22, 23]:

$$D \propto L^{2-d} \tag{2.18}$$

This holds for the case when the correlation length ξ is the shortest length scale of the system.

2.2.1. Multifractality of critical wave functions

Coming closer to the Anderson transition point either from the metallic or the insulating side, the electron wave functions start to fluctuate and acquire a particular, self-similar structure. On the metallic side, the extended spatial support of the wave functions shrinks to a certain area of the sample called fractal. An extended state can thus be considered as being made up of several such fractals forming a kind of mosaic within the correlation length (multifractal metal). This structure is referred to as being multifractal [24]. On the insulating side of the localization transition point, the wave functions within their localization areas bear a quite similar structure compared to the one of a fractal of the multifractal metal (multifractal insulator) (see Fig. 2.2 and Ref. [25]).

Such multifractal structures of the wave functions, as mentioned above, are characterized by an anomalous scaling behavior, i.e. when one considers different moments of an observable $O^{[q]}(c\lambda)$, the scaling dimension is not just a linear function of q but has a rather complicated dependence [26] such that

$$O^{[q]}(c\lambda) = c^{\Delta_O(q)} O^{[q]}(\lambda)\,. \tag{2.19}$$

Let us consider in the following the various moments of the wave function intensities $|\psi(\vec{x})|^2$ in dimensions d (see [13])

$$\langle P_q \rangle = L^d \langle |\psi(\vec{x})|^{2q} \rangle\,, \tag{2.20}$$

where P_q is defined as

$$P_q = \int d^d x\, |\psi(\vec{x})|^{2q} \tag{2.21}$$

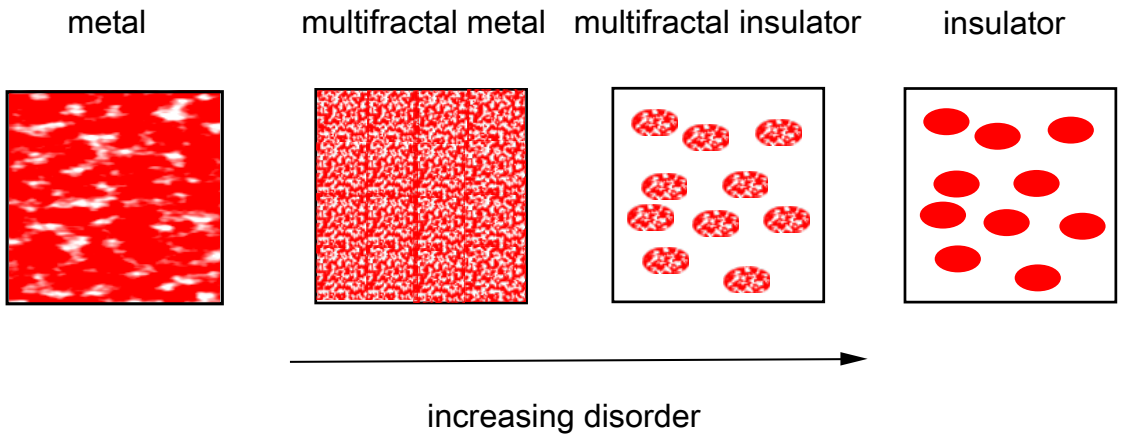


Figure 2.2.: Schematic demonstration of the multifractal structure evolving at the Anderson transition point. The red areas refer to higher wave function amplitudes.

and $\langle \ldots \rangle$ denotes averaging over the wave function intensities. The specific case of P_2 is called inverse participation ratio (IPR). The moments describe the effective fraction of space the wave functions occupy. One can immediately distinguish between two limiting cases. In the purely metallic regime, the qth moment can be estimated as follows:

$$\langle P_q \rangle_{\mathrm{me}} = L^d \langle |\psi(\vec{x})^{2q}| \rangle \sim L^d L^{-dq} = L^{-d(q-1)} \,, \tag{2.22}$$

which reflects how it scales with the length L. This scaling behavior is because the electron states are delocalized and the probability to find an electron in a tiny fraction of space around the point $\vec{x}$ is $|\psi(\vec{x})|^2 \sim L^{-d}$. For the completely insulating regime, $\langle P_q \rangle$, one can argue that

$$\langle P_q \rangle_{\mathrm{in}} \sim 1 \,. \tag{2.23}$$

This highlights the fact that all wave functions are localized and thus confined to an area of volume ξ^d ($\xi \ll L$). Only in this area, the wave function intensities are essentially different from zero and they are not sensitive to the system size. For the intermediate region with multifractal (MF) structure, one can write

$$\langle P_q \rangle_{\mathrm{MF}} \sim L^{-d_q(q-1)} \tag{2.24}$$

incorporating the two limiting cases (see [13]). d_q is called fractal dimension with

$$d_q = 0 \qquad \text{insulator} \tag{2.25}$$

$$0 < d_q < d \qquad \text{multifractal region} \tag{2.26}$$

$$d_q = d \qquad \text{metal} \,. \tag{2.27}$$

Usually, a continuous set of scaling exponents

$$\tau_q = d_q(q-1) \tag{2.28}$$

is introduced. One can now come up with the scaling or anomalous dimension Δ_q splitting τ_q in its normal (metallic) and anomalous parts

$$\tau_q = d(q-1) + \Delta_q = \tau_q^{\mathrm{me}} + \Delta_q \tag{2.29}$$

with $\tau_q^{\mathrm{me}} = d(q-1)$. One obtains for Δ_q

$$\Delta_q = d_q(q-1) - \tau_q^{\mathrm{me}} = -\tau_q^{\mathrm{me}} \left(1 - \frac{d_q}{d}\right) . \tag{2.30}$$

The case $q = 2$ with

$$\Delta_2 = d_2 - d \tag{2.31}$$

will play an important role in the following.

Correlations of critical wave functions

One can use the scaling of the averaged inverse participation ratio $\langle P_2 \rangle$ to study correlations $C(|\vec{x} - \vec{x}'|)$ of critical wave functions separated in space. One can, for example, write for the correlations of one and the same wave function intensity at different space points

$$C(|\vec{x} - \vec{x}'|) = \langle |\psi(\vec{x})|^2 |\psi(\vec{x}')|^2 \rangle = \langle |\psi^2(\vec{x}) \psi^2(\vec{x}')| \rangle . \tag{2.32}$$

It can be assumed that this should equal to $\langle P_2 \rangle$ for slightly different space points $r = |\vec{x} - \vec{x}'| \sim 1$, i.e. with a substitution $\vec{x}' \longrightarrow \vec{x}$. Indeed, it was shown by F. Wegner (see Ref. [27] and references therein) that the correlation function scales like

$$C(r) = \langle |\psi(\vec{x})|^2 |\psi(\vec{x}')|^2 \rangle \sim \left(\frac{r}{L}\right)^{-\eta} . \tag{2.33}$$

Here, the definition $\Delta_2 = -\eta$ was used. Furthermore, the assumption that the wave functions are essentially uncorrelated at a distance $r \sim L$ was put in.

Nearly the same scaling relations hold for correlations of different wave functions separated by an amount of energy $\omega = |\varepsilon_i - \varepsilon_j|$ with L being substituted by $L_\omega = \left(\frac{\Delta}{\omega}\right)^{\frac{1}{d}}$ ($L_\omega < L$). L_ω is the edge length of a cube where the energy separation ω is of the order of the mean level spacing Δ. This assumption is referred to as dynamical scaling hypothesis [27]. Following the same reasoning and substituting L_ω for L when $L_\omega < L$, one can obtain the correlation function of two wave functions at different energies

$$C(\omega) = \langle |\psi_i(\vec{x})|^2 |\psi_j(\vec{x})|^2 \rangle = \langle |\psi_i^2(\vec{x}) \psi_j^2(\vec{x})| \rangle \sim L_\omega^{-\eta} \propto \left(\frac{E_0}{\omega}\right)^{1-\frac{d_2}{d}} . \tag{2.34}$$

The latter estimate is valid for $\Delta \ll \omega \ll E_0$ with E_0 being an upper cut-off of critical scaling. J. T. Chalker has suggested [23] that the dynamical scaling holds true for any correlations of multifractal wave functions separated in space and energy:

$$C(r,\omega) \sim \left(\frac{r}{L_\omega}\right)^{-\eta} \quad (r > l\,,\; L_\omega < L)\,. \tag{2.35}$$

l is the mean free path. To give some examples of such correlations at the critical point where the correlation length ξ diverges, let us consider the local density of states (LDOS) correlation function

$$\begin{aligned} C_{\mathrm{LDOS}}(r,\omega) &= \langle \rho(\vec{x},E)\rho(\vec{x}',E')\rangle \\ &= \sum_{i,j} \langle |\psi_i(\vec{x})|^2 |\psi_j(\vec{x}')|^2 \delta(E-\varepsilon_i)\delta(E'-\varepsilon_j)\rangle \end{aligned} \tag{2.36}$$

with $\omega = |E - E'|$ and the two-point correlation function or dynamic structure factor [23]

$$S(r,\omega) = \langle \sum_{i,j} \psi_i(\vec{x})\overline{\psi_i(\vec{x}')}\psi_j(\vec{x}')\overline{\psi_j(\vec{x})}\delta(E-\varepsilon_i)\delta(E'-\varepsilon_j)\rangle\,. \tag{2.37}$$

Both correlation functions scale, according to [23], like

$$\left.\begin{matrix} C_{\mathrm{LDOS}}(r,\omega) \\ S(r,\omega) \end{matrix}\right\} \sim \left(\frac{r}{L_\omega}\right)^{-\eta}\,. \tag{2.38}$$

As one can see, the relative phases of the wave functions contribute to the correlations in Eq. (2.37) in contrast to Eq. (2.36). Though they do not change the scaling (2.38) if Chalker's idea holds true.

2.2.2. Weak and strong multifractality, critical PLBRMT

To sum up, the wave functions at the Anderson transition point undergo strong fluctuations acquire a certain structure and obey particular scaling relations. This shows up in terms of multifractality which can be weak or strong. The degree of strength is related to the dimensionality of the system under consideration, i.e. weak multifractality corresponds to small dimensions and strong multifractality appears in systems with higher dimensions $d \gg 1$. The strength of multifractality characterizes the behavior of the critical wave functions: weakly multifractal wave functions are quasi-metallic (multifractal metal) and strongly multifractal ones are quasi-insulating (multifractal insulator).

It has turned out that random matrix theory (RMT), ranging from the original Wigner-Dyson RMT, applicable to disordered metals with ergodic wave functions,

to banded RMTs which are useful in describing localized and especially disordered systems at criticality, is a suitable candidate to describe the properties of disordered systems with global symmetries (orthogonal, unitary etc.). The so-called power-law banded random matrix theory (PLBRMT) has in particular proven to be able to describe disordered systems at the Anderson transition point exhibiting weak or strong multifractality (see chapter 4). PLBRMT is characterized by a parameter $\mathcal{B}$ which corresponds to the bandwidth of power-law banded random matrices, i.e. within this bandwidth, the matrix elements are essentially different from zero and outside of it, the matrix elements are decaying in a power-law fashion. The magnitude of the bandwidth reflects the two multifractal regimes

- $\mathcal{B} \gg \mathbf{1}$: multifractal metal
- $\mathcal{B} \ll \mathbf{1}$: multifractal insulator.

It has been shown by a perturbative approach called supersymmetric virial expansion for the orthogonal and unitary symmetry class in [27, 28, 29] that the parameter $\mathcal{B}$ is directly connected with the fractal dimension d_2 of the wave function support in the limit of strong multifractality. Up to the first order of the supersymmetric virial expansion, the fractal dimension d_2 reads

$$d_2^{\text{orth}} = \sqrt{2}\mathcal{B} + O(\mathcal{B}^2)\,, \quad \text{orthogonal} \tag{2.39}$$

$$d_2^{\text{unit}} = \frac{\pi\mathcal{B}}{\sqrt{2}} + O(\mathcal{B}^2)\,, \quad \text{unitary.} \tag{2.40}$$

This indicates that the fractal dimension d_2 shrinks to zero in the strong multifractality regime. The supersymmetric virial expansion is going to be the subject of the calculations done for this diploma thesis.

3. Statement of the problem

We apply the supersymmetric version of a virial expansion to the propagation probabilities for the Gaussian unitary and Gaussian orthogonal ensemble of the almost diagonal random matrix theory to analyze universal properties of disordered systems either at or close to the point of Anderson localization. These correlation functions are the basis to calculate response functions of disordered systems. Besides, it is shown with accuracy of the leading term of the virial expansion that the dynamical scaling assumption at the Anderson transition point made by J. T. Chalker is valid and that it is not sensitive to the phases of wave functions. As it is known, this has not done before.

4. Critical RMT and supersymmetric virial expansion

A short introduction into the random matrix formalism is given in this chapter with main focus on the almost diagonal power-law banded random matrix theory. Moreover, the supersymmetric virial expansion, which is the calculational tool used in this diploma thesis, is developed.

4.1. RMT as a toy model for complex quantum systems

Random matrix theory is a mathematical formalism which helps to describe universal properties of large complex quantum systems with underlying global symmetries. It enables us to treat somehow randomised physical quantities on a statistical footing and consider ensemble averages. In this thesis, we are going to dwell on the Gaussian non-invariant random matrix theories.

The well-known Wigner-Dyson random matrix models are characterized by Gaussian shaped probability distributions with constant variances. These models have proven to be a good description for disordered metals. It is able to describe the statistics of the energy levels and the extended wave functions. In contrast to that, there are non-invariant RMTs like the so-called banded and power-law banded RMT (PLBRMT) which have index-dependent variances, i.e. the variances are some functions of the matrix element indices. Concerning the special case of the almost diagonal PLBRMT, all hopping entries, i.e. all off-diagonal matrix elements are parametrically small. The banded RMT and the PLBRMT are suitable to describe the insulating regime of disordered systems and all critical features at the Anderson transition point like multifractal wave functions and long-ranged correlations.

4.2. Invariant RMT: Wigner-Dyson random matrix model

One considers the Hamiltonian of a system as $N \times N$ matrix. Its entries H_{ij} are indepent and randomly Gaussian distributed variables. Hence, the probability distribution function of such a random matrix H factorizes

$$P(H) = \prod_{i,j} P(H_{ij}) \,, \tag{4.1}$$

where the product runs over all independent matrix elements. Additionally, it is required that $P(H)$ is invariant under canonical basis transformations, i.e. for instance orthogonal and unitary transformations, T with respect to H

$$P(H') = P(H) \,, \quad H' = \tilde{T} H T \,. \tag{4.2}$$

T corresponds either to an orthogonal or unitary matrix with $\tilde{T} = T^{T(\dagger)}$. With H being a real symmetric or Hermitian matrix, these assumptions lead to the Wigner-Dyson Gaussian ensembles of random matrices: Gaussian orthogonal (GOE, $\beta = 1$) and Gaussian unitary ensemble (GUE, $\beta = 2$) (see [30], chapter 4). In both cases, the probability distribution function reads

$$P(H) \propto e^{-\frac{\mathrm{tr} H^2}{2\nu(\beta)}} \tag{4.3}$$

with $\nu(\beta)$ being the site-independent variance of the matrix elements $\nu(\beta) = \langle |H_{ij}|^2 \rangle = \frac{1}{\beta}$ and distinguishing between the Gaussian ensembles. It was assumed that all the matrix elements fluctuate around zero on average ($\langle H_{ii} \rangle = 0$). This random matrix model first used by E. P. Wigner to analyze the energy level statistics of complex nuclei is referred to as Wigner-Dyson random matrix theory (WDRMT) [9, 10, 31]. It provides a statistical description of the energy level fluctuations in large quantum systems so that one need not take into account the influence of every single interaction in the system on the interesting quantities, but considers quantities averaged over ensembles of random matrices. The fields of physical application of WDRMT range from nuclear physics to quantum chaos [30] and mesoscopic physics, especially disordered metallic systems.

4.3. Basic results obtained in the framework of WDRMT

4.3.1. Eigenvalue distribution and level repulsion

Let us consider a change of variables that makes the random matrix H diagonal and thus casts the probability distribution $P(H)$ for each independent matrix

element H_{ij} (see Eq.(4.3) into one for the eigenvalues or eigenenergies ε_i of the random Hamiltonian matrix H

$$P(H) = P(\{H_{ij}\}) \longrightarrow P(\{\varepsilon_i\}) = P(\varepsilon)\,. \tag{4.4}$$

The normalization condition

$$1 = \int dH P(H) = \int d\varepsilon \mathcal{J} P(\varepsilon) \tag{4.5}$$

must hold anyway. The integration measures dH and $d\varepsilon$ refer to integrating over all independent random variables H_{ij} and ε_i respectively. $\mathcal{J}$ is the Jacobian of the variable transformation. One can realize how $\mathcal{J}$ in general looks like when considering, for simplicity, the case of a 2×2 random matrix from the Gaussian orthogonal ensemble. So, H can be chosen to be real and symmetric. We can write the independent matrix elements of H in terms of the eigenenergies ϵ_1 and ϵ_2

$$H_{11} = \varepsilon_1 \cos^2\phi + \varepsilon_2 \sin^2\phi \tag{4.6}$$
$$H_{22} = \varepsilon_1 \sin^2\phi + \varepsilon_2 \cos^2\phi \tag{4.7}$$
$$H_{12} = (\varepsilon_1 - \varepsilon_2)\sin\phi\cos\phi\,. \tag{4.8}$$

It is now possible to write H as

$$H = OEO^T\,, \quad E = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}\,, \quad O = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\,. \tag{4.9}$$

O is an orthogonal matrix. The Jacobian $\mathcal{J}$ of this particular transformation reads

$$\mathcal{J} = |\varepsilon_1 - \varepsilon_2|\,. \tag{4.10}$$

The probability distribution for the eigenenergies ε_1 and ε_2 thus becomes

$$P(\varepsilon) \propto |\varepsilon_1 - \varepsilon_2| e^{-\frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2)}\,. \tag{4.11}$$

All the simple steps done here can be generalized to other symmetry classes, e.g. unitary symmetry class ($\beta = 2$), and arbitrary matrix sizes N (see chapter 4 in [30]). As a result, one obtains

$$P(\varepsilon) \propto \prod_{i<j} |\varepsilon_i - \varepsilon_j|^\beta e^{-\sum_i \frac{\varepsilon_i^2}{2\nu(\beta)}}\,. \tag{4.12}$$

All these considerations show that the probability to find an energy level close to another one vanishes. This effect is called "level repulsion".

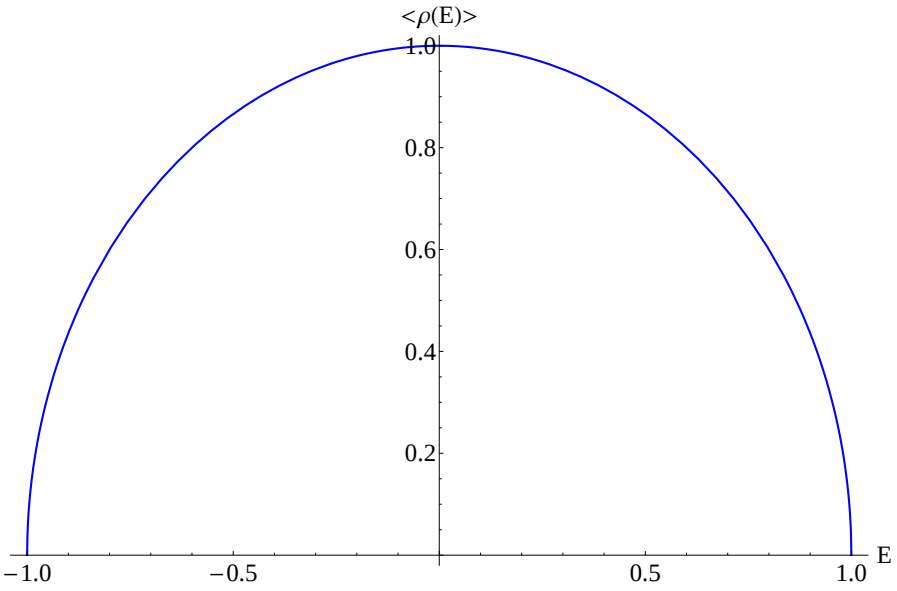


Figure 4.1.: Visualization of Wigner's semi-circle.

4.3.2. Mean density of states and Wigner's semi-circle law

The density of states is defined as

$$\rho(E) = \mathrm{tr}\delta(E - \hat{H})\,. \tag{4.13}$$

$\hat{H}$ is meant to be the Hamilton operator of a system as usual. Wigner calculated the averaged density of states $\langle\rho(E)\rangle$ and obtained for it what is nowadays called Wigner's semi-circle law [31, 30]

$$\langle\rho(E)\rangle \sim \sqrt{1 - E^2}\,. \tag{4.14}$$

Here, E was chosen to be dimensionless. This result is valid for all the mentioned Gaussian ensembles. Fig. 4.3.2 shows a plot of the semi-circle.

4.3.3. Level-level correlation function

The probability density to find two energy levels separated by an amount of energy ω is given by the two-level or level-level correlation function defined as follows:

$$R_2(\omega) = \frac{\langle\rho(E + \frac{\omega}{2})\rho(E - \frac{\omega}{2})\rangle}{\langle\rho(E)\rangle^2}\,. \tag{4.15}$$

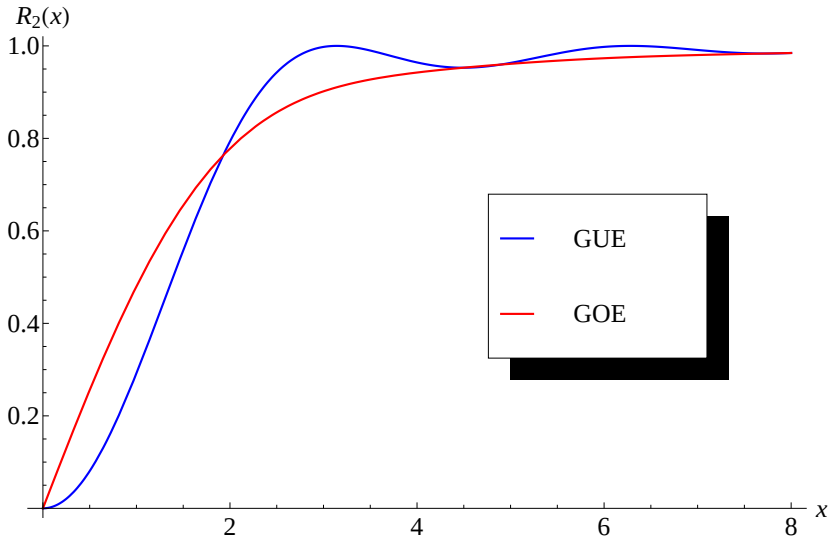


Figure 4.2.: Level-level correlation function $R_2(x)$ for the Gaussian unitary and Gaussian orthogonal ensemble.

It is shown in [8, 9, 10] that $R_2(\omega)$ takes different forms for the unitary and orthogonal ensembles (see Fig. 4.3.3):

$$R_2^{\text{unit}}(x) = 1 - \left(\frac{\sin x}{x}\right)^2 \tag{4.16}$$

$$R_2^{\text{orth}}(x) = R_2^{\text{unit}} - \left(\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right)\int_1^\infty dt\,\frac{\sin(xt)}{t}\,, \tag{4.17}$$

where x represents the rescaled energy separation $x = \pi\frac{\omega}{\Delta}$ with Δ being the mean level spacing. It is noticeable that the probability density $R_2(x)$ in units of the mean level spacing saturates for increasing values of x. Or stated the other way round, the probability to find two energy levels close in energy to each other goes down on the scale of the mean level spacing.

4.4. From WDRMT to non-invariant random matrix models

The fields of application of the Wigner-Dyson random matrix theory are wide-ranging. They include purely mathematical problems as looking for the zeros of the Riemann ζ-function and physical problems ranging from the description of energy spectra of heavy nuclei to quantum chaos (kicked rotator) and the treatment

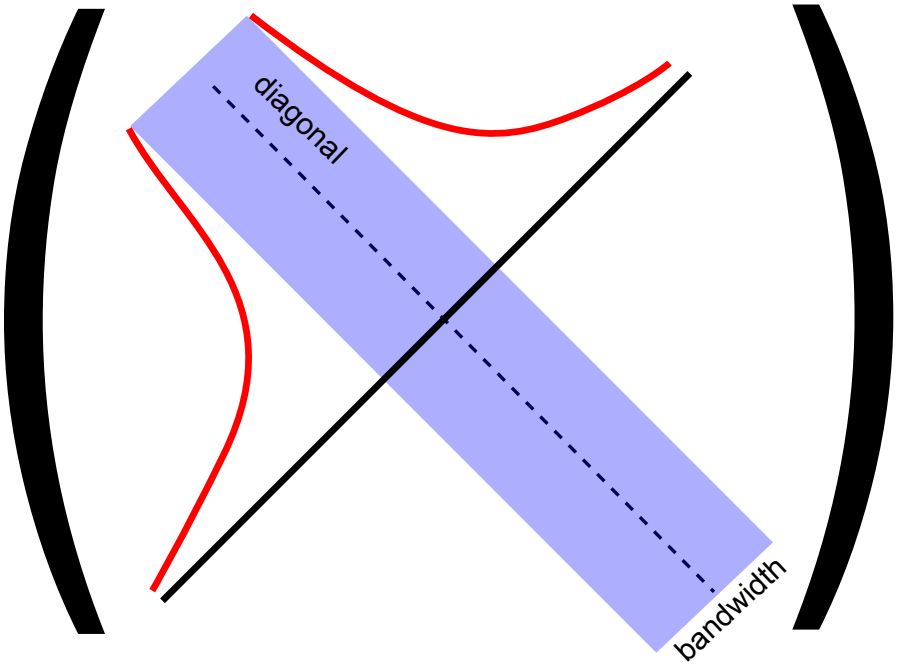


Figure 4.3.: Cartoon of the concept of banded random matrices.

of small disordered metallic grains in mesoscopic physics, e.g. quantum dots. Concerning the latter field of application, WDRMT manages to describe the statistics of ergodic (delocalized) wave functions, which is mainly thanks to the development of the so-called nonlinear supersymmetric σ-model by K. B. Efetov [8, 7]. This model represents a field-theoretical approach to disordered metals and is based on the combination of ordinary commuting and anti-commuting degrees of freedom called Grassmanns (see chapter A). It is capable to reproduce all results originally obtained by Wigner (see section 4.3 and [31]).

4.4.1. Non-invariant random matrix models

As opposed to invariant random matrix models like the Wigner-Dyson model, it is sometimes preferable to examine non-trivial correlations between different energy states. Therefore, one can define non-invariant Gaussian random matrix models which are characterized, unlike WDRMT, by variances with some functional dependence on the indices of the matrix entries $f(i, j)$

$$\langle |H_{ij}|^2 \rangle = f(i, j)\,. \tag{4.18}$$

Whereas the archetypical Wigner-Dyson model describes extended states, non-invariant RMTs are able to describe wave functions of other universality classes,

for example, multifractality at the Anderson transition point, which is the case for the critical power-law banded RMT (see below).

Certain unconventional non-invariant random matrix models have been developed to catch the features of systems where the wave functions are all localized [13, 14, 15, 16]. These models reside under the name banded random matrix models (BRMM). This name stems from the fact that the elements of the random matrices are essentially different from zero within a band of bandwidth $\mathcal{B}$ and the variances of the off-diagonal matrix elements are functions decaying with distance from the main diagonal

$$\langle |H_{i\neq j}|^2 \rangle = \mathcal{F}(|i-j|)\,. \tag{4.19}$$

The exact form of $\mathcal{F}(|i-j|)$ (e.g. exponential or power-law decrease) determines the physical situation and the corresponding key features one wants to analyze. Fig. 4.3 is supposed to show how the variances of the off-diagonal matrix elements decay outside of the band or, in other words, how the mean of the off-diagonal elements becomes sharply peaked around zero. A rather extreme representative of such RMTs is the diagonal RMT where the random matrices are diagonal reflecting complete localization. Further representatives of banded random matrices are the ones reflecting exponentially localized eigenstates. The matrix entries decrease exponentially with

$$\langle |H_{i\neq j}|^2 \rangle \sim e^{-\frac{|i-j|}{\mathcal{B}}}\,. \tag{4.20}$$

This banded random matrix model describes quasi-one-dimensional disordered wires [32]. A special type of banded RMT is the power-law banded RMT (PLBRMT) where the off-diagonal variances decrease according to a power law. This random matrix model and its capability to show all critical features of wave functions at the Anderson transition point will be the subject of the next section.

4.4.2. Critical power-law banded RMT

This RMT model is characterized by the following:

$$\langle H_{ij} \rangle = 0 \tag{4.21}$$

$$\langle H_{ii}^2 \rangle = \frac{1}{\beta} \tag{4.22}$$

$$\langle |H_{ij}|^2 \rangle = \mathcal{F}(|i-j|) = \frac{1}{\beta}\frac{1}{1+\frac{|i-j|^{2\alpha}}{\mathcal{B}^2}} \tag{4.23}$$

with β indicating the underlying Wigner-Dyson symmetry class. The off-diagonal matrix elements decay with a power law outside the band $\mathcal{B}$. The PLBRMT can be interpreted as an one-dimensional model with long-range hopping and powe-law

decreasing hopping amplitudes

$$\mathcal{F}(|i-j|) \sim \frac{1}{|i-j|^{2\alpha}}\,, \quad |i-j| \gg \mathcal{B}\,. \tag{4.24}$$

For $\alpha > 1$, the PLBRM model describes power-law localized wave functions. For the value $\alpha = 1$, it reflects criticality, i.e. it is able to reproduce all critical features at the Anderson transition point. This is the case for all values of $\mathcal{B}$ [13, 33]. Furthermore, the critical PLBRM model is capable to distinguish between the two multifractality regimes: strong ($\mathcal{B} \ll 1$) and weak ($\mathcal{B} \gg 1$).

RMT models like the critical PLBRMT for very small bandwidths $\mathcal{B} \ll 1$ where the off-diagonal matrix elements are parametrically smaller than the ones on the diagonal

$$H_{i\neq j} \sim \mathcal{B} \tag{4.25}$$

are usually referred to as almost diagonal [14, 15].

Analytical methods to study banded RMT with large and small bandwidth

In order to analyze and describe correlations and wave function statistics not only for the critical PLBRMT, but for a broader range of banded RMTs, the case of the quasi-metallic regime ($\mathcal{B} \gg 1$) can be mapped onto the nonlinear supersymmetric σ-model [8]. This supersymmetry approach allows one to find a solution using a saddle-point approximation which is a basic step needed in the derivation of the supersymmetric σ-model. Unfortunately, the application of this quite successful method fails at describing the quasi-insulating regime with $\mathcal{B} \ll 1$ because of the invalidity of the mentioned saddle-point approximation. Thus, another approach is here needed which brings us to the concept of the supersymmetric virial expansion in the next section. This method bears some resemblance to the classical virial expansion known from the theory of dilute, imperfect gases (see e.g. [34]).

4.5. Virial expansion

4.5.1. Idea behind a virial expansion

The supersymmetric virial expansion is a field-theoretical approach to calculate various correlation functions averaged over the ensemble of almost diagonal random matrices (disorder average). Such correlation functions (cf. Eqs. (2.36) and (2.37) in section 2.2.1) can normally be expressed in terms of retarded and advanced Green's functions

$$\left\langle G^R_{kl}\left(E+\frac{\omega}{2}\right) G^A_{mn}\left(E-\frac{\omega}{2}\right)\right\rangle\,. \tag{4.26}$$

$\langle \ldots \rangle$ represents the disorder average. The Green's functions in energy space are defined by [35]

$$G^R_{kl}(E) = \left(E - \hat{H} + i\eta\right)^{-1}_{kl} = \langle k| \left(E - \hat{H} + i\eta\right)^{-1} |l\rangle \tag{4.27}$$

$$G^A_{mn}(E) = \left(E - \hat{H} - i\eta\right)^{-1}_{mn} = \langle m| \left(E - \hat{H} - i\eta\right)^{-1} |n\rangle \tag{4.28}$$

with $\eta \longrightarrow +0$ being a small regularizing parameter. The cases where $k = l$, $m = n \neq k$ and $k = l = m = n$

$$\left\langle G^R_{kk}\left(E + \frac{\omega}{2}\right) G^A_{mm}\left(E - \frac{\omega}{2}\right)\right\rangle , \quad \left\langle G^R_{kk}\left(E + \frac{\omega}{2}\right) G^A_{kk}\left(E - \frac{\omega}{2}\right)\right\rangle \tag{4.29}$$

were investigated for the GUE and GOE in [36, 37]. In this diploma thesis, we are going to concentrate on the case $k = n$, $l = m \neq k$

$$\left\langle G^R_{km}\left(E + \frac{\omega}{2}\right) G^A_{mk}\left(E - \frac{\omega}{2}\right)\right\rangle . \tag{4.30}$$

The idea for the supersymmetric virial expansion for the almost diagonal RMT comes from the observation that the small and fast decaying off-diagonal part $\hat{V}$ of the Hamiltonian

$$\hat{H} = \hat{H}_d + \hat{V} \tag{4.31}$$

can be associated with interactions between localized states whereas the diagonal part $\hat{H}_d$ corresponds to completely localized wave functions.

In analogy with the expansion of thermodynamic quantities of the theory of imperfect, dilute gases with respect to the number of interacting or colliding particles where the small particle density serves as control parameter for the expansion (see [34]), one can try to expand the correlation functions of interest in the number of interacting localized states where the parameter $\mathcal{B} \ll 1$ controls this perturbation series. The bandwidth $\mathcal{B}$ being much smaller than one, but larger than the mean level spacing Δ, sets a new energy scale

$$\mathcal{B}\Delta \ll \Delta \quad (\Delta \ll \mathcal{B} \ll 1) . \tag{4.32}$$

This leads to a small probability of higher interactions of localized energy states. This idea was first implemented in a phenomenological real-space renormalization group (RG) approach to critical systems with long-range interactions by Levitov [38, 39, 40]. It has been used to study interactions of resonant energy levels where two-level interactions were assumed to be the main source of delocalization, i.e. in direct analogy with interactions of more than two particles in a dilute gas, the probability to have higher interactions of localized energy levels decreases and thus significant contributions become small. But, one major disadvantage of this RG approach is that one is nearly unable to take into account higher interactions of localized states and go beyond the leading term of the expansion of two interacting states. Besides, there is no rigorous control of accuracy.

4.5.2. Brief review of the Trotter virial expansion

A virial expansion can be generated by using the Trotter formula [41]

$$e^{\hat{A}+\hat{B}} = \lim_{n\longrightarrow\infty}\left(e^{\frac{\hat{A}}{n}}e^{\frac{\hat{B}}{n}}\right)^n . \tag{4.33}$$

This is referred to as Trotter virial expansion. Though it is in principle very well applicable [14, 15, 16], it has also some disadvantages. With increasing the number of interacting states taken into account, the combinatorics involved during the calculations grow tremendouly. In addition, the Trotter virial expansion is practically only useful to calculate spectral correlations which can be written in terms of powers of $\mathrm{tr}\, e^{i\hat{H}t}$ like the spectral form factor and the level compressibility (see [15, 16]). The basic procedure is as follows: once a correlation function can be expressed in terms of $\mathrm{tr}\, e^{i\hat{H}t}$, one applies the Trotter formula to circumvent the problem that the diagonal and off-diagonal parts of the Hamiltonian do not commute in general

$$\left[\hat{H}_d, \hat{V}\right] \neq 0\,. \tag{4.34}$$

This leads to

$$e^{i\hat{H}t} = e^{it\hat{H}_d + it\hat{V}} = \lim_{n\longrightarrow\infty}\left(e^{it\frac{\hat{H}_d}{n}}e^{it\frac{\hat{V}}{n}}\right)^n . \tag{4.35}$$

Contributions of m interacting states can now be considered by expanding m exponentials up to the first order in the interaction term $\hat{V}$

$$e^{it\frac{\hat{V}}{n}} = 1 + it\frac{\hat{V}}{n} + O(\hat{V}^2). \tag{4.36}$$

and neglecting the contributions of all the other $n-m$ off-diagonal exponentials by setting $\hat{V} \longrightarrow 0$. This is where the combinatorics come in because one has to allow for all possible combinations of choosing m exponential factors out of n. The combinatorial part which increases with the number of interacting states enormously is the reason why the Trotter virial expansion is finally just practicable to take interactions of two and three states into account.

The supersymmetric virial expansion we are going to present in the subsequent section can cope with many disadvantages and limitations of the Trotter virial expansion.

4.5.3. Supersymmetric virial expansion

Basic principle

The supersymmetric version of the virial expansion has so far been developed only for the almost diagonal RMT of the Gaussian unitary and Gaussian orthogonal

ensemble. In [36, 37], local density of states (LDOS) correlation functions were examined where the relative phases between wave functions do not play a significant role.

The starting point of the derivation is already the disorder-averaged representation of Green's functions in terms of superintegrals (see the appendix for more details)

$$\mathcal{G}_{pq,qp}(E,\omega) \stackrel{\text{def.}}{=} \left\langle G^R_{pq}\left(E+\frac{\omega}{2}\right) G^A_{qp}\left(E-\frac{\omega}{2}\right)\right\rangle \tag{4.37}$$

$$= -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]}\prod_{\substack{i,j=1\\ i<j}}^{N} e^{S[Q_i,Q_j]}\,. \tag{4.38}$$

$D\{Q\}$ is a compact notation for

$$D\{Q\} = \prod_{i=1}^{N} D\{Q_i\}\,. \tag{4.39}$$

and $\mathcal{P}^{RA}\mathcal{Q}^{AR}$ are so-called supersymmetry-breaking factors which include four Grassmann variables χ

$$\mathcal{P}^{RA}\mathcal{Q}^{AR} = -\chi^R_p\overline{\chi^R_q}\chi^A_q\overline{\chi^A_p}\,. \tag{4.40}$$

The Q_ks are $\frac{8}{\beta}\times\frac{8}{\beta}$ supermatrices which are basically constructed by the tensor product of ensemble specific supervectors $|\psi^R_k\rangle$, $|\psi^A_k\rangle$

$$Q_k = \frac{\beta}{2}\begin{pmatrix} |\psi^R_k\rangle\langle\psi^R_k| & |\psi^R_k\rangle\langle\psi^A_k| \\ |\psi^A_k\rangle\langle\psi^R_k| & |\psi^A_k\rangle\langle\psi^A_k| \end{pmatrix}\,. \tag{4.41}$$

(For the precise parametrization depending on the ensemble, GUE or GOE, see the appendix.) These supermatrices are associated with a site in a one-dimensional chain. $S_0[Q_i]$ represents a single-matrix action which corresponds to uncorrelated localized states. It has the form

$$S_0[Q_i] = iE\mathrm{Str}\,Q_i + i\frac{\Omega}{2}\mathrm{Str}(\Lambda_\beta Q_i) - \frac{1}{2\beta}(\mathrm{Str}\,Q_i)^2 \tag{4.42}$$

$$\Lambda_\beta = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}_{RA}\,,\quad \Omega = \omega + i\eta\,.$$

The cross-term action $S[Q_i,Q_j]$ reflects interactions between different states. It originally comes from disorder-averaging over the off-diagonal matrix elements which involve slight interactions between localized states represented by the elements on the main diagonal. It reads

$$S[Q_i,Q_j] = -2b_{ij}\mathrm{Str}(Q_iQ_j) \tag{4.43}$$

$$b_{ij} = \frac{1}{2\beta}\mathcal{F}(|i-j|) \equiv \frac{1}{2\beta}\mathcal{B}^2 g(|i-j|)\,. \tag{4.44}$$

In the last equation, $\mathcal{B}^2$ was pulled out of $\mathcal{F}$. The next step is to rewrite the cross-term exponentials in the following way:

$$e^{S[Q_i,Q_j]} \equiv 1 + f_{ij}\,, \tag{4.45}$$

where f_{ij} is a so-called Mayer function. This is absolutely analogous to the virial expansion known from statistical mechanics. This makes it possible to cast the product of the cross-term exponentials into a reordered sum of various combinations of the f_{ij}s

$$\prod_{\substack{i,j=1\\ i<j}}^{N} e^{S[Q_i,Q_j]} = \mathcal{V}^{(D)} + \sum_{m=2}^{\infty} \mathcal{V}^{(m)} \tag{4.46}$$

$$= \mathcal{V}^{(D)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)} + \cdots \tag{4.47}$$

with

$$\mathcal{V}^{(D)} = 1\,, \quad \mathcal{V}^{(2)} = \sum_{\substack{i,j=1\\ i<j}}^{N} f_{ij} \tag{4.48}$$

$$\mathcal{V}^{(3)} = \sum_{\substack{i,j,k=1\\ i<j<k}}^{N} \left(f_{ij}f_{ik}f_{jk} + f_{ij}f_{jk} + f_{ij}f_{ik} + f_{ik}f_{jk}\right)\,. \tag{4.49}$$

The $\mathcal{V}^{(m)}$s are referred to as the virial coefficients and bear information about how m sites or Q-matrices (or particles in the classical gas theory) out of N can interact with each other. Fig. 4.5.3 shows a cartoon how 3-particle interactions in a 3-particle classical gas can appear. One can now write Eq. (4.38) as

$$\mathcal{G}_{pq,qp}(E,\omega) = -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR} \prod_{i=1}^{N} e^{S_0[Q_i]} \prod_{\substack{i,j=1\\ i<j}}^{N} e^{S[Q_i,Q_j]} \tag{4.50}$$

$$= -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR} \prod_{i=1}^{N} e^{S_0[Q_i]} \left(1 + \sum_{m=2}^{\infty} \mathcal{V}^{(m)}\right)\,. \tag{4.51}$$

In this diploma thesis, we are going to consider only contributions up to the second virial coefficient $\mathcal{V}^{(2)}$ into account, which is also referred to as two-supermatrix approximation. How to proceed in principal to consider higher contributions from $\mathcal{V}^{(3)}$ (three-supermatrix approximation) can be found in Ref. [36].

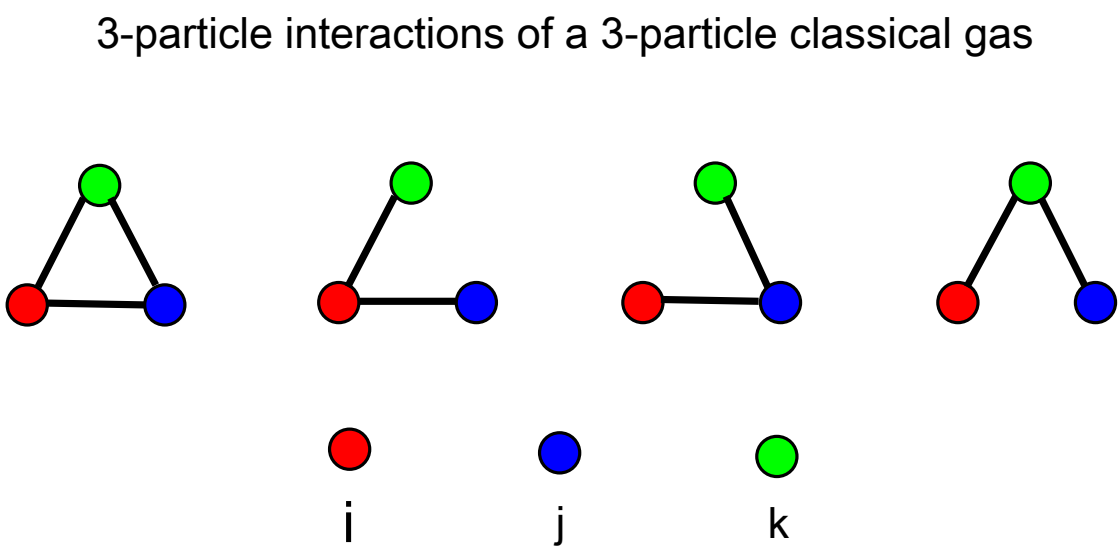


Figure 4.4.: Visualization of the virial coefficient $\mathcal{V}^{(3)}$ via 3-particle interactions in a 3-particle classical gas. The colored dots represent different particles and the black lines denote interactions.

Two-supermatrix approximation

So far, Eq. (4.51) is exact. We now merely consider leading terms up to the order of $\mathcal{V}^{(2)}$

$$\mathcal{G}_{pq,qp}(E,\omega) \simeq -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]}\left(1+\mathcal{V}^{(2)}\right) \tag{4.52}$$

$$= -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]} - \sum_{\substack{i,j=1\\ i<j}}^{N}\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\left(\prod_{i=1}^{N} e^{S_0[Q_i]}\right) f_{ij}(Q_i,Q_j)\,. \tag{4.53}$$

One can identify the zeroth order of the virial expansion

$$\mathcal{G}^{(D)}_{pq,qp}(E,\omega) = -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]} \tag{4.54}$$

(the superscript (D) refers to "diagonal" because it reflects completely localized states without any interactions) and the first order

$$\mathcal{G}^{(2)}_{pq,qp}(E,\omega) = -\sum_{\substack{i,j=1\\ i<j}}^{N} \int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\left(\prod_{i=1}^{N} e^{S_0[Q_i]}\right) f_{ij}(Q_i,Q_j) \tag{4.55}$$

such that

$$\mathcal{G}_{pq,qp}(E,\omega) = \mathcal{G}^{(D)}_{pq,qp} + \mathcal{G}^{(2)}_{pq,qp}\,. \tag{4.56}$$

This is the virial expansion up to the first virial coefficient $\mathcal{V}^{(2)}$.

Zeroth and first order term of the virial expansion

Let us take a closer look at the zeroth and first order of the virial expansion separately:

Zeroth order: It can be written in the form

$$\begin{aligned}\mathcal{G}^{(D)}_{pq,qp}(E,\omega) &= -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]} &(4.57)\\ &= -\int D\{Q_p\}\mathcal{P}^{RA}e^{S_0[Q_p]}\int D\{Q_q\}\mathcal{Q}^{AR}e^{S_0[Q_q]}\\ &\quad\times\prod_{i\neq p,q}\int D\{Q_i\}e^{S_0[Q_i]}\,, &(4.58)\end{aligned}$$

i.e. all integrals can be factorized. Now, we use the fact that the supersymmetry (cf. appendix A) is only broken for the integration over p and q variables due to the supersymmetry-breaking Grassmann factors $\mathcal{P}^{RA} = \chi_p^R\overline{\chi_p^A}$ and $\mathcal{Q}^{AR} = -\chi_q^A\overline{\chi_q^R}$ in front of the exponentials. Thus, all the other integrals where the supersymmetry is unbroken yield unity. As a consequence, one obtains

$$\mathcal{G}^{(D)}_{pq,qp}(E,\omega) = -\int D\{Q_p\}\mathcal{P}^{RA}e^{S_0[Q_p]}\int D\{Q_q\}\mathcal{Q}^{AR}e^{S_0[Q_q]}\,. \tag{4.59}$$

It can be shown that (see the calculations done for this thesis)

$$\mathcal{G}^{(D)}_{pq,qp}(E,\omega) = 0\,,\quad p\neq q\,. \tag{4.60}$$

A physical explanation for this is that the integrals over both p and q variables in Eq. (4.59) correspond to correlations of wave functions at one and the same point,

but different energies, and such localized wave functions are uncorrelated in the case of the diagonal RMT.

First order: The leading two-supermatrix contribution to the virial expansion reads

$$\mathcal{G}^{(2)}_{pq,qp}(E,\omega) = -\sum_{\substack{i,j=1\\ i<j}}^{N} \int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\left(\prod_{i=1}^{N} e^{S_0[Q_i]}\right) f_{ij}(Q_i,Q_j) \tag{4.61}$$

$$= -\sum_{\substack{i,j=1\\ i<j}}^{N} \int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\left(\prod_{i=1}^{N} e^{S_0[Q_i]}\right)\left(e^{S[Q_i,Q_j]}-1\right) . \tag{4.62}$$

Basically, the same argumentation as above for the the zeroth order term allows us to write

$$\mathcal{G}^{(2)}_{pq,qp}(E,\omega) = -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}e^{S_0[Q_p]+S_0[Q_q]}f_{pq} \tag{4.63}$$

$$-\sum_{i<p,q}\int D\{Q_p\}D\{Q_q\}D\{Q_i\}\mathcal{P}^{RA}\mathcal{Q}^{AR} \tag{4.64}$$

$$\times e^{S_0[Q_p]+S_0[Q_q]+S_0[Q_i]}\left(f_{ip}+f_{iq}\right) \tag{4.65}$$

$$-\sum_{i>p,q}\int D\{Q_p\}D\{Q_q\}D\{Q_i\}\mathcal{P}^{RA}\mathcal{Q}^{AR} \tag{4.66}$$

$$\times e^{S_0[Q_p]+S_0[Q_q]+S_0[Q_i]}\left(f_{pi}+f_{qi}\right) . \tag{4.67}$$

For all the integrals with functions f_{ij} the indices of which do not correspond to the ones contained in the supersymmetry-breaking factors, the supersymmetry is unbroken and, thus, these terms cancel

$$\int D\{Q_i\}D\{Q_j\}e^{S_0[Q_i]+S_0[Q_j]}\underbrace{\left(e^{S[Q_i,Q_j]}-1\right)}_{=f_{ij}} = 1-1 = 0\,. \tag{4.68}$$

The terms with functions f_{kl} where just one of the indices corresponds to p or q vanish, too. This is because the integral over the variables which bear an index different from the indices of f_{kl} yields zero, e.g.

$$\int D\{Q_p\}D\{Q_i\}\mathcal{P}^{RA}e^{S_0[Q_p]+S_0[Q_i]}f_{ip}\underbrace{\int D\{Q_q\}\mathcal{Q}^{AR}e^{S_0[Q_q]}}_{=0} = 0\,. \tag{4.69}$$

The integral over q variables is zero because it again corresponds to uncorrelated wave functions.

What remains at the end is

$$\mathcal{G}^{(2)}_{pq,qp}(E,\omega) = -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}e^{S_0[Q_p]+S_0[Q_q]}f_{pq}\,. \tag{4.70}$$

Thus, the two-supermatrix approximation of $\mathcal{G}_{pq,qp}$ reads

$$\mathcal{G}_{pq,qp}(E,\omega) \simeq \mathcal{G}^{(D)}_{pq,qp} + \mathcal{G}^{(2)}_{pq,qp} \tag{4.71}$$

$$= -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}e^{S_0[Q_p]+S_0[Q_q]}f_{pq}\,. \tag{4.72}$$

This will be the starting point of the calculations done for this thesis. We just consider the case $p \neq q$. The case $p = q$ corresponding to the LDOS-correlation function has been considered in [36, 37].

5. Calculations for the almost diagonal GUE and GOE

In this chapter, we present original calculations which were done during this thesis. The product of disorder-averaged Green's functions

$$\mathcal{G}_{pq,qp}(E,\omega) = \left\langle G^R_{pq}\left(E+\frac{\omega}{2}\right) G^A_{qp}\left(E-\frac{\omega}{2}\right)\right\rangle \tag{5.1}$$

$$= -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]}\prod_{\substack{i,j=1\\ i<j}}^{N} e^{S[Q_i,Q_j]} \tag{5.2}$$

is calculated for the almost diagonal RMT in the cases of GUE and GOE. The supersymmetric version of the virial expansion is applied and the results are calculated in the two-supermatrix approximation

$$\mathcal{G}_{pq,qp}(E,\omega) \simeq \mathcal{G}^{(D)}_{pq,qp} + \mathcal{G}^{(2)}_{pq,qp} \tag{5.3}$$

$$= -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR} e^{S_0[Q_p]+S_0[Q_q]}$$

$$-\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR} e^{S_0[Q_p]+S_0[Q_q]}\left(e^{S[Q_p,Q_q]}-1\right). \tag{5.4}$$

The calculations are referred to the band center $E=0$ such that $\mathcal{G}_{pq,qp}(E=0,\omega)\equiv \mathcal{G}_{pq,qp}(\omega)$ and the single-matrix action $S_0[Q_k]$ then reads

$$S_0[Q_k] = i\frac{\Omega}{2}\mathrm{Str}(\Lambda Q_k) - \frac{1}{2\beta}\left(\mathrm{Str}\, Q_k\right)^2, \quad \beta = 1,2\,. \tag{5.5}$$

For a deeper understanding where Eqs. (5.2) and (5.4) come from, the reader is referred to section 4.5.3 and the appendix.

5.1. Calculations for the GUE

5.1.1. First term of the virial expansion

We are going to show that the first term or zeroth order of the virial expansion is zero as mentioned in the last chapter. The first term reads

$$\mathcal{G}^{(D)}_{pq,qp}(\omega) = -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}e^{S_0[Q_p]+S_0[Q_q]}\,. \tag{5.6}$$

This expression can be factorized in the form

$$\mathcal{G}^{(D)}_{pq,qp}(\omega) = -\int D\{Q_p\}\mathcal{P}^{RA}e^{S_0[Q_p]}\int D\{Q_q\}\mathcal{Q}^{AR}e^{S_0[Q_q]}\,. \tag{5.7}$$

Thus, it suffices to consider one of the integrals and show that it vanishes. Let us consider the integral over p variables and apply the λ-parametrization in order to make the supermatrix Q_p block-diagonal (see appendix C):

$$\int D\{Q_p\}\mathcal{P}^{RA}e^{S_0[Q_p]} = \int\left(\frac{d\lambda d\varphi d\bar{\xi}d\xi}{\pi\lambda}\right)_p^{R,A}\lambda_p^{R,A}\,e^{i(\varphi_p^R-\varphi_p^A)}\,\tilde{\mathcal{P}}^{RA}\,e^{S_0[\lambda_p^R,\lambda_p^A]} \tag{5.8}$$

$$= \int\left(\frac{d\lambda d\varphi d\bar{\xi}d\xi}{\pi}\right)_p^{R,A}e^{i(\varphi_p^R-\varphi_p^A)}\,\tilde{\mathcal{P}}^{RA}\,e^{S_0[\lambda_p^R,\lambda_p^A]}\,. \tag{5.9}$$

The abbreviation $(\ldots)^{R,A}_{p,q}$ denotes the product of all retarded and advanced variables labeled by p and q, for example

$$\lambda^{R,A}_{p,q} = \lambda^R_p\lambda^R_q\lambda^A_p\lambda^A_q \tag{5.10}$$

$$\left(d\lambda d\varphi d\bar{\xi}d\xi\right)^{R,A}_{p,q} = \left(d\lambda_p d\varphi_p d\overline{\xi_p}d\xi_p d\lambda_q d\varphi_q d\overline{\xi_q}d\xi_q\right)^R \times\left(d\lambda_p d\varphi_p d\overline{\xi_p}d\xi_p d\lambda_q d\varphi_q d\overline{\xi_q}d\xi_q\right)^A\,. \tag{5.11}$$

Eq. (5.9) yields zero because the integrals contain the full set of Grassmannian variables and the factor $e^{i(\varphi_p^R-\varphi_p^A)}$. Thus, both integrals over $\{\xi\}$ and $\{\varphi\}$ give zero while integrals over unbounded commuting variables converge, i.e. there is no anomaly.

5.1.2. Leading term of the virial expansion

As a preliminary remark, we will drop all double indices ($\alpha_{ij}\longrightarrow\alpha$, $b_{ij}\longrightarrow b$, etc.) during the calculations for the leading term because we only deal with the indices p and q. The starting point is

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) = -\int D\{Q_p\}D\{Q_q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}e^{S_0[Q_p]+S_0[Q_q]}f(Q_p,Q_q) \tag{5.12}$$

with

$$f(Q_p, Q_q) = e^{S[Q_p,Q_q]} - 1 = \sum_{k=1}^{\infty} \frac{(-2b\mathrm{Str}(Q_pQ_q))^k}{k!}\,. \tag{5.13}$$

By using the λ-parametrization, we obtain

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) = -\frac{1}{\pi^4}\int (d\lambda d\varphi d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{i\theta}e^{S_0[\lambda^R_p,\lambda^A_p]+S_0[\lambda^R_q,\lambda^A_q]}f(\{\lambda\},\{\alpha\},\theta+\Delta) \tag{5.14}$$

with α, θ and Δ defined as in appendix C. Now, we can reduce the angle integration over all four angles to just one single angle by making a Taylor expansion in the nilpotent quantity Δ. In order to see this, we consider the integration with respect to one single angle φ and perform the Taylor expansion. One obtains

$$\begin{aligned}\int_0^{2\pi} d\varphi\, e^{i(\varphi+c)}f(\varphi+c+\Delta) = \int_0^{2\pi} d\varphi\, e^{i(\varphi+c)}\Big(& f(\varphi+c)+\Delta f'(\varphi+c)\\ &+\frac{\Delta^2}{2}f''(\varphi+c)\Big)\,.\end{aligned} \tag{5.15}$$

c contains all the other three angles. Next, we do the variable transformation

$$\theta = \varphi + c \tag{5.16}$$

and make use of the 2π-periodicity of the function f. We can write

$$\int_0^{2\pi} d\theta\, e^{i\theta}\left(f(\theta)+\Delta f'(\theta)+\frac{\Delta^2}{2}f''(\theta)\right) = \int_0^{2\pi} d\theta\, e^{i\theta}e^{-i\Delta}f(\theta)\,. \tag{5.17}$$

The function f now contains only one angle such that all the other three angles can be integrated out trivially. Furthermore, we integrated by parts to get the right hand side of Eq. (5.17). In fact, $\Delta^5 = 0$, but Eq. (5.17) can be generalized to higher nilpotents. Thus, we obtain

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega) &= -\frac{8}{\pi}\int d\theta(d\lambda d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{i\theta}e^{-i\Delta}e^{S_0[\lambda^R_p,\lambda^A_p]+S_0[\lambda^R_q,\lambda^A_q]}\\ &\quad\times f(\{\lambda\},\{\alpha\},\theta) \qquad (5.18)\\ &= -\frac{8}{\pi}\int d\theta(d\lambda d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\cos\theta\, e^{-i\Delta}e^{S_0[\lambda^R_p,\lambda^A_p]+S_0[\lambda^R_q,\lambda^A_q]}\\ &\quad\times f(\{\lambda\},\{\alpha\},\theta)\,. \qquad (5.19)\end{aligned}$$

In the last step, we used that f is an even function with respect to θ such that only the real part of $e^{i\theta}$ survives when integrating from $-\pi$ to π. Let us now apply

the RS-parametrization (see appendix C):

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) = -\frac{1}{2\pi}\int d\theta \left(\frac{dRdS}{\sqrt{S^2-R^2}}\right)_{p,q} (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\cos\theta\, e^{-i\Delta}e^{i\frac{\Omega}{2}(S_p+S_q)} \times e^{-\frac{(R_p^2+R_q^2)}{4}} f(\{R\},\{S\},\{\alpha\},\theta)\,. \tag{5.20}$$

In order to disentangle the integrals over R and S variables, we perform now a saddle-point approximation (see appendix B for more details) which gives

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq -2\int_0^\infty dS_{p,q}\int_0^{2\pi} d\theta \int (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\cos\theta\, e^{-i\Delta}(S_pS_q)^{-1}\, e^{i\frac{\Omega}{2}(S_p+S_q)} \times f(\{R=0\},\{S\},\{\alpha\},\theta) \tag{5.21}$$

with

$$f(\{R=0\},\{S\},\{\alpha\},\theta) = \sum_{k=1}^{\infty}\frac{(-2bS_pS_q)^k}{k!4^k}\left[4\sin^2\left(\frac{\theta}{2}\right) - 2\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right) \times \sin^2\left(\frac{\theta}{2}\right) - \frac{1}{2}\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\cos\theta\right]^k. \tag{5.22}$$

At this point, one is able to integrate out all the S variables using the integration formula

$$\int_0^\infty dS S^{k-1}e^{i\frac{\Omega}{2}S} = (-2)^k\frac{\Gamma(k)}{(i\Omega)^k}\,, \tag{5.23}$$

where $\Gamma(k) = (k-1)!$ is the Gamma function [42], and one obtains

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq -2\sum_{k=1}^{\infty}\left(\frac{2b}{\Omega^2}\right)^k\frac{(k-1)!}{k}\int_0^{2\pi} d\theta\int (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{-i\Delta}\cos\theta \times\left[4\sin^2\left(\frac{\theta}{2}\right) - 2\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^2\left(\frac{\theta}{2}\right) - \frac{1}{2}\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\cos\theta\right]^k. \tag{5.24}$$

Using the multinomial formula

$$(x_1+x_2+\cdots+x_m)^k = \sum_{j_1+j_2+\cdots+j_m=k}\frac{k!}{j_1!j_2!\cdots j_m!}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m} \tag{5.25}$$

and the property of all the α variables to be nilpotent, we can evaluate the kth power of the expression in square brackets:

$$\begin{aligned}\left[4\sin^2\left(\frac{\theta}{2}\right) - 2\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^2\left(\frac{\theta}{2}\right) - \frac{1}{2}\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\cos\theta\right]^k \\ = 8\sin^{2k}\left(\frac{\theta}{2}\right) - 4k\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^{2k}\left(\frac{\theta}{2}\right) \\ + k\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(2k\sin^{2k}\left(\frac{\theta}{2}\right) - \sin^{2k-2}\left(\frac{\theta}{2}\right)\right). \qquad (5.26)\end{aligned}$$

The Grassmanns can now be integrated out by performing the Taylor expansion of $e^{-i\Delta}$ and writing

$$\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\,e^{-i\Delta} = \tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR} - \frac{1}{2}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\left(\xi_q^R\overline{\xi_p^R} + \xi_p^A\overline{\xi_q^A}\right) + \frac{1}{4}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi_q^R\overline{\xi_p^R}\xi_p^A\overline{\xi_q^A}\,. \qquad (5.27)$$

Now, we split $\mathcal{G}^{(2)}_{pq,qp}(\omega)$ up into three terms:

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) = \mathcal{G}^{(2)}_{pq,qp}(\omega)_0 + \mathcal{G}^{(2)}_{pq,qp}(\omega)_2 + \mathcal{G}^{(2)}_{pq,qp}(\omega)_4\,. \qquad (5.28)$$

The indices 0, 2, 4 refer to the number of additional Grassmanns which now contribute to the supersymmetry-breaking factors. In the following, we will use the helpful formulae:

- **Supersymmetry-breaking factors**:

$$\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\overline{\alpha^R}\alpha^R = \tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi_q^R\overline{\xi_p^R} \qquad (5.29)$$

$$\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\overline{\alpha^A}\alpha^A = \tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi_p^A\overline{\xi_q^A} \qquad (5.30)$$

$$\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A = \tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi_q^R\overline{\xi_p^R}\xi_p^A\overline{\xi_q^A}\,. \qquad (5.31)$$

- **Angle integration**:

$$\int_0^{2\pi} d\theta\,\sin^{2k}\left(\frac{\theta}{2}\right) = 2\sqrt{\pi}\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}\,,\quad k\geq 0\,. \qquad (5.32)$$

- **Duplication formula for the Gamma function**:

$$\Gamma(2k) = \sqrt{2\pi}\,2^{2k-\frac{1}{2}}\Gamma(k)\Gamma\left(k+\frac{1}{2}\right). \qquad (5.33)$$

Let us now turn to the three terms of $\mathcal{G}^{(2)}_{pq,qp}(\omega)_m$ $(m = 0, 2, 4)$ separately:

$\mathbf{m = 0}$:

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega)_0 = -\frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{8b}{\Omega^2}\right)^k \frac{(k-1)!}{k}\int_0^{2\pi} d\theta \int (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\cos\theta \\ \times\left[8\sin^{2k}\left(\frac{\theta}{2}\right) - 4k\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^{2k}\left(\frac{\theta}{2}\right)\right. \\ \left. + k\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(2k\sin^{2k}\left(\frac{\theta}{2}\right) - \sin^{2k-2}\left(\frac{\theta}{2}\right)\right)\right] \qquad (5.34)\end{aligned}$$

$$= -2\pi\sum_{k=1}^{\infty}\left(\frac{2b}{\Omega^2}\right)^k \frac{2k^3-2k^2+1}{k(k+1)(2k-1)}\frac{\Gamma(2k)}{\Gamma(k)}\,. \qquad (5.35)$$

$\mathbf{m = 2}$:

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega)_2 = \frac{1}{8}\sum_{k=1}^{\infty}\left(\frac{8b}{\Omega^2}\right)^k \frac{(k-1)!}{k}\int_0^{2\pi} d\theta \int (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\left(\xi^R_q\overline{\xi^R_p} + \xi^A_p\overline{\xi^A_q}\right) \\ \times\cos\theta\left[8\sin^{2k}\left(\frac{\theta}{2}\right) - 4k\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^{2k}\left(\frac{\theta}{2}\right)\right. \\ \left. + k\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(2k\sin^{2k}\left(\frac{\theta}{2}\right) - \sin^{2k-2}\left(\frac{\theta}{2}\right)\right)\right] \qquad (5.36)\end{aligned}$$

$$= -4\pi\sum_{k=1}^{\infty}\left(\frac{2b}{\Omega^2}\right)^k \frac{\Gamma(2k)}{(k+1)\Gamma(k)}\,. \qquad (5.37)$$

$\mathbf{m = 4}$:

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega)_4 = -\frac{1}{16}\sum_{k=1}^{\infty}\left(\frac{8b}{\Omega^2}\right)^k \frac{(k-1)!}{k}\int_0^{2\pi} d\theta \int (d\bar{\xi}d\xi)^{R,A}_{p,q}\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi^R_q\overline{\xi^R_p}\xi^A_p\overline{\xi^A_q} \\ \times\cos\theta\left[8\sin^{2k}\left(\frac{\theta}{2}\right) - 4k\left(\overline{\alpha^R}\alpha^R + \overline{\alpha^A}\alpha^A\right)\sin^{2k}\left(\frac{\theta}{2}\right)\right. \\ \left. + k\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(2k\sin^{2k}\left(\frac{\theta}{2}\right) - \sin^{2k-2}\left(\frac{\theta}{2}\right)\right)\right] \qquad (5.38)\end{aligned}$$

$$= -2\pi\sum_{k=1}^{\infty}\left(\frac{2b}{\Omega^2}\right)^k \frac{\Gamma(2k)}{k(k+1)\Gamma(k)}\,. \qquad (5.39)$$

Combining all three terms yields

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq -4\pi\sum_{k=1}^{\infty}\left(\frac{2b}{\Omega^2}\right)^k \frac{k\Gamma(2k-1)}{\Gamma(k)}\,. \qquad (5.40)$$

This expression represents an asymptotic series with respect to Ω and is not very suitable for further considerations. Therefore, we go to the time representation of Eq. (5.40).

Time representation

We perform the Fourier transform of $\mathcal{G}^{(2)}_{pq,qp}(\omega)$

$$\mathcal{G}^{(2)}_{pq,qp}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi\mathcal{D}}\, e^{-i\omega t}\, \mathcal{G}^{(2)}_{pq,qp}(\omega)\,, \tag{5.41}$$

where $\mathcal{D}$ is the mean level spacing. Using the formula

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega + i\eta)^{2k}} = \frac{(-1)^k}{\Gamma(2k)} t^{2k-1} e^{-\eta t}\theta(t) \tag{5.42}$$

with $\theta(t)$ being the Heaviside step function, one obtains

$$\mathcal{G}^{(2)}_{pq,qp}(t) \simeq -\frac{4\pi}{\mathcal{D}} \frac{e^{-\eta t}\theta(t)}{t} \sum_{k=1}^{\infty} \frac{(-2bt^2)^k}{\Gamma(2k)} \frac{k\Gamma(2k-1)}{\Gamma(k)}\,. \tag{5.43}$$

This expression can be summed up (e.g. with Wolfram Mathematica) and we arrive at

$$\mathcal{G}^{(2)}_{pq,qp}(t) \simeq \frac{\pi}{\mathcal{D}}\theta(t)\left[\sqrt{2\pi b}\,\mathrm{erf}(\sqrt{2b}\,t) + 4b\,t e^{-2bt^2}\right], \tag{5.44}$$

where $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t dx\, e^{-x^2}$ denotes the error function. The regularizer η has been set to zero because it is not needed for the convergence of the Fourier backtransform any more. $\mathcal{G}^{(2)}_{pq,qp}(t)$ is bounded within the interval $[0,\infty)$ (see Fig. 5.1.). This is the first main result of the present thesis.

Energy representation

Going back to the energy representation

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) = \mathcal{D}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\mathcal{G}^{(2)}_{pq,qp}(t)\,, \tag{5.45}$$

and using the formula

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t}\,\sqrt{2\pi b}\,\mathrm{erf}(\sqrt{2b}\,t)\theta(t) = \sqrt{2\pi b}\Big(\int_0^{\infty} dt\, e^{i\omega t}\,\Big(\mathrm{erf}(\sqrt{2b}\,t) - 1\Big) + \int_0^{\infty} dt\, e^{i\omega t}\Big) \tag{5.46}$$

$$= \sqrt{2\pi^3 b}\,\delta(\omega) - \frac{\sqrt{2\pi b}}{\omega} e^{-\frac{\omega^2}{8b}}\,\mathrm{erfi}\left(\frac{\omega}{\sqrt{8b}}\right) + i\frac{\sqrt{2\pi b}}{\omega} e^{-\frac{\omega^2}{8b}}\,, \tag{5.47}$$

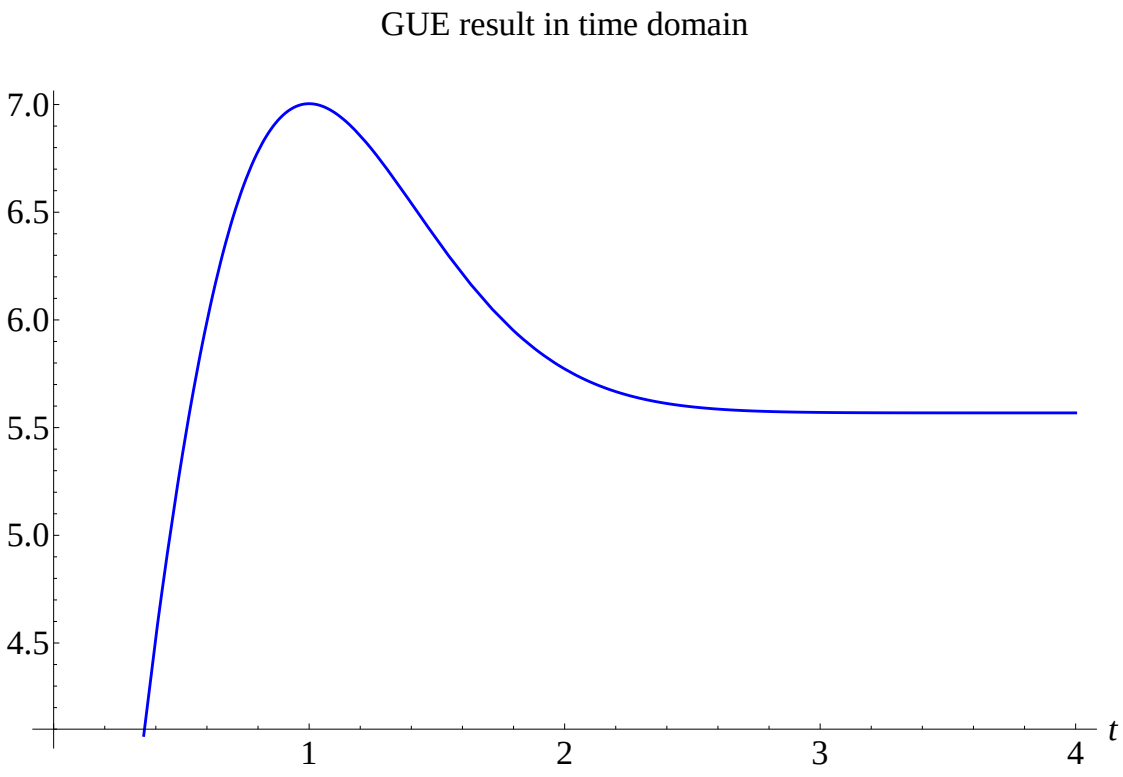


Figure 5.1.: Dimensionless plot of Eq. (5.44).

one gets for the energy representation

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq \pi^{\frac{3}{2}}\left[\pi\sqrt{2b}\,\delta(\omega) + \frac{1}{\sqrt{\pi}} - \left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right)\mathrm{erfi}\left(\frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}} + i\left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}}\right]. \tag{5.48}$$

or split into real and imaginary part

$$\mathrm{Re}\mathcal{G}^{(2)}_{pq,qp}(\omega) = \pi^{\frac{3}{2}}\left[\pi\sqrt{2b}\,\delta(\omega) + \frac{1}{\sqrt{\pi}} - \left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right) \times \mathrm{erfi}\left(\frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}}\right] \tag{5.49}$$

$$\mathrm{Im}\mathcal{G}^{(2)}_{pq,qp}(\omega) = \pi^{\frac{3}{2}}\left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}}. \tag{5.50}$$

The function $\mathrm{erfi}(x) = -i\mathrm{erf}(ix)$ denotes the imaginary error function. Eq. (5.48) is the second main result of the present thesis.

5.2. Calculations for the GOE

Concerning the orthogonal symmetry class, all the steps of the calculations are very similar to those of the GUE case. One can start considering the first and leading order of the virial expansion after the saddle-point approximation because the zeroth order vanishes for the same arguments as for the unitary symmetry class. Hence, the starting point of the GOE calculations is

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq & -\frac{1}{8\pi^3}\int_0^\infty dS_{p,q}\int_0^{2\pi} d\varphi^{R,A}_{p,q}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{i\phi^R}e^{-i\phi^A} \\ & \times \frac{e^{i\frac{\Omega}{2}(S_p+S_q)}}{S_pS_q} f(\{R=0\},\{S\},\{\alpha\},\phi^R+\Delta^R,\phi^A+\Delta^A) \qquad (5.51) \\ = & -\frac{1}{8\pi^3}\sum_{k=1}^{\infty}\frac{(-b)^k}{2^k k!}\int_0^\infty dS_{p,q}\int_0^{2\pi} d\varphi^{R,A}_{p,q}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR} \\ & \times e^{i\phi^R}e^{-i\phi^A}\left(S_pS_q\right)^{k-1}e^{i\frac{\Omega}{2}(S_p+S_q)}\Bigg[\cos\left(\phi^R+\Delta^R\right)\left(1-\frac{1}{2}\overline{\alpha^R}\alpha^R\right) \\ & -\cos\left(\phi^A+\Delta^A\right)\left(1-\frac{1}{2}\overline{\alpha^A}\alpha^A\right)\Bigg]^{2k} \qquad (5.52)\end{aligned}$$

with ϕ and Δ being defined for the retarded and advanced sector as in appendix C. In total analogy with the calculations for the GUE, we can reduce the angle integration for each sector to the integration over the relative angle ϕ whereas the other angles can be integrated out. Moreover, by making a Taylor expansion in Δ for each sector and integrating by parts, one can rewrite Eq. (5.52) in the form

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq & -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(-b)^k}{2^k k!}\int_0^\infty dS_{p,q}\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{-i\Delta^R}e^{i\Delta^A} \\ & \times \cos\phi^R\cos\phi^A\left(S_pS_q\right)^{k-1}e^{i\frac{\Omega}{2}(S_p+S_q)}\Bigg[\cos\phi^R\left(1-\frac{1}{2}\overline{\alpha^R}\alpha^R\right) \\ & -\cos\phi^A\left(1-\frac{1}{2}\overline{\alpha^A}\alpha^A\right)\Bigg]^{2k}. \qquad (5.53)\end{aligned}$$

5. Calculations for the almost diagonal GUE and GOE

The S variables can be integrated out like for the GUE and Eq. (5.53) reads

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(2b)^k}{k\Omega^{2k}}\Gamma(k)\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{-i\Delta^R}\,e^{i\Delta^A} \\ \times\cos\phi^R\cos\phi^A\left[\cos\phi^R\left(1-\frac{1}{2}\overline{\alpha^R}\alpha^R\right)-\cos\phi^A\left(1-\frac{1}{2}\overline{\alpha^A}\alpha^A\right)\right]^{2k}. \tag{5.54}$$

Next, we integrate out the Grassmann variables. For this purpose, one must apply the multinomial formula (see the GUE calculations) to evaluate the $2k$th power of the expression in square brackets:

$$\begin{aligned}&\left[\cos\phi^R\left(1-\frac{1}{2}\overline{\alpha^R}\alpha^R\right)-\cos\phi^A\left(1-\frac{1}{2}\overline{\alpha^A}\alpha^A\right)\right]^{2k}\\ &\quad=\left(\cos\phi^R-\cos\phi^A\right)^{2k}-k\overline{\alpha^R}\alpha^R\left(\cos\phi^R-\cos\phi^A\right)^{2k-1}\cos\phi^R\\ &\qquad+k\overline{\alpha^A}\alpha^A\left(\cos\phi^R-\cos\phi^A\right)^{2k-1}\cos\phi^A\\ &\qquad-\frac{k}{2}(2k-1)\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(\cos\phi^R-\cos\phi^A\right)^{2k-2}\cos\phi^R\cos\phi^A\,.\end{aligned} \tag{5.55}$$

To integrate out the Grassmanns and angles, it is convenient to split $\mathcal{G}^{(2)}_{pq,qp}(\omega)$ again up into three terms ($m=0,2,4$) corresponding to the number of additional Grassmanns joining the supersymmetry-breaking factors by making a Taylor expansion in Δ^R, Δ^A

$$\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}e^{-i\Delta^R}e^{i\Delta^A}=\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}-\frac{1}{2}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\left(\xi^R_q\overline{\xi^R_p}+\xi^A_p\overline{\xi^A_q}\right) \\ +\frac{1}{4}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\xi^R_q\overline{\xi^R_p}\xi^A_p\overline{\xi^A_q}\,. \tag{5.56}$$

Let us consider the three terms separately:

$\mathbf{m} = \mathbf{0}$:

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega)_0 = &-\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(2b)^k}{k\Omega^{2k}}\Gamma(k)\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\\ &\times\cos\phi^R\cos\phi^A\left[\cos\phi^R\left(1-\frac{1}{2}\overline{\alpha^R}\alpha^R\right)-\cos\phi^A\left(1-\frac{1}{2}\overline{\alpha^A}\alpha^A\right)\right]^{2k} \qquad (5.57)\\ = &-\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(2b)^k}{k\Omega^{2k}}\Gamma(k)\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\\ &\times\cos\phi^R\cos\phi^A\Big[\left(\cos\phi^R-\cos\phi^A\right)^{2k}\\ &-k\overline{\alpha^R}\alpha^R\left(\cos\phi^R-\cos\phi^A\right)^{2k-1}\cos\phi^R\\ &+k\overline{\alpha^A}\alpha^A\left(\cos\phi^R-\cos\phi^A\right)^{2k-1}\cos\phi^A\\ &-\frac{k}{2}(2k-1)\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\left(\cos\phi^R-\cos\phi^A\right)^{2k-2}\cos\phi^R\cos\phi^A\Big] \qquad (5.58)\end{aligned}$$

The Grassmanns can now be integrated out

$$\begin{aligned}\mathcal{G}^{(2)}_{pq,qp}(\omega)_0 = &\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{(2b)^k}{\Omega^{2k}}\Gamma(k)(2k-1)\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\overline{\alpha^R}\alpha^R\overline{\alpha^A}\alpha^A\\ &\times\left(\cos\phi^R-\cos\phi^A\right)^{2k-2}\cos^2\phi^R\cos^2\phi^A \qquad (5.59)\\ = &-\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{(2b)^k}{\Omega^{2k}}\Gamma(k)(2k-1)\int_0^{2\pi} d\phi^{R,A}\int \left(d\bar{\xi}d\xi\right)^{R,A}_{p,q}\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\\ &\times\left(\cos\phi^R-\cos\phi^A\right)^{2k-2}\cos^2\phi^R\cos^2\phi^A\,. \qquad (5.60)\end{aligned}$$

The angle integration is a bit more cumbersome, but can be done for the terms $\mathcal{G}^{(2)}_{pq,qp}(\omega)_2$ and $\mathcal{G}^{(2)}_{pq,qp}(\omega)_4$ analogously. Let us consider the integral

$$\begin{aligned}I_\phi = &\int_0^{2\pi} d\phi^{R,A}\left(\cos\phi^R-\cos\phi^A\right)^{2k-2}\cos^2\phi^R\cos^2\phi^A \qquad (5.61)\\ = &\sum_{j=0}^{2k-2}(-1)^j\binom{2k-2}{j}\int_0^{2\pi} d\phi^{R,A}\left(\cos\phi^R\right)^{2k-j}\left(\cos\phi^A\right)^{j+2}\,. \qquad (5.62)\end{aligned}$$

In the last step, the binomial formula

$$(a+b)^m = \sum_{j=0}^{m}\binom{m}{j}a^{m-j}b^j \qquad (5.63)$$

was used. With the help of the integral identity

$$\int_0^{2\pi} d\phi \, cos^j \phi = \frac{(-2)^j \left(1 + (-1)^j\right) \pi^2}{\Gamma(j+1)\Gamma^2\left(\frac{1}{2} - \frac{j}{2}\right)}, \quad j \geq 0 \tag{5.64}$$

one can carry out the angle integration

$$I_\phi = 4^{k+1}\pi^4 \sum_{j=0}^{2k-2} \binom{2k-2}{j} \frac{1 + (-1)^j}{\Gamma(2k+1-j)\Gamma^2\left(\frac{1}{2} - k + \frac{j}{2}\right)\Gamma(j+3)\Gamma^2\left(-\frac{j}{2} - \frac{1}{2}\right)}. \tag{5.65}$$

Here, only even values of j survive in the sum and doing the summation over j (e.g. with Wolfram Mathematica) yields

$$I_\phi = \pi 4^k \frac{2k^3 - 2k^2 + 1}{2k} \frac{\Gamma^2\left(k - \frac{1}{2}\right)}{\Gamma(k)\Gamma(k+2)}. \tag{5.66}$$

Thus, one obtains

$$\mathcal{G}^{(2)}_{pq,qp}(\omega)_0 = -\sum_{k=1}^{\infty} \frac{(8b)^k}{\Omega^{2k}} \frac{(2k-1)\left(2k^3 - 2k^2 + 1\right)}{8k} \frac{\Gamma^2\left(k - \frac{1}{2}\right)}{\Gamma(k+2)}. \tag{5.67}$$

The terms $m = 2$ and $m = 4$ are calculated in the same manner, so only the results are presented below:

$\mathbf{m = 2}$:

$$\begin{aligned} \mathcal{G}^{(2)}_{pq,qp}(\omega)_2 = \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{(2b)^k}{\Omega^{2k}} \Gamma(k) \int_0^{2\pi} d\phi^{R,A} \cos\phi^R \cos\phi^A \\ \times \Big[\left(\cos\phi^R - \cos\phi^A\right)^{2k-1} \cos\phi^R \\ - \left(\cos\phi^R - \cos\phi^A\right)^{2k-1} \cos\phi^A \Big] \end{aligned} \tag{5.68}$$

$$= -\sum_{k=1}^{\infty} \frac{(8b)^k}{\Omega^{2k}} \frac{\Gamma^2\left(k + \frac{1}{2}\right)}{\Gamma(k+2)}. \tag{5.69}$$

$\mathbf{m = 4}$:

$$\mathcal{G}^{(2)}_{pq,qp}(\omega)_4 = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{(2b)^k}{\Omega^{2k}} \frac{\Gamma(k)}{k} \int d\phi^{R,A} \cos\phi^R \cos\phi^A \left(\cos\phi^R - \cos\phi^A\right)^{2k} \tag{5.70}$$

$$= -\sum_{k=1}^{\infty} \frac{(8b)^k}{\Omega^{2k}} \frac{1}{2k} \frac{\Gamma^2\left(k + \frac{1}{2}\right)}{\Gamma(k+2)}. \tag{5.71}$$

Putting all three terms together then yields

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq -\sum_{k=1}^{\infty} \frac{(8b)^k}{\Omega^{2k}} \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)}{2\Gamma(k)} \,. \tag{5.72}$$

Time representation

In the time domain and after performing the k-summation, we end up with

$$\mathcal{G}^{(2)}_{pq,qp}(t) \simeq \frac{2\pi}{\mathcal{D}} \theta(t) I_0(bt^2) bt e^{-bt^2} \,, \tag{5.73}$$

where $I_0(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}t^2\right)^k}{(k!)^2}$ is the modified Bessel function of the first kind of the order zero. This is the third main result of the present thesis.

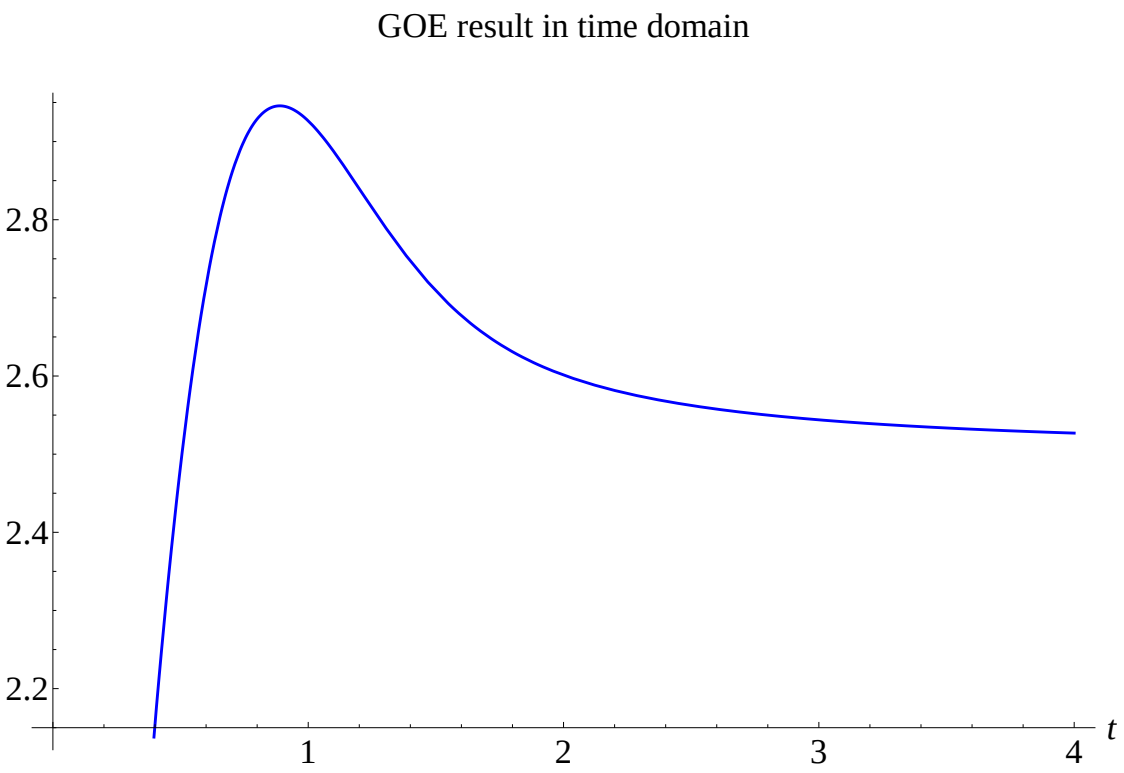


Figure 5.2.: Dimensionless plot of Eq. (5.73).

Energy representation

The Fourier backtransform to the energy domain gives

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq \sqrt{2\pi^3 b}\left[\delta(\omega) + \frac{|\omega|}{16b} e^{-\frac{\omega^2}{16b}} \left(I_1\left(\frac{\omega^2}{16b}\right) - I_0\left(\frac{\omega^2}{16b}\right)\right)\right] + i\sqrt{2\pi b}\frac{\omega}{16b} e^{-\frac{\omega^2}{16b}} \left(K_0\left(\frac{\omega^2}{16b}\right) + K_1\left(\frac{\omega^2}{16b}\right)\right), \tag{5.74}$$

or, dividing into real and imaginary part, we obtain

$$\mathrm{Re}\mathcal{G}^{(2)}_{pq,qp}(\omega) = \sqrt{2\pi^3 b}\left[\delta(\omega) + \frac{|\omega|}{16b} e^{-\frac{\omega^2}{16b}} \left(I_1\left(\frac{\omega^2}{16b}\right) - I_0\left(\frac{\omega^2}{16b}\right)\right)\right] \tag{5.75}$$

$$\mathrm{Im}\mathcal{G}^{(2)}_{pq,qp}(\omega) = \sqrt{2\pi b}\frac{\omega}{16b} e^{-\frac{\omega^2}{16b}} \left(K_0\left(\frac{\omega^2}{16b}\right) + K_1\left(\frac{\omega^2}{16b}\right)\right). \tag{5.76}$$

The functions $K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}$ $(\nu = 0, 1)$ in the imaginary part are modified Bessel functions of the second kind. Eq. (5.74) is the fourth and last main result obtained for this thesis.

5.3. Brief discussion of the results and sum rule

Both the GUE and GOE results bear astonishing resemblance with results previously obtained for the second term of the virial expansion of the local density of states correlation function $\langle\langle \mathcal{G}^{(2)}_{pp}(\omega)\rangle\rangle$ [36, 37]. One can notice that

$$\langle\langle \mathcal{G}^{(2)}_{pp}(\omega)\rangle\rangle \equiv -\sum_{\substack{q \\ q\neq p}} \mathcal{G}^{(2)}_{pq,qp}(\omega)\,. \tag{5.77}$$

This relation has a profound, physical reason. Let us consider the Fourier transform with respect to time of the probability $P(x,t)$ for a localized particle at time $t = 0$ to travel the distance $|x| = |p - q|$ between the space points p and q within the time interval $t > 0$

$$P(x,\omega) \propto \mathcal{G}_{pq,qp}(\omega)\,. \tag{5.78}$$

The probability is equivalent to the correlator we consider in this thesis (see [43]). It is known that the zero-mode value $(k = 0)$ of $P(k,\omega)$ in momentum representation, which is identical to the normalization condition of $P(x,\omega)$, yields

$$P(k = 0,\omega) = \int dx\, P(x,\omega) = \frac{i}{\omega}\,. \tag{5.79}$$

Eq. (5.79) reflects particle conservation and is a very profound feature. Thus, one may expect that this feature should appear in the leading order of the virial expansion (VE). So, when the virial expansion is applied up to the 2-supermatrix approximation, one obtains:

$$\sum_{p,q} \mathcal{G}_{pq,qp}(\omega) = \sum_{p} \mathcal{G}_{pp,pp}(\omega) + \sum_{\substack{p,q \\ q\neq p}} \mathcal{G}_{pq,qp}(\omega) \overset{\text{VE}}{\simeq} \langle\langle \mathcal{G}^{(D)}_{pp}(\omega)\rangle\rangle \propto i\frac{\rho(E=0)}{\omega}\,. \quad (5.80)$$

This is because the 2-supermatrix term cancels due to Eq. (5.77) and only the term $\langle\langle \mathcal{G}^{(D)}_{pp}(\omega)\rangle\rangle \propto i\frac{\rho(E=0)}{\omega}$ (see [36, 37]) survives since $\mathcal{G}^{(D)}_{pq,qp}(\omega) = 0$. Therefore, in order to see corrections, one must also take the 3-supermatrix terms ($\mathcal{V}^{(3)}$) into account. Corrections to Eq. (5.80) may result from corrections to the density of states ρ, but they are absent within the 2-supermatrix approximation [15].

6. Critical scaling analysis

We apply the critical power law banded RMT (PLBRMT) to the results obtained for the GOE ($\beta = 1$) and GUE ($\beta = 2$) in the last chapter. For the critical PLBRMT, the function $\mathcal{F}(|p-q|) \equiv \mathcal{B}^2 g(|p-q|)$ ($\mathcal{B} \ll 1$) decreases in a power law fashion

$$\mathcal{F}(|p-q|) \equiv \mathcal{B}^2 g(|p-q|) = \frac{\mathcal{B}^2}{\mathcal{B}^2 + |p-q|^2} \approx \frac{\mathcal{B}^2}{|p-q|^2}\,, \quad \mathcal{B} \ll 1\,. \tag{6.1}$$

Thus, one must substitute

$$b = \frac{1}{2\beta}\mathcal{F}(|p-q|) \approx \frac{1}{2\beta}\frac{\mathcal{B}^2}{|p-q|^2} = \frac{1}{2\beta}\frac{\mathcal{B}^2}{|x|^2} \tag{6.2}$$

for b with $x = p - q$. We perform a scaling analysis of $\mathcal{G}^{(2)}_{pq,qp}(x,\omega)$ and show that the dynamical scaling hypothesis (see section 2.2.1) holds true for the correlation function at criticality we investigate.

6.1. Dynamical scaling hypothesis

In section 2.2.1, it was pointed out that every correlation function $C(|x|,\omega)$ of wave functions at the Anderson transition point which are separated in space and energy scales in the following way:

$$C(|x|,\omega) \sim \left(\frac{|x|}{L_\omega}\right)^{-(1-d_2)}\,, \quad l < |x| < L_\omega\,. \tag{6.3}$$

d_2 is the second fractal dimension, l is the mean free path and L_ω reads

$$L_\omega = \frac{\mathcal{B}}{\omega} \tag{6.4}$$

with respect to the critical RMT. The scaling relation in Eq. (6.3) is the so-called dynamical scaling hypothesis [13, 27, 23]. In the following, a scaling analysis of the GUE and GOE results is done for the critical region $|x| < L_\omega$.

6.2. Scaling behavior

6.2.1. GUE

Applying the critical PLBRMT to the GUE results of the last chapter ($\omega \neq 0$) yields

$$\mathrm{Re}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) = \pi\left[1-\sqrt{\pi}\left(\frac{\mathcal{B}}{2\tilde{\omega}|x|}+\frac{\tilde{\omega}}{\mathcal{B}}|x|\right)\,\mathrm{erfi}\left(\frac{\tilde{\omega}}{\mathcal{B}}|x|\right)\,e^{-\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2}\right] \tag{6.5}$$

$$\mathrm{Im}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) = \pi^{\frac{3}{2}}\left(\frac{\mathcal{B}}{2\tilde{\omega}|x|}+\frac{\tilde{\omega}}{\mathcal{B}}|x|\right)\,e^{-\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2}\,. \tag{6.6}$$

The definition $\tilde{\omega} = \frac{\omega}{\beta\sqrt{2}}$ has been introduced. In the limit $d_2 \longrightarrow 0$, which accounts for the fact that we deal with the critical almost diagonal RMT $\mathcal{B} \ll 1$, i.e. the strong multifractality regime, and

$$d_2 \propto \mathcal{B} \tag{6.7}$$

(see e.g. [37]), one expects, according to the dynamical scaling hypothesis, that

$$\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) \sim \left(\frac{|x|}{L_{\tilde{\omega}}}\right)^{-1} = \frac{\mathcal{B}}{\tilde{\omega}|x|} \tag{6.8}$$

for $|x| < L_{\tilde{\omega}}$.

Critical region

We see that for $|x| < L_{\tilde{\omega}}$ the real part of $\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega})$ vanishes in a power law fashion

$$\mathrm{Re}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) \sim \left(\frac{\tilde{\omega}}{\mathcal{B}}|x|\right)^2\,, \tag{6.9}$$

i.e. there is no critical region for the real part and it does not bear any information about the fractality of the wave functions. However, the imaginary part scales like

$$\mathrm{Im}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) \sim \frac{\mathcal{B}}{\tilde{\omega}|x|}\,, \tag{6.10}$$

which is in accordance with Eq. (6.8).

6.2.2. GOE

The same scaling analysis works for the critical GOE. Here, the real and imaginary part of the correlator acquire the form

$$\mathrm{Re}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) = \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{|\tilde{\omega}|}{\mathcal{B}}|x|\, e^{-\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2} \left(I_1\left(\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2\right) - I_0\left(\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2\right)\right) \tag{6.11}$$

$$\mathrm{Im}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) = \sqrt{\frac{\pi}{2}}\,\frac{\tilde{\omega}}{2\mathcal{B}}|x|\, e^{-\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2} \left(K_0\left(\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2\right) + K_1\left(\frac{\tilde{\omega}^2}{\mathcal{B}^2}|x|^2\right)\right). \tag{6.12}$$

In the critical region, the real part vanishes with

$$\mathrm{Re}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) \sim -\frac{\tilde{\omega}}{\mathcal{B}}|x|\,. \tag{6.13}$$

It is again the imaginary part which shows critical scaling

$$\mathrm{Im}\,\mathcal{G}^{(2)}_{pq,qp}(x,\tilde{\omega}) \sim \frac{\mathcal{B}}{\tilde{\omega}|x|}\,. \tag{6.14}$$

To conclude, the dynamical scaling hypothesis obviously holds true for both the GUE and GOE results.

7. Conclusion

In this thesis, we have studied the propagation probability of critical disordered systems, i.e. systems at the Anderson transition point, which are either strongly multifractal or critical insulators. We have especially examined the propagation probability for the orthogonal and unitary symmetry classes. As far as it is known at the stage of the development of this thesis, such correlation functions have not been considered before and our results are the first basic step for linear response theory. The response functions can be straightforwardly obtained from our results Eqs. (5.44), (5.48) and (5.73), (5.74).

In order to investigate universal properties of disordered systems which are at or close to the point of the localization transition in the strong multifractality regime, we have used almost diagonal random matrix theory. This type of RMT is characterized by parametrically small off-diagonal matrix elements $H_{i\neq j} \sim \mathcal{B}$ ($\mathcal{B} \ll 1$). A specific representative of almost diagonal RMT which has been applied in this thesis is the (critical) power law banded random matrix theory (PLBRMT). This model describes a one-dimensional chain with long-range hopping and incorporates all features of the wave functions at or close to the Anderson transition point. The already existing non-linear supermatrix σ-model, which applies to the case of weak multifractality ($\mathcal{B} \gg 1$) very well, fails to apply to systems in the strong multifractality regime. Therefore, we have used a supersymmetric version of a virial expansion, which is a perturbative approach accounting for the fact that contributions of terms taking higher and higher numbers of interacting localized states into account decrease. Using this method, the propagation probabilities for the orthogonal and unitary symmetry classes have been calculated up to the leading term of the virial expansion.

To briefly sum up what the several chapters in this thesis were about, in chapter 2, we gave an introduction to the field of Anderson physics. We presented some early attempts to analyze disordered systems undergoing a transition from localized to delocalized states and took a closer look at the features of wave functions at the critical point of the Anderson transition. These features range from a critical scaling behavior to weak and strong fractality of the wave functions. Furthermore, it was mentioned that the Anderson transition can be regarded as second-order phase transition with a non-trivially defined order parameter which is rather an order parameter function. The first part of chapter 3 dealt with the development of the field of random matrix theory and its application to disorderd systems. The

main focus was on the critical power law banded RMT describing low-dimensional and high-dimensional critical systems by a very broad ($\mathcal{B} \gg 1$) or narrow ($\mathcal{B} \ll 1$) bandwith $\mathcal{B}$ outside of which interactions of localized states decay in a power law fashion. The second part reviewed the ideas for the virial expansion to examine high-dimensional critical systems because, as already mentioned, the method called non-linear supermatrix σ-model fails to apply to these critical systems with very narrow bandwidth. In chapter 4, our results for the unitary and orthogonal symmetry classes of the almost diagonal RMT were presented. We have found (see Eqs. (5.44), (5.48) and (5.73), (5.74)):

unitary symmetry class:

$$\mathcal{G}^{(2)}_{pq,qp}(t) \simeq \frac{\pi}{\Delta}\theta(t)\left[\sqrt{2\pi b}\,\mathrm{erf}(\sqrt{2b}\,t) + 4b\,te^{-2bt^2}\right] \tag{7.1}$$

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq \pi^{\frac{3}{2}}\left[\pi\sqrt{2b}\,\delta(\omega) + \frac{1}{\sqrt{\pi}} - \left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right)\mathrm{erfi}\left(\frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}} + i\left(\frac{\sqrt{2b}}{\omega} + \frac{\omega}{\sqrt{8b}}\right)e^{-\frac{\omega^2}{8b}}\right]. \tag{7.2}$$

orthogonal symmetry class:

$$\mathcal{G}^{(2)}_{pq,qp}(t) \simeq \frac{2\pi}{\Delta}\theta(t)I_0(bt^2)bte^{-bt^2} \tag{7.3}$$

$$\mathcal{G}^{(2)}_{pq,qp}(\omega) \simeq \sqrt{2\pi^3 b}\left[\delta(\omega) + \frac{|\omega|}{16b}e^{-\frac{\omega^2}{16b}}\left(I_1\left(\frac{\omega^2}{16b}\right) - I_0\left(\frac{\omega^2}{16b}\right)\right)\right] + i\sqrt{2\pi b}\frac{\omega}{16b}e^{-\frac{\omega^2}{16b}}\left(K_0\left(\frac{\omega^2}{16b}\right) + K_1\left(\frac{\omega^2}{16b}\right)\right). \tag{7.4}$$

Furthermore, we have found a sum rule indicating that there are no corrections to the space-integrated propagation probability within the 2-supermatrix approximation of the supersymmetric virial expansion, i.e. when considering the zero-mode of the propagation probability to get from point p to point $q \neq p$:

$$\sum_{p,q}\mathcal{G}_{pq,qp}(\omega) \simeq i\frac{\rho(E=0)}{\omega}. \tag{7.5}$$

This is due to particle conservation and due to the fact that corrections are expected to contribute when considering higher terms of the virial expansion. In the last chapter, applying the critical PLBRMT, the results have been compared with scaling predictions made by J. T. Chalker [23]. It has been shown with accuracy

of the leading term of the virial expansion that the dynamical scaling hypothesis holds true, which demonstrates that it is not sensitive to phases of wave functions. To the best of our knowledge, this result has never been obtained before.

The next steps will be to compare our results with numerical calculations and to relate them to the critical conductance. Additionally, we intend to study a crossover between the conductivity at the Anderson transition point and Mott's conductivity.

As an outlook to the future, there is still plenty of work to do concerning the theoretical description of localization. From an analytical point of view, it is always favorable to have exact methods. As it was mentioned in this thesis, the non-linear supermatrix σ-model provides an exact solution at least for the situation of multifractal metals or weak multifractality, i.e. for the critical power law banded RMT with bandwidth $\mathcal{B} \gg 1$. But, concerning the case of a multifractal insulator and the case of strong multifractality where this model ceases to be valid, there exist only perturbative approaches such as the supersymmetric virial expansion used here. To close the gap between the weak and strong multifractality regimes, a supermatrix field-theoretical approach named superbosonization, which has been suggested in [44], appears to be a promising candidate. It has been applied to calculate the density of states for an almost diagonal random matrix ensemble. At the next step, one can use superbosonization for multipoint correlation functions using the recently suggested idea of analytic continuation [45].

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A. A short introduction to supermathematics

This chapter is meant to give a brief survey of the mathematics we will be dealing with in this thesis.

In supermathematics, one deals with ordinary (commuting) variables and anti-commuting variables at once, so-called Grassmann variables or Grassmanns [46]. In analogy to boson and fermion fields in field theory, the mentioned variables are also referred to bosonic (commuting) and fermionic (anti-commuting) variables, respectively.

A.1. Grassmanns and their properties

The Grassmanns χ form an anti-commuting algebra, i. e. different elements of this algebra χ_i and χ_j anti-commute

$$\{\chi_i, \chi_j\} = \chi_i\chi_j + \chi_j\chi_i = 0\,. \tag{A.1}$$

The curly brackets denote the anti-commutator. As a consequence, all Grassmanns are nilpotent

$$\chi_i^2 = 0\,. \tag{A.2}$$

It is also possible to define the complex conjugate of Grassmanns

$$\overline{\chi_i\chi_j} = \overline{\chi_i}\,\overline{\chi_j}\,. \tag{A.3}$$

It is further defined to have some kind of norm which does not change under complex conjugation

$$\overline{\overline{\chi}\chi} = \overline{\chi}\chi\,. \tag{A.4}$$

This implies that

$$\overline{\overline{\chi}} = -\chi \tag{A.5}$$

for all Grassmanns. It should be mentioned that $\overline{\chi}$ itself anti-commute with χ. Even functions of Grassmann variables as their arguments can be defined. Let, as

an example, χ and ξ be two Grassmanns, then a function of these two variables can be made sense of through its Taylor expansion

$$f(\chi,\xi) = f(\chi,\xi=0) + \frac{\partial}{\partial\xi} f(\chi,\xi=0)\xi \tag{A.6}$$

$$= f(\chi=0,\xi=0) + \frac{\partial}{\partial\chi} f(\chi=0,\xi=0)\chi + \frac{\partial}{\partial\xi} f(\chi=0,\xi=0)\xi + \frac{\partial}{\partial\xi\partial\chi} f(\chi=0,\xi=0)\chi\xi\,, \tag{A.7}$$

where the derivative with respect to anti-commuting variables was used. It has the properties defined as follows:

$$\frac{\partial}{\partial\chi}\chi = 1\,, \quad \frac{\partial}{\partial\chi}\xi = 0 \tag{A.8}$$

$$\left\{\frac{\partial}{\partial\chi}, \frac{\partial}{\partial\xi}\right\} = 0 \tag{A.9}$$

$$\left\{\frac{\partial}{\partial\chi}, \chi\right\} = 0 = \left\{\frac{\partial}{\partial\chi}, \xi\right\}\,. \tag{A.10}$$

So, a function with anti-commuting arguments can in general be viewed as a series of all combinations of containing Grassmanns $\{\chi\}$

$$f(\{\chi\}) = \sum_{\substack{i_k=0,1 \\ k=1,2,\ldots,N}} a_{i_1 i_2 \cdots i_N} \chi_1^{i_1} \chi_2^{i_2} \cdots \chi_N^{i_N}\,. \tag{A.11}$$

A.2. Supervectors

Supervectors are objects consisting of commuting and anti-commuting variables. They can be viewed as the direct product of two vectors, the one containing only commuting variables s and the other one only Grassmanns χ

$$\Psi = \begin{pmatrix} s \\ \chi \end{pmatrix}\,. \tag{A.12}$$

The transpose of Ψ is defined as usual

$$\Psi^T = (s, \chi)\,. \tag{A.13}$$

Together with the transpose, the adjoint vector reads

$$\Psi^\dagger = (\overline{s}, \overline{\chi})\,. \tag{A.14}$$

The squared norm of Ψ thus gets

$$|\Psi|^2 = \Psi^\dagger \Psi = \overline{s} s + \overline{\chi} \chi \,, \tag{A.15}$$

which is invariant under complex conjugation due to the definition of the complex conjugate.

A.3. Supermatrices

Apart from supervectors, one can also call for supermatrices which have the form

$$Q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \,, \tag{A.16}$$

where a, b and σ, ρ are ordinary matrices containing commuting variables and anti-commuting variables, respectively. This definition is because Q is to map an element from superspace to superspace

$$\Psi' = Q\Psi \,, \tag{A.17}$$

where Ψ' is again a supervector. The transpose of a supermatrix is defined as

$$Q^T = \begin{pmatrix} a^T & \rho^T \\ -\sigma^T & b^T \end{pmatrix} \tag{A.18}$$

in order to make a supermatrix be unchanged when applying the Hermitian conjugate two times

$$\left(Q^\dagger\right)^\dagger = Q \,. \tag{A.19}$$

Of course, one is also able to diagonalize supermatrices in the following way:

$$Q = UDV^\dagger \,, \tag{A.20}$$

where U, V are two different unitary supermatrices and D is diagonal. In the special case of Hermitian supermatrices, both U and V coincide.

The trace of a supermatrix Q (see Eq. (A.16) called supertrace is defined as

$$\mathrm{Str}\, Q = \mathrm{tr}\, a - \mathrm{tr}\, b \tag{A.21}$$

in order to equip it with the same invariance under cyclic permutations as the trace of ordinary matrices. The superdeterminant of Q is computed as follows:

$$\mathrm{Sdet}\, Q = \frac{\det\left(a - \sigma b^{-1} \rho\right)}{\det b} \,. \tag{A.22}$$

It has the same properties as the determinant of conventional matrices.

To avoid confusions regarding the calculations done for this thesis, it should be pointed out that one must differentiate between two supermatrix representations:

- **boson-fermion**
  This representation refers to the definition of supermatrices described above where the diagonal blocks consist of commuting variables and the off-diagonal blocks contain anticommuting ones.

- **retarded-advanced**
  The situation is now a bit different. The supermatrices in this representation have the shape

$$Q|_{RA} = \begin{pmatrix} Q^{RR} & Q^{RA} \\ Q^{AR} & Q^{AA} \end{pmatrix} \tag{A.23}$$

  and their supertrace reads

$$\operatorname{Str} Q|_{RA} = \operatorname{Str} Q^{RR} + \operatorname{Str} Q^{AA} \,. \tag{A.24}$$

  The matrices Q^{kl} $(k, l \in \{R, A\})$ are supermatrices in boson-fermion representation (cf. Eq. (A.16)).

All the subsequent definitions and explanations on supermathematics will rely on the first representation.

A.4. Integral calculus

A.4.1. Integrals over Grassmanns

The whole integral calculus with respect to anti-commuting variables is based just on two definitions

$$\int d\chi = \int d\overline{\chi} = 0 \,, \quad \int d\chi \, \chi = \int d\overline{\chi} \, \overline{\chi} = 1 \,. \tag{A.25}$$

The same anti-commutation relations hold for the differentials as in Eqs. (A.9) and (A.10). One sees that differentiation and integration are closely connected to each other and can principally be used interchangeably. As a result of the definitions (A.25), one can deduce the property that the integral over a function of Grassmanns is "translation" invariant

$$\int d\chi \, f(\chi + \xi) = \int d\chi \, f(\chi) \,. \tag{A.26}$$

Linear transformation of variables

Sometimes, it can be useful to change the integration variables from one set of Grassmanns to another by a linear transformation of the form

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_N \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{N1} & & & \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} = A\xi \,. \tag{A.27}$$

The integral then has to transform in the following way to keep up the definitions (A.25):

$$\int d\chi_1 d\chi_2 \cdots d\chi_N f(\chi) = (\det A)^{-1} \int d\xi_1 d\xi_2 \cdots d\xi_N f(A\xi) \,. \tag{A.28}$$

The Jacobian $\mathcal{J}$ of this transformation is therefore

$$\mathcal{J} = (\det A)^{-1} \,. \tag{A.29}$$

Gaussian Grassmann integrals

This kind of integrals plays an important role for the calculations in this thesis. It is known from their counterparts with ordinary commuting variables that the integral equation

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{d^2 s_i}{\pi} \, e^{-s^\dagger A s} = (\det A)^{-1} \tag{A.30}$$

holds. A can be an arbitrary complex $N \times N$ matrix under the condition $\operatorname{Re} A > 0$. The integration measure $d^2 s_i$ stands for $d(\operatorname{Re} s_i) d(\operatorname{Im} s_i)$. It is also possible to go from an integration over real and imaginary parts to one over the actual complex variables and their conjugates such that

$$d^2 s_i = d(\operatorname{Re} s_i) d(\operatorname{Im} s_i) = \frac{d\overline{s_i} ds_i}{2i} \,. \tag{A.31}$$

Nearly the same integral equation (A.30) can be found for Grassmann variables with one exception:

$$\int \prod_{i=1}^{N} d\overline{\chi_i} d\chi_i \, e^{-\chi^\dagger A \chi} = \det A \,. \tag{A.32}$$

The determinant on the right hand side of this equation stands in the numerator and not in the denominator. Eq. (A.32) is proven for arbitrary matrices in [30]. Let

us here consider the case of a Hermitian matrix H. We know that any Hermitian matrix can be diagonalized by a unitary transformation U in the form

$$H = UEU^\dagger\,, \tag{A.33}$$

where E is diagonal and contains all the eigenvalues of H. Thus, Eq. (A.32) becomes

$$\int \prod_{i=1}^{N} d\overline{\chi_i} d\chi_i \, e^{-\chi^\dagger H \chi} = \int \prod_{i=1}^{N} d\overline{\xi_i} d\xi_i \, e^{-\xi^\dagger E \xi} \tag{A.34}$$

$$= \int \prod_{i=1}^{N} d\overline{\xi_i} d\xi_i \, e^{-\overline{\xi_i} E_i \xi_i} \tag{A.35}$$

$$= \int \prod_{i=1}^{N} d\overline{\xi_i} d\xi_i \, \left(1 - E_i \overline{\xi_i} \xi_i\right) \tag{A.36}$$

$$= \prod_{i=1}^{N} E_i = \det H\,. \tag{A.37}$$

In the first line, the change of variables was made:

$$\chi^\dagger U = \xi^\dagger\,, \quad U^\dagger \chi = \xi\,. \tag{A.38}$$

The Jacobian of such a transformation is always unity. There is another property of Gaussian Grassmann integrals analogous to the ordinary case. The matrix elements of an inverse matrix can be expressed through the formula

$$\left(A^{-1}\right)_{kl} = (\det A)^{-1} \int \prod_{i=1}^{N} d\overline{\chi_i} d\chi_i \, \chi_k \overline{\chi_l} \, e^{-\chi^\dagger A \chi}\,. \tag{A.39}$$

This can be shown when we denote the left hand side of Eq. (A.32) by I and take the derivative of its logarithm with respect to the matrix element A_{lk}

$$\frac{\partial}{\partial A_{lk}} \ln I = (\det A)^{-1} \int \prod_{i=1}^{N} d\overline{\chi_i} d\chi_i \, \chi_k \overline{\chi_l} \, e^{-\chi^\dagger A \chi}\,. \tag{A.40}$$

Simultaneously, one has to take the same derivative of the logarithm of the right hand side

$$\frac{\partial}{\partial A_{lk}} \ln \det A = \left(A^{-1}\right)_{kl}\,. \tag{A.41}$$

Hence, it follows that

$$\left(A^{-1}\right)_{kl} = \frac{\partial}{\partial A_{lk}} \ln I \tag{A.42}$$

and Eq. (A.39) holds true.

A.4.2. Superintegrals

When one now combines integration over commuting and anti-commuting variables, this leads to the notion of superintegrals

$$\int_D \prod_{i=1}^{N} ds_i \int \prod_{j=1}^{M} d\chi_j \, f(s_1, s_2, \ldots, s_N, \chi_1, \chi_2, \ldots, \chi_N) \,. \tag{A.43}$$

This class of integrals is well-defined as long as the integrals over the commuting sector do not diverge, i.e. the integrand with respect to the commuting variables must be bounded within the integration domain D and go sufficiently fast to zero at the boundaries of it. Such requirements are not needed for the integrals over Grassmanns since they only yield zero or one according to the definitions (A.25).

Gaussian superintegrals

By combining the conventional Gaussian integral and the one for Grassmann variables, one immediately obtains

$$\int d\Psi^\dagger d\Psi \, e^{-\Psi^\dagger F \Psi} = \mathrm{Sdet}\, F = 1 \,, \tag{A.44}$$

where Ψ is a supervector of the form (A.12) and $d\Psi^\dagger d\Psi = \prod_{i=1}^{N} \frac{d^2 s_i}{\pi} d\overline{\chi_i} d\chi_i$. F is a supermatrix containing only the matrix A in the upper left and bottom right block

$$F = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,. \tag{A.45}$$

As a result, we can rewrite Eq. (A.39) in terms of a superintegral

$$\left(A^{-1}\right)_{kl} = \int d\Psi^\dagger d\Psi \, \chi_k \overline{\chi_l} \, e^{-\Psi^\dagger F \Psi} \,. \tag{A.46}$$

We can conclude that the "supersymmetry" is preserved for Eq. (A.44) unless no Grassmanns which break this symmetry are in front of the Gaussian exponential.

A.4.3. Change of variables

In some situations, it can be favorable to choose another set of integration variables which may simplify the integrals tremendously. This change of variables can of course be performed either with respect to the sector of commuting or anti-commuting variables. Changing variables in the sector of anti-commuting variables can be done in the standard way described in section A.4.1. However,

one sometimes has to choose parametrizations that mix commuting variables and anti-commuting ones such that the old variables are functions of both types of variables:

$$s_i = s_i\left(\{\lambda\},\{\xi\}\right)\,, \quad \chi_i = \chi_i\left(\{\lambda\},\{\xi\}\right)\,, \quad i = 1,2,\ldots,N\,. \tag{A.47}$$

$\{\lambda\}$ and $\{\xi\}$ denote the full sets of new variables. Such a change of variables is now carried out by the so-called Berezinian B named after F. A. Berezin, who is said to be the originator of supermathematics. The Berezinian is analogous to the Jacobian of ordinary variables, which arises from the change of the integration measure

$$d\Psi^\dagger d\Psi \longrightarrow B d\tilde{\Psi}^\dagger d\tilde{\Psi}\,. \tag{A.48}$$

The Berezinian is the superdeterminant of a matrix containing the partial derivatives of the old variables with respect to the new ones and can be calculated as follows:

$$B = \mathrm{Sdet}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \tag{A.49}$$

with

$$a_{ij} = \frac{\partial s_i}{\partial \lambda_j}\,, \quad \alpha_{ij} = \frac{\partial s_i}{\partial \xi_j} \tag{A.50}$$

$$\beta_{ij} = \frac{\partial \chi_i}{\partial \lambda_j}\,, \quad b_{ij} = \frac{\partial \chi_i}{\partial \xi_j}\,. \tag{A.51}$$

B. Representation of Green's functions in terms of superintegrals

This appendix is meant to give an outline of the calculations done for this diploma thesis.

B.1. Superintegral representation

The retarded and advanced Green's functions $\hat{G}^R\left(E+\frac{\omega}{2}\right)$ and $\hat{G}^A\left(E-\frac{\omega}{2}\right)$ at different energies are defined as

$$\hat{G}^R\left(E+\frac{\omega}{2}\right)=\left(\left(E+\frac{\Omega}{2}\right)\hat{\mathbb{1}}-\hat{H}\right)^{-1}, \quad \hat{G}^A\left(E-\frac{\omega}{2}\right)=\left(\left(E-\frac{\Omega}{2}\right)\hat{\mathbb{1}}-\hat{H}\right)^{-1} \tag{B.1}$$

with $\Omega=\omega+i\eta$ and $\eta\longrightarrow 0^+$ being a regularizing parameter as usual. According to Eq. (A.46), it is possible to express every arbitrary matrix element of the Green's functions in terms of Gaussian superintegrals

$$G^R_{pq}\left(E+\frac{\omega}{2}\right)=\int d\Psi^{R\dagger}d\Psi^R\,\chi^R_p\overline{\chi^R_q}\,e^{i\Psi^{R\dagger}F^R\Psi^R} \tag{B.2}$$

$$G^A_{qp}\left(E-\frac{\omega}{2}\right)=\int d\Psi^{A\dagger}d\Psi^A\,\chi^A_q\overline{\chi^A_p}\,e^{-i\Psi^{A\dagger}F^A\Psi^A}\,. \tag{B.3}$$

Here, $F^{R/A}$ is a supermatrix

$$F^R=\begin{pmatrix}\left(E+\frac{\Omega}{2}\right)\mathbb{1}-H & 0\\ 0 & \left(E+\frac{\Omega}{2}\right)\mathbb{1}-H\end{pmatrix} \tag{B.4}$$

$$F^A=\begin{pmatrix}\left(E-\frac{\Omega}{2}\right)\mathbb{1}-H & 0\\ 0 & \left(E-\frac{\Omega}{2}\right)\mathbb{1}-H\end{pmatrix}. \tag{B.5}$$

The regularizing parameter in Ω ensures convergence of the integrals over commuting variables. The product of the Green's functions can be written in the form

$$G^R_{pq}\left(E+\frac{\omega}{2}\right)G^A_{qp}\left(E-\frac{\omega}{2}\right)=-\int d\Psi^\dagger d\Psi\,\mathcal{P}^{RA}\mathcal{Q}^{AR}\,e^{i\Psi^\dagger F\Psi} \tag{B.6}$$

using

$$\mathcal{P}^{RA}\mathcal{Q}^{AR} = -\chi_p^R\overline{\chi_q^R}\chi_q^A\overline{\chi_p^A} \tag{B.7}$$

$$F = \begin{pmatrix} F^R & 0 \\ 0 & -F^A \end{pmatrix} \tag{B.8}$$

$$\Psi = \begin{pmatrix} \Psi^R \\ \Psi^A \end{pmatrix} . \tag{B.9}$$

B.2. Disorder average and supermatrix parametrization

In this thesis, the disorder-averaged product of the Green's functions in Eq. (B.6) is considered, i.e. we assume H to be a random $N \times N$ matrix from the almost diagonal Gaussian unitary and orthogonal ensemble introduced in chapter 4 and perform the ensemble average. The probability distribution of the diagonal and off-diagonal matrix elements for the GUE and the GOE respectively read

- **GUE**: H is a complex Hermitian matrix

$$P(H_{ii}) = \frac{1}{\sqrt{\pi}} e^{-H_{ii}^2} , \quad P(H_{i<j}) = \frac{1}{2\pi b_{ij}} e^{-\frac{|H_{ij}|^2}{2b_{ij}}} \tag{B.10}$$

$$b_{ij} = \frac{1}{4}\mathcal{F}(|i-j|) \equiv \frac{1}{4}\mathcal{B}^2 g(|i-j|) , \quad \mathcal{B} \ll 1 . \tag{B.11}$$

- **GOE**: H is real and symmetric

$$P(H_{ii}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{H_{ii}^2}{2}} , \quad P(H_{i<j}) = \frac{1}{\sqrt{2\pi b_{ij}}} e^{-\frac{H_{ij}^2}{2b_{ij}}} \tag{B.12}$$

$$b_{ij} = \frac{1}{2}\mathcal{F}(|i-j|) \equiv \frac{1}{2}\mathcal{B}^2 g(|i-j|) , \quad \mathcal{B} \ll 1 . \tag{B.13}$$

Let us start with the GUE.

B.2.1. Gaussian unitary ensemble

Let us analyze the exponential in the expression for the product of Green's functions. In terms of commuting and anti-commuting variables, it looks like as follows:

$$\begin{aligned} e^{i\Psi^\dagger F \Psi} = \exp\Bigg\{ & i\sum_{i,j}\left[\left(E+\frac{\Omega}{2}\right)\delta_{ij} - H_{ij}\right]\left(\overline{s_i^R}s_j^R + \overline{\chi_i^R}\chi_j^R\right) \\ & - i\sum_{i,j}\left[\left(E-\frac{\Omega}{2}\right)\delta_{ij} - H_{ij}\right]\left(\overline{s_i^A}s_j^A + \overline{\chi_i^A}\chi_j^A\right)\Bigg\} . \end{aligned} \tag{B.14}$$

We define now (just for notational convience)

$$R_{ij} = \overline{s_i^R} s_j^R + \overline{\chi_i^R} \chi_j^R \tag{B.15}$$

$$A_{ij} = \overline{s_i^A} s_j^A + \overline{\chi_i^A} \chi_j^A \,. \tag{B.16}$$

So, one can obtain

$$e^{i\Psi^\dagger F \Psi} = \exp\left\{ iE \sum_i (R_{ii} - A_{ii}) + i\frac{\Omega}{2} \sum_i (R_{ii} + A_{ii}) - i \sum_i H_{ii} (R_{ii} - A_{ii}) \right.$$
$$\left. - i \sum_{i \neq j} H_{ij} (R_{ij} - A_{ij}) \right\}. \tag{B.17}$$

Disorder averaging according to the probability distributions for the matrix elements from the GUE yields

$$\left\langle e^{i\Psi^\dagger F \Psi} \right\rangle = \exp\left\{ iE \sum_i (R_{ii} - A_{ii}) + i\frac{\Omega}{2} \sum_i (R_{ii} + A_{ii}) - \frac{1}{4} \sum_i (R_{ii} - A_{ii}) \right.$$
$$\left. - \sum_{i<j} \frac{b_{ij}}{2} \left((R_{ij} - A_{ij} + R_{ji} - A_{ji})^2 - (R_{ij} - A_{ij} - R_{ji} + A_{ji})^2 \right) \right\}. \tag{B.18}$$

The next step is to introduce supervectors

$$|\psi_i^R\rangle = \begin{pmatrix} s_i^R \\ \chi_i^R \end{pmatrix}, \quad |\psi_i^A\rangle = \begin{pmatrix} s_i^A \\ \chi_i^A \end{pmatrix} \tag{B.19}$$

which yield supermatrices Q_i defined as the direct product of the supervectors $|\psi_i^R\rangle$ and $|\psi_i^A\rangle$ (see [36])

$$Q_i = \begin{pmatrix} |\psi_i^R\rangle\langle\psi_i^R| & -|\psi_i^R\rangle\langle\psi_i^A| \\ |\psi_i^A\rangle\langle\psi_i^R| & -|\psi_i^A\rangle\langle\psi_i^A| \end{pmatrix} = \begin{pmatrix} Q_i^{RR} & -Q_i^{RA} \\ Q_i^{AR} & -Q_i^{AA} \end{pmatrix}. \tag{B.20}$$

This enables us to write Eq. (B.18) in terms of supertraces of Q-matrices

$$\left\langle e^{i\Psi^\dagger F \Psi} \right\rangle = \exp\left\{ \sum_i \left(iE \mathrm{Str}\, Q_i + i\frac{\Omega}{2} \mathrm{Str}(\Lambda Q_i) - \frac{1}{4} (\mathrm{Str}\, Q_i)^2 \right) \right.$$
$$\left. - \sum_{i<j} 2 b_{ij} \mathrm{Str}(Q_i Q_j) \right\} \tag{B.21}$$

with

$$\Lambda = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{B.22}$$

Here, it was used that

$$R_{ii} - A_{ii} = \langle \psi_i^R | \psi_i^R \rangle - \langle \psi_i^A | \psi_i^A \rangle = \mathrm{Str}\left(|\psi_i^R\rangle\langle\psi_i^R| \right) + \mathrm{Str}\left(-|\psi_i^A\rangle\langle\psi_i^A| \right) \tag{B.23}$$

$$= \mathrm{Str}\, Q_i^{RR} + \mathrm{Str}\, Q_i^{AA} = \mathrm{Str}\, Q_i \tag{B.24}$$

$$R_{ii} + A_{ii} = \mathrm{Str}(\Lambda Q_i) \tag{B.25}$$

and

$$(R_{ij} - A_{ij} + R_{ji} - A_{ji})^2 - (R_{ij} - A_{ij} - R_{ji} + A_{ji})^2 = 4\mathrm{Str}(Q_i Q_j)\,. \tag{B.26}$$

The last equation can be obtained by very straightforward calculations. Now, we are able to transform the disorder-averaged product of Green's functions into an integral over supermatrices Q_i

$$\left\langle G_{pq}^R \left(E + \frac{\omega}{2} \right) G_{qp}^A \left(E - \frac{\omega}{2} \right) \right\rangle = - \int D\{Q\} \mathcal{P}^{RA} \mathcal{Q}^{AR} \prod_{i=1}^{N} e^{S_0[Q_i]} \prod_{\substack{i,j=1 \\ i<j}}^{N} e^{S[Q_i,Q_j]} \tag{B.27}$$

with the help of the definitions

$$S_0[Q_i] = iE\mathrm{Str}\, Q_i + i\frac{\Omega}{2}\mathrm{Str}(\Lambda Q_i) - \frac{1}{4}\left(\mathrm{Str}\, Q_i\right)^2 \tag{B.28}$$

$$S[Q_i, Q_j] = -2b_{ij}\mathrm{Str}(Q_i Q_j) \tag{B.29}$$

$$D\{Q\} = \prod_{i=1}^{N} D\{Q_i\} = \prod_{i=1}^{N} \frac{d\overline{s_i^R} ds_i^R}{2\pi i} d\overline{\chi_i^R} d\chi_i^R \frac{d\overline{s_i^A} ds_i^A}{2\pi i} d\overline{\chi_i^A} d\chi_i^A\,. \tag{B.30}$$

The expression Eq. (B.27) is the starting point of all the considerations in this thesis.

B.2.2. Gaussian orthogonal ensemble

All the steps till the disorder average is carried out are identical to those for the GUE. So, let us begin with disorder averaging. The averaged exponential becomes

$$\left\langle e^{i\Psi^\dagger F \Psi} \right\rangle = \exp\left\{ iE \sum_i (R_{ii} - A_{ii}) + i\frac{\Omega}{2} \sum_i (R_{ii} + A_{ii}) - \frac{1}{2} \sum_i (R_{ii} - A_{ii}) \right.$$
$$\left. - \sum_{i<j} \frac{b_{ij}}{2} (R_{ij} - A_{ij} + R_{ji} - A_{ji})^2 \right\}. \tag{B.31}$$

For the GOE case, we make use of a supermatrix parametrization developed and used in [37]. We define the supervectors

$$|\psi_i^R\rangle = \begin{pmatrix} s_i^R \\ \chi_i^R \\ \overline{s_i^R} \\ \overline{\chi_i^R} \end{pmatrix}, \quad |\psi_i^A\rangle = \begin{pmatrix} s_i^A \\ \chi_i^A \\ \overline{s_i^A} \\ \overline{\chi_i^A} \end{pmatrix}. \tag{B.32}$$

The Q-matrices read

$$Q_i = \frac{1}{2}\begin{pmatrix} |\psi_i^R\rangle\langle\psi_i^R| & -|\psi_i^R\rangle\langle\psi_i^A| \\ |\psi_i^A\rangle\langle\psi_i^R| & -|\psi_i^A\rangle\langle\psi_i^A| \end{pmatrix} = \begin{pmatrix} Q_i^{RR} & -Q_i^{RA} \\ Q_i^{AR} & -Q_i^{AA} \end{pmatrix} \tag{B.33}$$

Eq. (B.33) yields

$$R_{ii} - A_{ii} = \mathrm{Str}\, Q_i \tag{B.34}$$

$$R_{ii} + A_{ii} = \mathrm{Str}(\Lambda Q_i), \quad \Lambda = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{B.35}$$

$$(R_{ij} - A_{ij} + R_{ji} - A_{ji})^2 = 4\mathrm{Str}(Q_i Q_j). \tag{B.36}$$

Note that, in contrast to the GUE where the Q-matrices are 4×4 matrices, the Q-matrices for the GOE are doubled in size, i.e. 8×8 matrices which a direct consequence of the underlying symmetry class. Again using the definitions

$$S_0[Q_i] = iE\mathrm{Str}\, Q_i + i\frac{\Omega}{2}\mathrm{Str}(\Lambda Q_i) - \frac{1}{2}\left(\mathrm{Str}\, Q_i\right)^2 \tag{B.37}$$

$$S[Q_i, Q_j] = -2b_{ij}\mathrm{Str}(Q_i Q_j) \tag{B.38}$$

$$D\{Q\} = \prod_{i=1}^{N} D\{Q_i\} = \prod_{i=1}^{N} \frac{d\overline{s_i^R} ds_i^R}{2\pi i} d\overline{\chi_i^R} d\chi_i^R \frac{d\overline{s_i^A} ds_i^A}{2\pi i} d\overline{\chi_i^A} d\chi_i^A \tag{B.39}$$

with the difference that $\frac{1}{4}\left(\mathrm{Str}\, Q_i\right)^2$ for the GUE has become $\frac{1}{2}\left(\mathrm{Str}\, Q_i\right)^2$, we are able to express the ensemble average of the product of the Green's functions in terms of superintegrals over supermatrices

$$\left\langle G_{pq}^R\left(E + \frac{\omega}{2}\right) G_{qp}^A\left(E - \frac{\omega}{2}\right)\right\rangle = -\int D\{Q\}\mathcal{P}^{RA}\mathcal{Q}^{AR}\prod_{i=1}^{N} e^{S_0[Q_i]} \prod_{\substack{i,j=1 \\ i<j}}^{N} e^{S[Q_i,Q_j]}. \tag{B.40}$$

B.3. Saddle-point approximation

During the calculations in this thesis, a crucial step, namely a saddle-point approximation, is performed in order to disentangle the integration variables and make the integrals much easier to carry out. We come across an integral of the kind

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} F\left(\sqrt{b}x, \sqrt{b}y\right) , \quad b \ll 1 . \tag{B.41}$$

assuming that the function F is slow as concerns oscillations. We rescale the variables x and y by $\sqrt{b}$ such that

$$\tilde{x} = \sqrt{b}x \tag{B.42}$$

$$\tilde{y} = \sqrt{b}y \tag{B.43}$$

and the integral expression is transformed into

$$\int_{-\infty}^{\infty} \frac{d\tilde{x}}{\sqrt{b}}\, e^{-\frac{\tilde{x}^2}{b}} F(\tilde{x}, \tilde{y}) . \tag{B.44}$$

The characteristic scale of $\tilde{x}$ is $\sqrt{b}$, i.e. the main contribution comes from values around $\tilde{x} \approx 0$ and it is possible to perform a saddle-point approximation:

$$\int_{-\infty}^{\infty} \frac{d\tilde{x}}{\sqrt{b}}\, e^{-\frac{\tilde{x}^2}{b}} \left(F(0, \tilde{y}) + O(\tilde{x}^2)\right) . \tag{B.45}$$

$F(\tilde{x}, \tilde{y})$ has been expanded in a Taylor series around $\tilde{x} \approx 0$. It is obvious that all odd orders vanish. Since $\tilde{x} \sim \sqrt{b}$, we neglect all higher orders as well. This leads to the approximation

$$\int_{-\infty}^{\infty} \frac{d\tilde{x}}{\sqrt{b}}\, e^{-\frac{\tilde{x}^2}{b}} \left(F(0, \tilde{y}) + O(\tilde{x}^2)\right) \simeq \int_{-\infty}^{\infty} \frac{d\tilde{x}}{\sqrt{b}}\, e^{-\frac{\tilde{x}^2}{b}} F(0, \tilde{y}) \tag{B.46}$$

$$= \sqrt{\pi} F(0, \tilde{y}) . \tag{B.47}$$

Afterwards, the scaling with $\sqrt{b}$ is made undone and we arrive at

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} F\left(\sqrt{b}x, \sqrt{b}y\right) \simeq \sqrt{\pi} F(0, \sqrt{b}y) . \tag{B.48}$$

Unfortunately, the situation in the virial expansion is a little more intricate. We have to deal with multiple integrals and the function F entangles the integration variables in a complicated way. Furthermore, F contains an oscillating part $F_{\mathrm{osc}}(y) \sim e^{i\Omega y}$ ($\Omega = \omega + i\eta$, $\eta \longrightarrow 0^+$) which decides about the applicability of the saddle-point approximation as the typical scale of y is $y \sim \omega^{-1}$, i.e. the saddle-point approximation should only be valid for not too fast oscillations ($\omega \ll 1$) [36]. Nevertheless, there is numerical evidence (e.g. see Fig. B.3 and [36, 37]) that this approximation is also applicable in a very broad energy range of our interest.

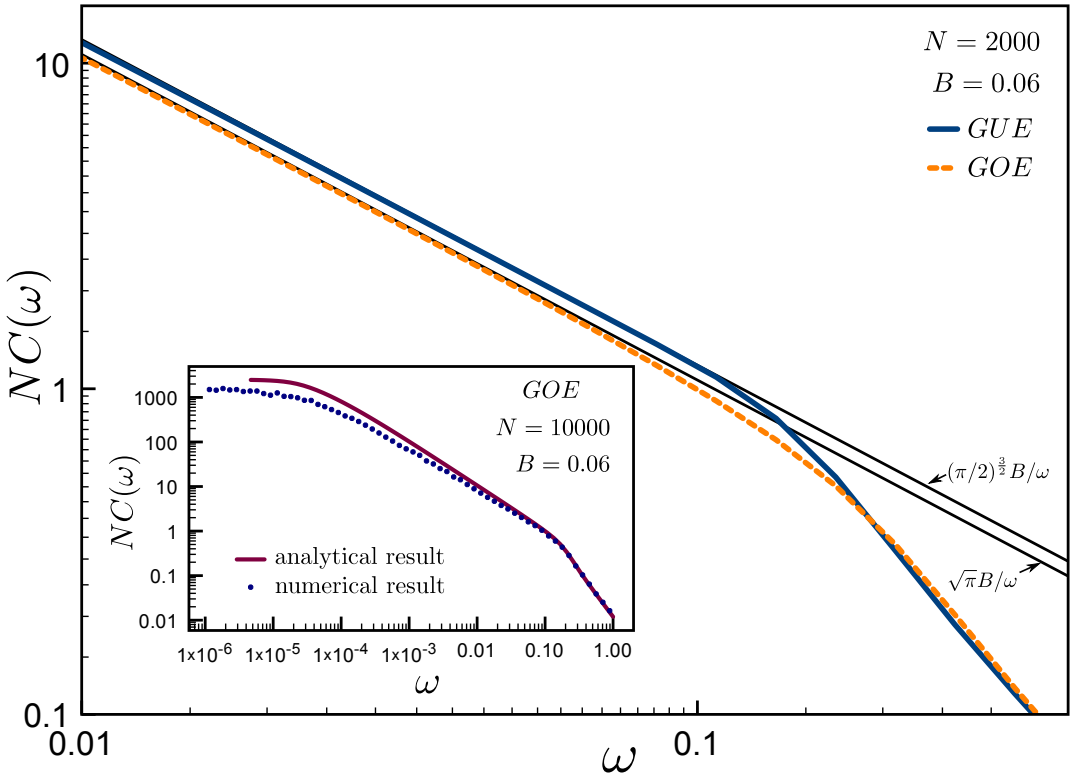

Figure B.1.: Numerical evidence for the broad range of applicability of the saddle-point approximation (this figure has been taken from [37], figure 1)

C. Parametrizations of integration variables

C.1. λ-parametrization

This parametrization was used in a sllightly different form in [47]. One defines

$$s = \lambda e^{i\varphi}\left(1 - \frac{1}{2}\bar{\xi}\xi\right) \tag{C.1}$$

$$\chi = \lambda e^{i\varphi}\xi \tag{C.2}$$

$$\lambda \in [0,\infty)\,, \qquad \varphi \in [0, 2\pi]\,. \tag{C.3}$$

ξ is a new Grassmann variable. The advantage of this parametrization is that it makes the Q-matrices block-diagonal. Again, one must differentiate between the GUE and GOE.

C.1.1. Gaussian unitary ensemble

With this variable transformation, which was previously used in [36], it is possible to write Q as

$$Q = UDU^{\dagger}\,. \tag{C.4}$$

U is a unitary supermatrix

$$U = \begin{pmatrix} u^R & 0 \\ 0 & u^A \end{pmatrix}\,, \qquad U^{\dagger}U = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{C.5}$$

$$u^{R/A} = \begin{pmatrix} 1 - \frac{1}{2}\overline{\xi^{R/A}}\xi^{R/A} & -\overline{\xi^{R/A}} \\ \xi^{R/A} & 1 + \frac{1}{2}\overline{\xi^{R/A}}\xi^{R/A} \end{pmatrix}\,, \quad u^{R/A\,\dagger}u^{R/A} = \mathbb{1}\,. \tag{C.6}$$

The matrices $u^{R/A}$ are ordinary unitary supermatrices. D is a block-diagonal supermatrix

$$D = \begin{pmatrix} D^{RR} & D^{RA} \\ D^{AR} & D^{AA} \end{pmatrix} = \begin{pmatrix} \left(\lambda^R\right)^2 & 0 & -\lambda^R\lambda^A e^{i\phi} & 0 \\ 0 & 0 & 0 & 0 \\ \lambda^R\lambda^A e^{-i\phi} & 0 & -\left(\lambda^A\right)^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.7}$$

with $\phi = \varphi^R - \varphi^A$. Thus, the supertrace of a single Q-matrix has the form

$$\mathrm{Str}\, Q = \mathrm{Str}\left(UDU^\dagger\right) = \left(\lambda^R\right)^2 - \left(\lambda^A\right)^2 \tag{C.8}$$

$$\mathrm{Str}(\Lambda Q) = \left(\lambda^R\right)^2 + \left(\lambda^A\right)^2 . \tag{C.9}$$

The supertrace of the product of two different supermatrices is hard to calculate, but can be simplified using the formula taken from [48]

$$u^\dagger(\xi_i)u(\xi_j) = u(\xi_j - \xi_i)e^{\frac{1}{2}(\overline{\xi_i}\xi_j - \overline{\xi_j}\xi_i)} \tag{C.10}$$

for retarded and advanced variables respectively. The us are the unitary supermatrices for the Grassmanns ξ defined in Eq. (C.6). With the formula in Eq. (C.10), one obtains

$$\mathrm{Str}(Q_iQ_j) = \mathrm{Str}\left(U_iD_iU_i^\dagger U_jD_jU_j^\dagger\right) \tag{C.11}$$

$$\begin{aligned} &= \left(\lambda_i^R\right)^2\left(\lambda_j^R\right)^2(1 - \overline{\alpha_{ij}^R}\alpha_{ij}^R) + \left(\lambda_i^A\right)^2\left(\lambda_j^A\right)^2(1 - \overline{\alpha_{ij}^A}\alpha_{ij}^A) \\ &\quad - 2\lambda_i^R\lambda_j^R\lambda_i^A\lambda_j^A(1 - \frac{1}{2}\overline{\alpha_{ij}^R}\alpha_{ij}^R)(1 - \frac{1}{2}\overline{\alpha_{ij}^A}\alpha_{ij}^A)\cos(\theta_{ij} + \Delta_{ij}) , \end{aligned} \tag{C.12}$$

where

$$\alpha_{ij} = \xi_i - \xi_j \tag{C.13}$$

$$\theta_{ij} = \phi_i - \phi_j \tag{C.14}$$

$$\Delta_{ij} = \frac{i}{2}\left(\overline{\xi_i^R}\xi_j^R - \overline{\xi_j^R}\xi_i^R - \left(\overline{\xi_i^A}\xi_j^A - \overline{\xi_j^A}\xi_i^A\right)\right) . \tag{C.15}$$

The Berezinian B of this variable transformation for one set of variables (retarded or advanced) is

$$B = \frac{2i}{\lambda} . \tag{C.16}$$

Therefore, the integration measure with respect to one single supermatrix becomes

$$D\{Q_i\} \longrightarrow \left(\frac{d\lambda d\varphi d\overline{\xi}d\xi}{\pi\lambda}\right)_i^{R,A} = \frac{d\lambda_i^R d\varphi_i^R d\overline{\xi_i^R}d\xi_i^R}{\pi\lambda_i^R}\frac{d\lambda_i^A d\varphi_i^A d\overline{\xi_i^A}d\xi_i^A}{\pi\lambda_i^A} \tag{C.17}$$

The supersymmetry breaking factors transform into

$$\mathcal{P}^{RA}\mathcal{Q}^{AR} \longrightarrow \lambda_p^R\lambda_p^A\lambda_q^R\lambda_q^A\,\tilde{\mathcal{P}}^{RA}\tilde{\mathcal{Q}}^{AR}\,e^{i\theta_{pq}} = -\lambda_p^R\lambda_p^A\lambda_q^R\lambda_q^A\,\xi_p^R\overline{\xi_q^R}\xi_q^A\overline{\xi_p^A}\,e^{i\theta_{pq}} . \tag{C.18}$$

C.1.2. Gaussian orthogonal ensemble

What changes for the GOE is that the rank of the Q-matrices is twice as big as that for the GUE. One must deal with 8×8 supermatrices, which makes the diagonalization procedure more complicated. Yet, it can be achieved (see [37]) by defining the block-diagonal 8×8 unitary supermatrix

$$U = \begin{pmatrix} u^R e^{i\varphi^R} & 0 & 0 & 0 \\ 0 & \overline{u^R} e^{-i\varphi^R} & 0 & 0 \\ 0 & 0 & u^A e i\varphi^A & 0 \\ 0 & 0 & 0 & \overline{u^A} e^{-i\varphi^A} \end{pmatrix} \tag{C.19}$$

with the same $u^{R/A}$ defined in the preceeding section and using the symmetric and orthogonal matrix

$$\sigma = \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma' \end{pmatrix}, \quad \sigma' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.20}$$

$$\sigma^T = \sigma\,, \quad \sigma^T = \sigma^{-1} \quad \Rightarrow \quad \sigma\sigma = \mathbb{1}\,. \tag{C.21}$$

As a result, one can write for Q

$$Q = U\sigma D\sigma U^\dagger \tag{C.22}$$

with D being the desired block-diagonal matrix which can be written as a tensor product

$$D = \begin{pmatrix} \left(\lambda^R\right)^2 & -\lambda^R\lambda^A \\ \lambda^A\lambda^R & -\left(\lambda^A\right)^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.23}$$

For the supertraces of one single supermatrix Q and the product of two different supermatrices, we thus obtain

$$\mathrm{Str}\, Q = \mathrm{Str}\left(U\sigma D\sigma U^\dagger\right) = \left(\lambda^R\right)^2 - \left(\lambda^A\right)^2 \tag{C.24}$$

$$\mathrm{Str}(\Lambda Q) = \left(\lambda^R\right)^2 + \left(\lambda^A\right)^2 \tag{C.25}$$

$$\begin{aligned} \mathrm{Str}(Q_iQ_j) &= \mathrm{Str}\left(U_j^\dagger U_i\sigma D_i\sigma U_i^\dagger U_j\sigma D_j\sigma\right) \\ &= \Bigg[\lambda_i^R\lambda_j^R\cos\left(\phi_{ij}^R + \Delta_{ij}^R\right)\left(1 - \frac{1}{2}\overline{\alpha_{ij}^R}\alpha_{ij}^R\right) \\ &\quad - \lambda_i^A\lambda_j^A\cos\left(\phi_{ij}^A + \Delta_{ij}^A\right)\left(1 - \frac{1}{2}\overline{\alpha_{ij}^A}\alpha_{ij}^A\right)\Bigg]^2. \end{aligned} \tag{C.26}$$

ϕ_{ij} and Δ_{ij} are defined for each sector (retarded and advanced) as follows:

$$\phi_{ij} = \varphi_i - \varphi_j \tag{C.27}$$

$$\Delta_{ij} = \frac{i}{2}\left(\overline{\xi_i}\xi_j - \overline{\xi_j}\xi_i\right) . \tag{C.28}$$

C.2. RS-parametrization

We change variables from λ^R, λ^A to R, S:

$$R = \left(\lambda^R\right)^2 - \left(\lambda^A\right)^2 \tag{C.29}$$

$$S = \left(\lambda^R\right)^2 + \left(\lambda^A\right)^2 \tag{C.30}$$

$$R \in (-\infty, \infty), \qquad S \in [0, \infty) \tag{C.31}$$

$$\lambda^R = \sqrt{\frac{S+R}{2}}, \qquad \lambda^A = \sqrt{\frac{S-R}{2}} . \tag{C.32}$$

In this parametrization, the supertraces of a single supermatrix Q are the same for the GUE and GOE

$$\operatorname{Str} Q = R \tag{C.33}$$

$$\operatorname{Str}(\Lambda Q) = S . \tag{C.34}$$

The supertraces of the product of two supermatrices read for the GUE and GOE:

$$\begin{aligned}\operatorname{Str}(Q_i Q_j)_{\mathrm{GUE}} = {} & \frac{1}{4}(S_i + R_i)(S_j + R_j)(1 - \overline{\alpha_{ij}^R}\alpha_{ij}^R) \\ & + \frac{1}{4}(S_i - R_i)(S_j - R_j)(1 - \overline{\alpha_{ij}^A}\alpha_{ij}^A) - \frac{1}{2}\sqrt{(S_i^2 - R_i^2)(S_j^2 - R_j^2)} \\ & \times (1 - \frac{1}{2}\overline{\alpha_{ij}^R}\alpha_{ij}^R)(1 - \frac{1}{2}\overline{\alpha_{ij}^A}\alpha_{ij}^A)\cos(\theta_{ij} + \Delta_{ij})\end{aligned} \tag{C.35}$$

$$\begin{aligned}\operatorname{Str}(Q_i Q_j)_{\mathrm{GOE}} = {} & \Bigg[\frac{1}{2}\sqrt{(S_i + R_i)\,(S_j + R_j)}\cos\left(\phi_{ij}^R + \Delta_{ij}^R\right)\left(1 - \frac{1}{2}\overline{\alpha_{ij}^R}\alpha_{ij}^R\right) \\ & - \frac{1}{2}\sqrt{(S_i - R_i)\,(S_j - R_j)}\cos\left(\phi_{ij}^A + \Delta_{ij}^A\right)\left(1 - \frac{1}{2}\overline{\alpha_{ij}^A}\alpha_{ij}^A\right)\Bigg]^2 .\end{aligned} \tag{C.36}$$

The Jacobian of this variable transformation is

$$\mathcal{J} = \frac{1}{4\sqrt{S^2 - R^2}} . \tag{C.37}$$

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Selbstständigkeitserklärung

Hiermit erkläre ich, die vorgelegte Arbeit selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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