Master's Thesis

Quantum Phase Transition to Incommensurate $2k_F$ Charge Density Wave Order

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Masterarbeit

Quantenphasenübergang zu inkommensurabler $2k_F$ Ladungsdichtewellenordnung

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Abstract

We study the problem of quantum phase transitions to incommensurate $\mathbf{Q} = 2k_F$ charge density wave order in two dimensional metals, where the CDW wave vector $\mathbf{Q}$ connects two points on the Fermi surface with parallel tangents. In contrast to previous works, which came to differing conclusions about the order of the phase transition, we use a controlled, perturbative renormalization group analysis based on the work by Dalidovich and Lee [1]. We calculate contributions to the boson and fermion self-energies to one-loop order in dimensional regularization and renormalize the theory using the minimal substraction scheme. We identify a stable fixed point corresponding to a second order phase transition with a flattening of the Fermi surface at the hot-spots, following from the scaling form of the two-point fermion correlation function at the critical point. Finally, we consider the possibility of superconductivity in the vicinity of the quantum critical point.

Contents

1 Introduction

Phase transitions and their theoretical description have been a big and very important topic in physics over the last decades. Many different approaches were introduced to study them and deep insights could be gained about the physics happening at critical points.

A rather simple, but nonetheless very powerful tool is given by mean field theory, where the problem of many interacting particles is reduced to a single-particle problem moving in an effective background field caused by the other constituents of the system. A good understanding of systems like the Ising model or of complex effects like superconductivity are provided by this approach, the validity of mean field theory is, however, limited in dimensions $d < 4$ [2].

Another concept leading to a better understanding of the nature of phase transitions is the concept of order and the associated order parameter. The configuration of the ground state of a system acquires a certain order when the temperature is lowered below the critical temperature T_c, which is reflected by a zero valued order parameter above T_c and a non-zero order parameter below T_c. Considering the order parameter as a dynamical field, the phenomenological entity fully characterizing the properties of the phase transition is the Ginzburg-Landau functional, which is obtained by expanding the free energy of the system in powers and gradients of the order parameter, respecting certain mathematical and physical constraints such as the internal symmetries of the system, like rotational or translational invariance. The Ginzburg-Landau functional has the advantage of taking local fluctuations of the order parameter into account and thus provides an extension of simple mean field theory [3].

It seems somewhat surprising that phase transitions are described by the order parameter alone, to a large extent irrespective of the microscopic properties of the system under consideration. This approach, however, indeed is applicable to a vast variety of phase transitions. Not only for thermal phase transition, where the transition is driven by thermal fluctuations and takes place at a critical temperature, but also for quantum phase transitions, the Ginzburg-Landau functional is an excellent starting point for investigating critical properties [4].

Quantum phase transitions are induced by other control parameters than temperature, for example the relative strength of interaction terms in the Hamiltonian, and can in principle take place at $T = 0$. They are not just theoretically possible, but were already realized experimentally, for example the quantum phase transition from a superfluid to a Mott insulator [5].

The theoretical treatment of quantum phase transitions, however, can be fairly involved in the presence of a Fermi surface in the underlying system. An often used description of low-energy particle- and hole-like quasi-particle excitations and their stability in the vicinity of the Fermi surface is provided by Landau's Fermi liquid picture. The range of validity is, however, limited and famously breaks down in one dimension, where it has to be replaced by the Tomonaga-Luttinger liquid. But even in two dimensions, the Fermi liquid picture does not apply under certain circumstances, for example in the vicinity of

quantum phase transitions associated with symmetry breaking in two dimensional metals [4]. This is connected to the invalidity of the Hertz approach, where the low-energy fermionic quasi-particle excitations near the Fermi surface are integrated out to obtain an effective Ginzburg-Landau functional of the order parameter alone [6]. Thus, it is necessary to treat the low-energy fermionic excitations on equal footing as the order parameter fluctuations. We will come back to the Hertz approach later in the thesis.

One of these quantum phase transition, where the order parameter fluctuations and the low-energy excitations of the fermions have to be treated equally, is the transition to charge density wave order in two dimensional metals, where the ground state spontaneously breaks the translational invariance of the system via the modulation of the electron density with a certain wave vector $\mathbf{Q}$. Such order was observed experimentally for materials such as $SmTe_3$ [7] and $TbTe_3$ [8], where the CDW ordering wave vector $\mathbf{Q}$ is incommensurate with the underlying lattice.

In this thesis, we study the case where the incommensurate CDW ordering wave vector $\mathbf{Q} = 2k_F$ connects two points on the Fermi surface with parallel tangents. Earlier theoretical works are given for example in [9] and [10], but lead to disagreeing results about the nature of the phase transition. The goal of this thesis is to resolve the problem using a different approach, namely a controlled, perturbative renormalization group analysis based on the work by Dalidovich and Lee [1].

But before starting with the main part, we give a brief introduction to charge density waves in the next chapter and get to know dimensional regularization and the minimal substraction scheme by renormalizing the ϕ^4-theory in the third chapter.

2 Charge Density Waves

To understand what charge density waves actually are, we first introduce the concept of phonons, which are quasi-particles connected to the distortion of lattices. In the second part, we retrace Peierl's argumentation for a phase transition from a normal metal to a CDW ordered phase, the so called Peierls transition, before investigating the stability of this new phase via mean field theory. Finally, we summarize the results of research in the subject of CDW's in two dimensional metals.

2.1 Phonons

This introduction of phonons is a brief summary of the main results of chapters 9.1 and 9.4 of [11].

Let's consider a one dimensional monoatomic chain of length L with N atoms separated by the lattice constant a, such that $L = Na$. We assume periodic boundary conditions,

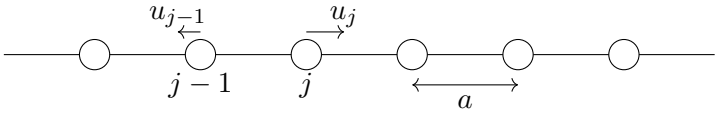


Figure 1: Monoatomic one-dimensional chain with lattice constant a.

i.e. the atom on lattice site $j = N + 1$ corresponds to the atom on lattice site $j = 1$. The atoms position can deviate from their equilibrium position, which is specified by being a multiple of a, s.t. the coordinate of the j^{th} atom is given by $R_j = a \times j + u_j$, where u_j is the displacement from the equilibrium point. Assuming that the ions interact via a potential $V(\{u_j\})$, which is minimized by the equilibrium configuration, we can expand this potential in powers of the displacement

$$V(\{u_j\}) = V_0 + \frac{1}{2} \sum_{i,j} \underbrace{\left(\frac{\partial^2 V}{\partial u_i \partial u_j} \right) \Big|_{u=0}}_{:=D_{ij}} u_i u_j + \mathcal{O}\left(u^3\right), \tag{2.1}$$

where the symmetric matrix D_{ij} is called the dynamical matrix. Taking translational invariance into account and considering the interaction to be only between nearest neighbours, the potential takes the form

$$V(\{u_j\}) = V_0 + \frac{\kappa}{2} \sum_j \left(u_j - u_{j+1}\right)^2, \tag{2.2}$$

where κ is a measure for the interaction strength. Ignoring the constant contribution V_0, the whole Hamiltonian of the system then reads

$$H = \sum_{j=1}^{N} \frac{p_j}{2m} + \frac{\kappa}{2} \sum_{j=1}^{N} \left(u_j - u_{j+1}\right)^2 \tag{2.3}$$

with m the mass of an atom and $p_j = m\dot{u}_j$.

Quantization of this Hamiltonian starts as usual with imposing canonical commutation relations $[\hat{u}_i, \hat{p}_j] = i\hbar\delta_{i,j}$, where displacement and momentum are now operators. These can be rewritten in terms of creation operators $a^\dagger$ and annihilation operators a as

$$\hat{u}_j = \sum_q e^{iqaj}\sqrt{\frac{\hbar}{2Nm\omega_q}}\left(a_q + a_{-q}^\dagger\right), \quad \hat{p}_j = -i\sum_q e^{-iqaj}\sqrt{\frac{m\hbar\omega_q}{2N}}\left(a_{-q} - a_q^\dagger\right), \tag{2.4}$$

where $\omega_q^2 = \frac{4\kappa}{m}\sin^2\left(\frac{qa}{2}\right)$. The quantum mechanical Hamiltonian

$$H = \sum_q \hbar\omega_q\left(a_q^\dagger a_q + \frac{1}{2}\right) \tag{2.5}$$

then describes harmonic oscillators with quantized energies which are multiples of $\hbar\omega_q$. Hence, lattice vibrations of the one dimensional monoatomic chain can be described in terms of free bosons with energy quanta $\hbar\omega_q$, which are called phonons. The artificially introduced operators $a_q^\dagger$ and a_q then create/annihilate phonons with wave number q. The displacement operator $\hat{u}_j$ is important later, since it will be connected to the order parameter of CDW's.

2.2 Peierls Transition

Peierls discovered that a one dimensional monoatomic chain with lattice constant a, as considered before, should be unstable against periodic lattice distortions. His argumentation in [12] goes as follows. Let's assume that every atom in the chain contributes one electron to the valence band, s.t. the valence band is half filled (see left diagram of Fig. 2). In this case,

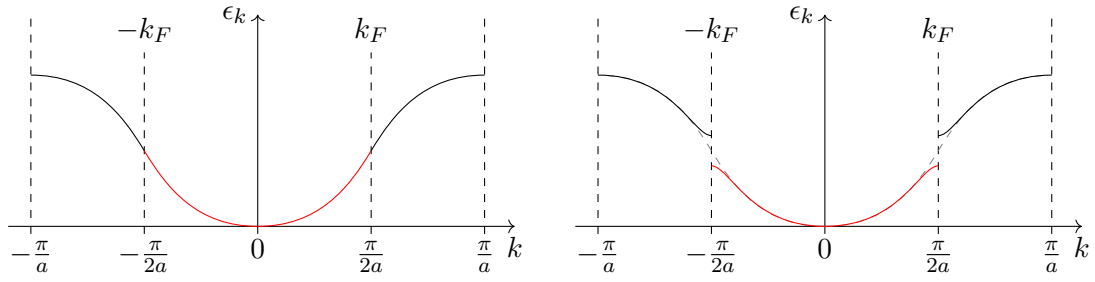


Figure 2: The left diagram shows the filled electron energy states (red) in the unperturbed system. The right diagram shows the opening of the gap in the energy spectrum due to lattice distortions and the resulting lowering of the electron energies.

the reciprocal lattice vector is given by $\frac{2\pi}{a}$. Now let's displace every second atom by the same amount, which introduces a new periodicity $2a$ to the lattice, instead of the previous periodicity a (see Fig. 3). The shift in the electrons energy can be calculated in degenerate perturbation theory and leads to an opening of a gap at $\pm k_F$. The occupied electron states at $\pm k_F$ are shifted downwards, whereas the empty electron states are shifted upwards (see right diagram of Fig. 2), which leads to a decrease in the energy of the electrons. The new reciprocal lattice vector is now given by $\frac{\pi}{a} = 2k_F$. That the decrease in the electrons energy

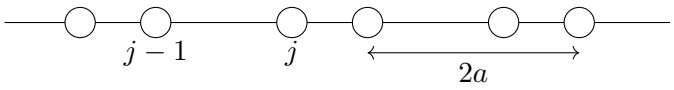


Figure 3: New lattice periodicity after displacing every second atom by the same amount.

is indeed larger in magnitude than the energy used to distort the lattice, can be understood in terms of degenerate perturbation theory as in [12], but also in terms of mean field theory as shown in the next section. Hence the system actually is unstable towards introducing a new periodicity in the lattice. Since lattice distortions are connected to phonons, we can say that in the state of new periodicity the phonon states corresponding to $q = 2k_F$ are macroscopically occupied.

2.3 Mean Field Theory of Charge Density Waves

In this section we review the results of chapter 33.4 of [13], but with some additional, explanatory calculations. Note that the simplifications used in the following are only applicable for the case where the band is initially half filled.

The mean field treatment of the one dimensional model starts with a simplified Fröhlich Hamiltonian which describes electrons with energy ϵ_k and phonons of energy $\hbar\omega_q$ interacting with a constant interaction strength g:

$$\mathcal{H} = \sum_{k,\sigma} \epsilon_k c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_q \hbar\omega_q a^\dagger_q a_q + \frac{g}{\sqrt{L}} \sum_{k,q,\sigma} c^\dagger_{k+q,\sigma} c_{k,\sigma} \left(a_q + a^\dagger_{-q} \right). \tag{2.6}$$

We suppress the spin index σ in further calculations, since it only contributes a factor of two in some places. In the state of CDW order which we want to describe, the lattice has a new periodicity corresponding to $q = \pm 2k_F$, so it is reasonable to say that the main phononic contribution to the Hamiltonian comes from phonons with these wave numbers. Since these phonon states should be macroscopically occupied in the new phase, we further assume $\langle a_{\pm 2k_F} \rangle \neq 0$ and $\langle a^\dagger_{\pm 2k_F} \rangle \neq 0$. As an order parameter we thus choose

$$\Delta = |\Delta| \, e^{i\phi} = \frac{g}{\sqrt{L}} \left(\langle a_{2k_F} \rangle + \langle a^\dagger_{-2k_F} \rangle \right), \tag{2.7}$$

which is directly related to the displacement operator $\hat{u}_j$ in (2.4). This can be seen by considering the modes $q = \pm 2k_F$ only, which leads to

$$\begin{aligned}
\langle \hat{u}_j \rangle &= \sqrt{\frac{\hbar}{2mN\omega_{2k_F}}} e^{i2k_F aj} \left(\langle a_{2k_F} \rangle + \langle a^\dagger_{-2k_F} \rangle \right) + \sqrt{\frac{\hbar}{2mN\omega_{-2k_F}}} e^{-i2k_F aj} \left(\langle a_{-2k_F} \rangle + \langle a^\dagger_{2k_F} \rangle \right) \\
&= \sqrt{\frac{\hbar}{2mN\omega_{2k_F}}} \left[e^{i2k_F aj} \left(\langle a_{2k_F} \rangle + \langle a^\dagger_{-2k_F} \rangle \right) + c.c. \right] \\
&= \sqrt{\frac{\hbar L}{2mN\omega_{2k_F}}} \frac{2\,|\Delta|}{g} \cos\left(2k_F aj + \phi\right).
\end{aligned} \tag{2.8}$$

Replacing the phonon operators by their expectation value and taking only the phonon modes $q = \pm 2k_F$ into account, we get the mean field Hamiltonian

$$\begin{aligned}\mathcal{H} &= \sum_k \epsilon_k c_k^\dagger c_k + \hbar\omega_{2k_F}\langle a_{2k_F}^\dagger\rangle\langle a_{2k_F}\rangle + \hbar\omega_{-2k_F}\langle a_{-2k_F}^\dagger\rangle\langle a_{-2k_F}\rangle \\ &+ \frac{g}{\sqrt{L}}\sum_k c_{k+2k_F}^\dagger c_k\left(\langle a_{2k_F}\rangle + \langle a_{-2k_F}^\dagger\rangle\right) + \frac{g}{\sqrt{L}}\sum_k c_{k-2k_F}^\dagger c_k\left(\langle a_{-2k_F}\rangle + \langle a_{2k_F}^\dagger\rangle\right) \\ &= \sum_k \epsilon_k c_k^\dagger c_k + L\hbar\omega_{2k_F}\frac{|\Delta|^2}{2g^2} + \sum_k c_{k+k_F}^\dagger c_{k-k_F}\Delta + \sum_k c_{k-k_F}^\dagger c_{k+k_F}\Delta^*. \end{aligned} \tag{2.9}$$

We use one further simplification: Since the interesting physics happens near $\pm k_F$ in the electron energy spectrum, we linearize ϵ_k at these points and introduce $c_{k_F+k} := c_{+,k}$ and $c_{-k_F+k} := c_{-,k}$ as independent particles. The energies near $\pm k_F$ are then given by

$$\epsilon_{k\pm k_F} \approx \epsilon_F \pm \hbar v_F k := \epsilon_k^\pm \tag{2.10}$$

and the Hamiltonian in its full simplification reads

$$\mathcal{H} = \sum_k \begin{pmatrix} c_{+,k}^\dagger & c_{-,k}^\dagger \end{pmatrix} \begin{pmatrix} \epsilon_k^+ & \Delta \\ \Delta^* & \epsilon_k^- \end{pmatrix} \begin{pmatrix} c_{+,k} \\ c_{-,k} \end{pmatrix} + L\hbar\omega_{2k_F}\frac{|\Delta|^2}{2g^2}. \tag{2.11}$$

The electronic term can be diagonalized using a Bogoliubov transformation. We introduce new fermionic particle operators via

$$\begin{pmatrix} \alpha_k^\dagger & \beta_k^\dagger \end{pmatrix} = \begin{pmatrix} c_{+,k}^\dagger & c_{-,k}^\dagger \end{pmatrix} \begin{pmatrix} u_k & -v_k \\ v_k^* & u_k \end{pmatrix} \tag{2.12}$$

with a real parameter u_k and a complex parameter v_k, which have to fulfill $u_k^2 + |v_k|^2 = 1$ for the new operators to obey fermionic anti-commutation relations. Using the basis of the new operators, the electronic term of the Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}_{el} = \sum_k &\begin{pmatrix} \alpha_k^\dagger & \beta_k^\dagger \end{pmatrix} \\ &\begin{pmatrix} u_k^2\epsilon_k^+ + |v_k|^2\epsilon_k^- + u_k v_k^*\Delta + v_k u_k\Delta^* & -(\epsilon_k^+ - \epsilon_k^-)u_k v_k + u_k^2\Delta - v_k^2\Delta^* \\ -(\epsilon_k^+ - \epsilon_k^-)u_k v_k^* + u_k^2\Delta^* - v_k^2\Delta & |v_k|^2\epsilon_k^+ + u_k^2\epsilon_k^- - u_k v_k^*\Delta - u_k v_k\Delta^* \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \end{aligned} \tag{2.13}$$

We can determine the parameters u_k and v_k by setting the off-diagonal elements to zero, i.e. we need to solve

$$-(\epsilon_k^+ - \epsilon_k^-)u_k v_k + u_k^2\Delta - v_k^2\Delta^* = 0. \tag{2.14}$$

Inserting the ansatz

$$u_k = \cos\theta_k, \qquad v_k = |v_k|\, e^{i\phi} = \sin\theta_k e^{i\phi} \tag{2.15}$$

in the above equation, we find

$$\theta_k = \frac{1}{2}\arctan\left(\frac{|\Delta|}{\xi_k}\right), \tag{2.16}$$

where $\xi_k = \frac{1}{2}\left(\epsilon_k^+ - \epsilon_k^-\right)$. Thus the parameters read

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta|^2}}\right), \qquad |v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta|^2}}\right), \tag{2.17}$$

which yields for the upper diagonal term of the matrix

$$E_k^+ = u_k^2\epsilon_k^+ + |v_k|^2\,\epsilon_k^- + 2u_k\,|v_k|\,|\Delta| = \frac{\epsilon_k^+ + \epsilon_k^-}{2} + \sqrt{\left(\frac{\epsilon_k^+ - \epsilon_k^-}{2}\right)^2 + |\Delta|^2} \tag{2.18}$$

and for the lower diagonal term

$$E_k^- = |v_k|^2\,\epsilon_k^+ + u_k^2\epsilon_k^- - 2u_k\,|v_k|\,|\Delta| = \frac{\epsilon_k^+ + \epsilon_k^-}{2} - \sqrt{\left(\frac{\epsilon_k^+ - \epsilon_k^-}{2}\right)^2 + |\Delta|^2}. \tag{2.19}$$

The Hamiltonian in the new basis is then given by

$$\mathcal{H} = \sum_k \left(E_k^+\alpha_k^\dagger\alpha_k + E_k^-\beta_k^\dagger\beta_k\right) + L\hbar\omega_{2k_F}\frac{|\Delta|^2}{2g^2}. \tag{2.20}$$

In the electronic energy spectrum, a gap of magnitude $2\,|\Delta|$ has opened at $k = \pm k_F$, which corresponds to the right diagram of Fig. 2. The α-particles hence describe electrons in the upper band, whereas the β-particles describe electrons in the lower band.

In the mean field treatment, the energies of electrons near $\pm k_F$ indeed get shifted downwards through the opening of a gap, but we still need to show that the total energy of the CDW ordered phase is lower than in the normal phase. Considering the now fully filled lower band, the total energy of the electrons can be obtained by

$$E_e = 2\sum_{k=-k_c}^{k_c} E_k^-, \tag{2.21}$$

where the factor of two comes from the spin degeneracy and we've taken the sum over a finite band width $2D = 2\hbar v_F k_c$. Using the linearized expressions for $\epsilon_k^\pm$, the energy of the lower band reads

$$E_k^- = \epsilon_F - \sqrt{(\hbar v_F k)^2 + |\Delta|^2}. \tag{2.22}$$

Ignoring the constant term $\propto \epsilon_F$ and converting the sum into an integral, we get

$$E_e = -2\frac{L}{2\pi}\int_{-k_c}^{k_c} dk\sqrt{(\hbar v_F k)^2 + |\Delta|^2} = -\frac{L}{\hbar\pi v_F}\int_{-D}^{D} d\epsilon\sqrt{\epsilon^2 + |\Delta|^2}$$

$$= -\frac{L}{\hbar\pi v_F}\left[D\sqrt{D^2+|\Delta|^2}+\Delta^2\text{arsinh}\left(\frac{D}{|\Delta|}\right)\right]$$
$$\overset{g\ll 1}{\approx} -\frac{L}{\hbar\pi v_F}\left[D^2+|\Delta|^2\ln\left(\frac{2D}{|\Delta|}\right)\right]. \tag{2.23}$$

The main contribution to the electronic energy difference near $\pm k_F$ between the normal phase and the CDW ordered phase $\Delta E = E_{\text{normal phase}} - E_e$ then comes from the term $\propto -|\Delta|^2\ln(|\Delta|)$, s.t. the energy decreases. This decrease is always larger in magnitude than the increase due to the lattice distortion, which is $\propto |\Delta|^2$ as seen in the phononic part of the Hamiltonian. Thus we conclude that the one-dimensional monoatomic chain is indeed unstable towards a CDW order.

The expression for the gap $|\Delta|$ can be deduced by minimizing the total energy w.r.t. $|\Delta|$. The total energy reads

$$E_{\text{total}} = -\frac{L}{\hbar\pi v_F}\left[D^2+|\Delta|^2\ln\left(\frac{2D}{|\Delta|}\right)\right]+L\hbar\omega_{2k_F}\frac{|\Delta|^2}{2g^2} \tag{2.24}$$

and differentiating this expression yields

$$\frac{\partial E_{\text{total}}}{\partial|\Delta|} = -\frac{2L}{\hbar\pi v_F}|\Delta|\ln\left(\frac{2D}{|\Delta|}\right)+\frac{L}{\hbar\pi v_F}|\Delta|+L\hbar\omega_{2k_F}\frac{|\Delta|}{g^2}=0. \tag{2.25}$$

Neglecting the electronic term $\propto|\Delta|$, we obtain

$$|\Delta| = 2De^{-\frac{1}{\lambda}} \tag{2.26}$$

with the dimensionless coupling $\lambda = \frac{2g^2}{\hbar^2\pi\omega_{2k_F}}$.

To justify the name charge density wave, we can calculate the general form of the electron density given by the expression

$$\rho(x) = -e\sum_\sigma\left\langle\psi_\sigma^\dagger(x)\psi_\sigma(x)\right\rangle_{\Psi_0}, \tag{2.27}$$

where $\psi_\sigma(x)$ is the electronic field operator

$$\psi_\sigma(x) = \frac{1}{\sqrt{L}}\sum_k c_{k,\sigma}e^{ikx} \approx \frac{1}{\sqrt{L}}\sum_k\left(c_{k+k_F,\sigma}e^{i(k+k_F)x}+c_{k-k_F,\sigma}e^{i(k-k_F)x}\right)$$
$$= \frac{1}{\sqrt{L}}\sum_k\left[\left(u_k e^{ik_Fx}+v_k^* e^{-ik_Fx}\right)\alpha_{k,\sigma}+\left(-v_k e^{ik_Fx}+u_k e^{-ik_Fx}\right)\beta_{k,\sigma}\right]e^{ikx} \tag{2.28}$$

and the expectation value is evaluated w.r.t. to the new ground state $|\Psi_0\rangle$, i.e. all states in the lower band are filled and all states in the upper band are empty. When evaluating the product $\psi^\dagger\psi$, we get terms $\propto\alpha^\dagger\alpha$, $\alpha^\dagger\beta$, $\beta^\dagger\alpha$ and $\beta^\dagger\beta$. Since the upper band is empty, we have $\alpha|\Psi_0\rangle = \langle\Psi_0|\alpha^\dagger = 0$, s.t. the only term contributing is $\propto\beta^\dagger\beta$. Using

$$\rho(x)\propto\sum_{k,q}\left\langle\beta_k^\dagger\beta_q\right\rangle_{\Psi_0}\propto\sum_{k,q}\left\langle\beta_k^\dagger\beta_k\right\rangle_{\Psi_0}\delta_{k,q}, \tag{2.29}$$

we obtain

$$\begin{aligned}\rho(x) &= -\frac{e}{L}\sum_{k,\sigma}\left(-v_k^* e^{-ik_F x} + u_k e^{ik_F x}\right)\left(-v_k e^{ik_F x} + u_k e^{-ik_F x}\right)\left\langle \beta_{k,\sigma}^\dagger \beta_{k,\sigma}\right\rangle_{\Psi_0} \\ &= -\frac{e}{L}\sum_{k,\sigma}\left\langle \beta_{k,\sigma}^\dagger \beta_{k,\sigma}\right\rangle_{\Psi_0} + \frac{2e}{L}\sum_{k,\sigma} u_k\left|v_k\right|\left\langle \beta_{k,\sigma}^\dagger \beta_{k,\sigma}\right\rangle_{\Psi_0}\cos\left(2k_F x + \phi\right) \\ &= \rho_0 + \rho_1\cos\left(2k_F x + \phi\right), \end{aligned} \tag{2.30}$$

thus clarifying the name charge density wave.

In this thesis we will consider only incommensurate CDW's, i.e. the wavelength λ of the CDW is not a rational multiple of the lattice constant a.

2.4 Quantum Phase Transitions to Incommensurate CDW Order in Two Dimensional Metals

All the above calculations were done for a one dimensional monoatomic chain. In two dimensional systems however, phase transitions to CDW order are possible due to quantum fluctuations. Examples for materials in which CDW order has been observed were already given in the introduction. However, for some materials it is still unclear what kind of underlying mechanism drives a quantum phase transition to incommensurate $2k_F$ CDW order, whether it is the coupling of electrons to phonons, strong electron correlations or something even different.

We will consider incommensurate CDW's with an ordering wave vector $\mathbf{Q} = 2k_F$, which connects two points on the Fermi surface with parallel tangents (see Fig. 4). Theoretical

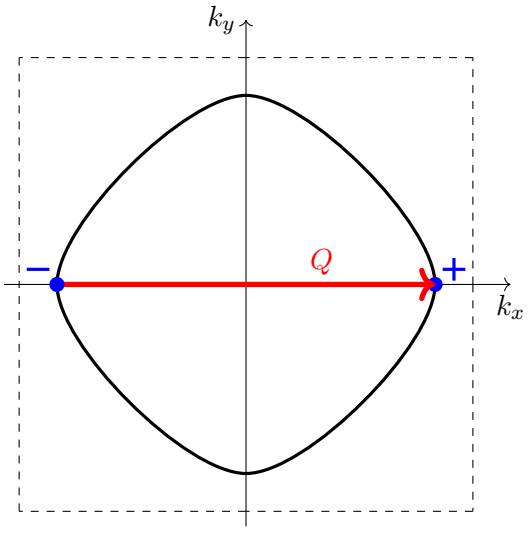


Figure 4: The CDW wave vector $\mathbf{Q} = 2k_F$ connects two points on the Fermi surface with parallel tangents.

research about such systems was done in various papers, as for example in [9]. A $1/N$ expansion was used to obtain infrared divergencies of the leading order diagrams correcting

for example the fermion self-energy and the interaction. As an parameter for the commensurability of the system Altshuler et al. used $\Delta G = |\mathbf{Q} - \mathbf{G}/2|$, where $\mathbf{G}$ is a reciprocal lattice vector, and distinguished two different regimes: One with large momenta, where the infrared cutoff is large compared to ΔG and one with small momenta and therefore a small cutoff in comparison to ΔG. In the large momentum regime, Altshuler et al. found logarithmic divergencies which could be summed to power laws, indicating a second order phase transition. In the small momentum regime, however, the divergencies were stronger than just logarithmic. The authors therefore concluded that for a commensurate wave vector $\mathbf{Q}$ the phase transition to CDW order is of second order, whereas for incommensurate $\mathbf{Q}$ strong fluctuations reduce the transition to first order.

Recently, the problem was revisited by [10], where the fermion self-energy in one-loop approximation in the fluctuation propagator, dressed to one-loop order as well, was calculated. They found non-Fermi liquid behavior at the hot-spots in the imaginary part of the fermion self-energy in the form of a $|\omega|^{2/3}$-dependence and therefore confirmed the result of a previous work [14]. For the real part of the fermion self-energy, Sýkora et al. obtained logarithmically diverging corrections to the fermion dispersion, which indicated, by resumming them to power laws, that the Fermi surface at the hot-spots is flattened. However, a self-consistency calculation, where the polarization function was computed with the renormalized fermion Green's function, showed that the peak of the RPA susceptibility is shifted away from the $2k_F$ line which is inconsistent with the original assumptions. Sýkora et al. concluded, in contrast to the previous work by Altshuler et al., that the quantum phase transition might still be of second order with a wave vector $\mathbf{Q}$ shifted away from $2k_F$, or with $\mathbf{Q} = 2k_F$ and a sufficiently flat Fermi surface at the hot-spots.

In this master's thesis, we resolve this problem by performing a controlled, perturbative renormalization group analysis based on [1], where dimensional regularization and the minimal substraction scheme is used to get access to the critical behavior of quantum phase transitions. But before diving into this subject, we briefly get familiar to dimensional regularization and the minimal substraction scheme by renormalizing the famous ϕ^4-theory to one-loop order.

3 Renormalization of ϕ^4-Theory

In quantum field theories, the corrections to correlation functions due to interactions are filled with diverging integrals. Over the past decades, many different tools were developed to control these divergencies and obtain finite results. One method to extract the infinities is the dimensional regularization by 't Hooft and Veltman [15]. The theory can then be made finite by renormalization in the minimal substraction scheme [16].

In dimensional regularization, the originally diverging integrals are evaluated in arbitrary, continuous dimensions d and afterwards expanded in small $\epsilon = d_c - d$, where d_c is the dimension in which the interaction constant becomes dimensionless. The divergencies then manifest themselves in $1/\epsilon$-poles and can be cancelled by substracting appropriate terms in the original action to get finite results, thus the name minimal substraction scheme.

When using an energy functional describing a system in the vicinity of its critical point, these $1/\epsilon$-poles carry the crucial information about the critical exponents characterizing the phase transition. The values of the critical exponents can be obtained by tuning the dimension away from d_c to the physically meaningful dimension through an appropriate choice of $\epsilon = d_c - d$.

In this chapter, which is based on [17], we get familiar with the concepts of dimensional regularization and the minimal substraction scheme by renormalizing the well-known ϕ^4-theory. We will consider corrections to the two- and four-point functions at one-loop level only, i.e. corrections where the $1/\epsilon$-poles are $\propto \lambda$ with λ the interaction constant.

3.1 Model and Free Propagator

Our treatment of the renormalization of ϕ^4-theory starts with the energy functional

$$E[\phi] = \int d^d x \left\{ \frac{1}{2}\left[(\partial_x \phi(x))^2 + m^2 \phi^2(x) \right] + \frac{\lambda}{4!}\phi^4(x) \right\} = \underbrace{\frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \left(\mathbf{k}^2 + m^2\right) \phi(k)\phi(-k)}_{=E_0}$$
$$+ \underbrace{\frac{\lambda}{4!}\int \frac{d^d k_1 d^d k_2 d^d k_3}{(2\pi)^3} \phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1-k_2-k_3)}_{=E_{\text{int}}}. \tag{3.1}$$

In the further calculations we will use the notation $\int_k = \int \frac{d^d k}{(2\pi)^d}$. The partition function is defined as the sum over all degrees of freedom, here given by the field ϕ, weighted with a factor $e^{-\beta E}$. Setting $\beta = \frac{1}{k_B T} = 1$ for simplicity, the functional integral for the normalized partition function therefore reads

$$Z = \frac{1}{Z_f} \int \mathcal{D}\phi \; e^{-E[\phi]}, \tag{3.2}$$

where Z_f is the partition function of the free theory

$$Z_f = \int \mathcal{D}\phi \; e^{-E_0[\phi]}. \tag{3.3}$$

The propagator of the field ϕ can be evaluated by the expression

$$\langle\phi(q_1)\phi(q_2)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(q_1)\phi(q_2)\ e^{-E[\phi]}, \tag{3.4}$$

which is usually not solvable exactly. One exception is given by energy functionals quadratic in ϕ, as our free theory. The expectation values w.r.t. the free theory are denoted by a subscript

$$\langle\phi(q_1)\phi(q_2)\rangle_0 = \frac{1}{Z_f}\int \mathcal{D}\phi\ \phi(q_1)\phi(q_2)\ e^{-E_0[\phi]} = G_0(q_1)\ \delta_{q_1+q_2,0}, \tag{3.5}$$

where the free propagator is given by $G_0(k) = G_0(-k) = \frac{1}{\mathbf{k}^2+m^2}$ and we abbreviated $\delta_{q_1+q_2,0} = (2\pi)^d\delta^{(d)}(q_1+q_2)$.

Higher order correlation functions are obtained using Wick's theorem, i.e. n-point functions are a sum of all possible combinations using free two-point functions. As an example, the free four-point function reads

$$\begin{aligned}\big\langle\phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4)\big\rangle_0 =\ & \langle\overbrace{\phi(q_1)\phi(q_2)}\overbrace{\phi(q_3)\phi(q_4)}\rangle_0 + \langle\overbrace{\phi(q_1)\underbrace{\phi(q_2)\phi(q_3)}\phi(q_4)}\rangle_0 \\ +\ \langle\overbrace{\phi(q_1)\underbrace{\phi(q_2)\phi(q_3)}\phi(q_4)}\rangle_0 =\ & G_0(q_1)\delta_{q_1+q_2,0}G_0(q_3)\delta_{q_3+q_4,0} + G_0(q_1)\delta_{q_1+q_3,0}G_0(q_2)\delta_{q_2+q_4,0} \\ +\ & G_0(q_1)\delta_{q_1+q_4,0}G_0(q_2)\delta_{q_2+q_3,0}.\end{aligned} \tag{3.6}$$

3.2 Feynman Diagrams and One-Loop Corrections

Up to now, the calculations were done w.r.t. the free theory, correlation functions however are corrected in the presence of interactions. These corrections are usually arranged in a perturbation expansion in the interaction, e.g. the first order correction to the two-point function can be obtained by Taylor expanding the interacting part of the exponential function in (3.4)

$$\begin{aligned}\langle\phi(q_1)\phi(q_2)\rangle &= \frac{1}{Z}\int \mathcal{D}\phi\ \phi(q_1)\phi(q_2)\left[1 - E_{\text{int}} + \mathcal{O}\left(\lambda^2\right)\right]\ e^{-E_0} \\ &= \frac{Z_f}{Z}\left[\langle\phi(q_1)\phi(q_2)\rangle_0 - \langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0\right] + \mathcal{O}\left(\lambda^2\right) \\ &= \frac{Z_f}{Z}\Big[\ \langle\phi(q_1)\phi(q_2)\rangle_0\left(1 - \langle E_{\text{int}}\rangle_0\right) - \langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con}\ \Big] + \mathcal{O}\left(\lambda^2\right).\end{aligned} \tag{3.7}$$

The superscript *con* denotes connected terms, i.e. the fields are contracted with E_{int} and not with each other. The prefactor can be simplified to

$$\begin{aligned}\frac{Z_f}{Z} &= Z_f\left[\int \mathcal{D}\phi\ e^{-E_0-E_{\text{int}}}\right]^{-1} = Z_f\left[\int \mathcal{D}\phi\ \left[1 - E_{\text{int}} + \mathcal{O}\left(\lambda^2\right)\right]e^{-E_0}\right]^{-1} \\ &= \frac{Z_f}{Z_f - Z_f\left\langle E_{\text{int}}\right\rangle_0 + \mathcal{O}\left(\lambda^2\right)} = \frac{1}{1 - \left\langle E_{\text{int}}\right\rangle_0 + \mathcal{O}\left(\lambda^2\right)} = 1 + \left\langle E_{\text{int}}\right\rangle_0 + \mathcal{O}\left(\lambda^2\right),\end{aligned} \tag{3.8}$$

which in total leads to

$$\begin{aligned}\langle\phi(q_1)\phi(q_2)\rangle &= \left(1+\langle E_{\text{int}}\rangle_0\right)\Big[\,\langle\phi(q_1)\phi(q_2)\rangle_0\left(1-\langle E_{\text{int}}\rangle_0\right)-\langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con}\Big]+\mathcal{O}\left(\lambda^2\right)\\ &= \langle\phi(q_1)\phi(q_2)\rangle_0-\langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con}+\mathcal{O}\left(\lambda^2\right). \end{aligned} \tag{3.9}$$

The cancellation of any term $\propto \left\langle E_{\text{int}}^p\right\rangle_0$ is in fact true for any order of perturbation theory and for any correlation function, s.t. only connected terms can contribute to the n-point functions.

The first order term

$$\begin{aligned}&-\langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con}\\ &=-\frac{\lambda}{4!}\int_{k_1,k_2,k_3}\langle\phi(q_1)\phi(q_2)\phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1-k_2-k_3)\rangle_0^{con}\end{aligned} \tag{3.10}$$

can be calculated using Wick's theorem, but all different contractions will give the same result. The number of possible connected contractions is 4×3, since we have four possibilities to contract $\phi(q_1)$ with a field coming from the interaction and remaining three possibilities for contracting $\phi(q_2)$, yielding

$$\begin{aligned}-\langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con} &= -\frac{\lambda}{2}\int_{k_1,k_2,k_3}\langle\overbrace{\phi(q_1)\underbrace{\phi(q_2)\phi(k_1)}\phi(k_2)\overbrace{\phi(k_3)\phi}(-k_1-k_2-k_3)\rangle_0^{con}\\ &= -\frac{\lambda}{2}\int_{k_1,k_2,k_3}G_0(q_1)\delta_{q_1+k_1,0}\;G_0(q_2)\delta_{q_2+k_2,0}\;G_0(k_3)\delta_{k_1+k_2,0}\\ &= G_0^2(q_1)\delta_{q_1+q_2,0}\left(-\frac{\lambda}{2}\int_{k_3}G_0(k_3)\right)=G_0^2(q_1)\delta_{q_1+q_2,0}\;\Sigma.\end{aligned} \tag{3.11}$$

Σ is called the self-energy and will be evaluated later.

Such terms can also be represented by Feynman diagrams. For the free propagator we use a line

$$\langle\phi(q)\phi(-q)\rangle_0 = \overset{q}{\longrightarrow} = G_0(q) \tag{3.12}$$

and for the interaction term a vertex


$= -\lambda.$ (3.13)

Then the first order correction to the free propagator is represented diagrammatically by

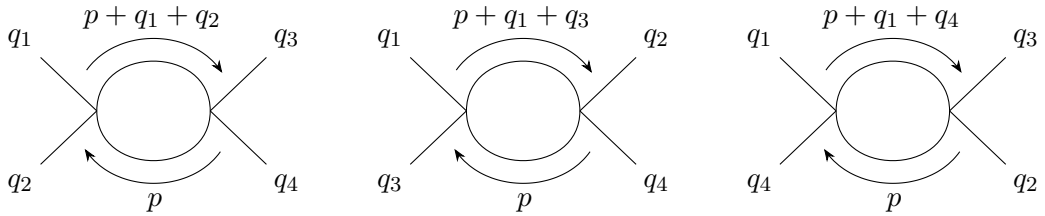
$-\langle\phi(q_1)\phi(q_2)E_{\text{int}}\rangle_0^{con} =$ 

. (3.14)

In addition to writing down the propagators and the vertex factors, we need to integrate over internal momenta, ensure momentum conservation via a δ-function and consider symmetry factors. The symmetry factors are obtained by a factor of $\frac{1}{4!}$ for every vertex multiplied by the possible ways to draw a certain Feynman diagram. For the diagram above, we have

four possibilities for the first external leg, three possibilities for the second external leg and just one possibility to connect the remaining two legs to a loop, hence leading to the factor $\frac{4\times 3}{4!} = \frac{1}{2}$.

The one-loop corrections to the vertex factor are given by the three diagrams



(3.15)

corresponding to the mathematical expression

$$G_0(q_1)G_0(q_2)G_0(q_3)G_0(q_4)\delta_{q_1+q_2+q_3+q_4,0}\underbrace{\left(\frac{\lambda^2}{2}\int_p G_0(p)G_0(p+q)\right)}_{:=V}, \tag{3.16}$$

with q either $q = q_1 + q_2$, $q = q_1 + q_3$ or $q = q_1 + q_4$.

3.3 Calculation of One-Loop Integrals

The one-loop integrals

$$\Sigma = -\frac{\lambda}{2}\int_p G_0(p) = -\frac{\lambda}{2}\int_p \frac{1}{\mathbf{p}^2 + m^2} \tag{3.17}$$

and

$$V = \frac{\lambda^2}{2}\int_p G_0(p)G_0(p+q) = \frac{\lambda^2}{2}\int_p \frac{1}{\mathbf{p}^2 + m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2 + m^2} \tag{3.18}$$

are typical examples for the divergencies at large momenta occurring in quantum field theories. This can be easily seen by going to hyperspherical coordinates

$$\Sigma \propto \int dr \frac{r^{d-1}}{r^2 + m^2}, \tag{3.19}$$

which is obviously UV-divergent for $d \geq 2$.

In the introduction to this chapter we already stated that the divergencies of the integrals can be extracted by calculating them in arbitrary dimensions and afterwards expanding the result in small deviations from the dimension in which the interaction is dimensionless. We chose natural units $\hbar = c = 1$ as well as $\beta = \frac{1}{k_B T} = 1$, s.t. the energy functional $E[\phi]$ should be dimensionless. Since momenta have mass dimension $[\mathbf{k}] = 1$, the fields have to have mass dimension $[\phi] = -\frac{d}{2} - 1$. The interaction term then has zero mass dimension when $[\lambda] = 4 - d$ and we introduce a dimensionless coupling constant by substituting $\lambda \to \lambda\mu^{4-d}$ with an arbitrary mass scale μ. Thus, we will expand the results of the above integrals in small $\epsilon = 4 - d$.

Before evaluating Σ and V, we turn to

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^a}{\left(\mathbf{k}^2 + A\right)^b}, \tag{3.20}$$

which can be easily solved analytically for arbitrary dimensions using the Schwinger parametrization

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty du\ u^{n-1} e^{-uA}. \tag{3.21}$$

Turning to hyperspherical coordinates, the integral I can be written as

$$I = \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \int_0^\infty dr\ r^{d-1} \frac{\left(r^2\right)^a}{\left(r^2 + A\right)^b}. \tag{3.22}$$

Substituting $y = r^2$ with $dr = \frac{dy}{2\sqrt{y}}$ leads to

$$\begin{aligned} I &= \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \int_0^\infty dy\ y^{\frac{d}{2}+a-1} \frac{1}{(y+A)^b} \\ &= \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \Gamma(b)} \int_0^\infty du\ u^{b-1} e^{-uA} \int_0^\infty dy\ y^{\frac{d}{2}+a-1} e^{-uy} \\ &= \frac{\Gamma\left(\frac{d}{2}+a\right)}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \Gamma(b)} \int_0^\infty du\ u^{b-a-\frac{d}{2}-1} e^{-uA}, \end{aligned} \tag{3.23}$$

which evaluates to the final result

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^a}{\left(\mathbf{k}^2 + A\right)^b} = \frac{\Gamma\left(\frac{d}{2}+a\right) \Gamma\left(b-a-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \Gamma(b)} A^{\frac{d}{2}+a-b}. \tag{3.24}$$

Given this, we can immediately write down the solution to the integral of the self energy

$$\Sigma = -\frac{\lambda \mu^\epsilon}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\mathbf{p}^2 + m^2} = -\frac{\lambda \mu^\epsilon}{2} \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(m^2\right)^{\frac{d}{2}-1}, \tag{3.25}$$

yielding with $\epsilon = 4 - d$

$$\Sigma = \frac{m^2 \lambda}{(4\pi)^2 \epsilon} + \text{finite} \tag{3.26}$$

by expanding around $\epsilon = 0$.

To solve the integral V, we make use of the Feynman parametrization

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{1-\alpha}(1-x)^{1-\beta}}{\left[xA + (1-x)B\right]^{\alpha+\beta}}, \tag{3.27}$$

such that

$$V = \frac{\lambda^2\mu^{2\epsilon}}{2}\int_p\int_0^1 dx \frac{1}{\left[x(\mathbf{p}+\mathbf{q})^2 + xm^2 + (1-x)\mathbf{p}^2 + (1-x)m^2\right]^2}$$
$$= \frac{\lambda^2\mu^{2\epsilon}}{2}\int_0^1 dx \int_p \frac{1}{\left(\mathbf{p}^2 + 2x\mathbf{p}\mathbf{q} + x\mathbf{q}^2 + m^2\right)^2}. \tag{3.28}$$

After shifting $\mathbf{p} \to \mathbf{p} - x\mathbf{q}$, the expression further simplifies

$$V = \frac{\lambda^2\mu^{2\epsilon}}{2}\int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{1}{\left[\mathbf{p}^2 + x(1-x)\mathbf{q}^2 + m^2\right]^2}$$
$$= \frac{\lambda^2\mu^{2\epsilon}}{2}\int_0^1 dx \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\left[x(1-x)\mathbf{q}^2 + m^2\right]^{\frac{d}{2}-2} = \lambda\mu^\epsilon\frac{\lambda}{(4\pi)^2\epsilon} + \text{finite}. \tag{3.29}$$

We had to keep one factor μ^ϵ since the pole in ϵ will modify the interaction term $\propto \lambda\mu^\epsilon$ in the energy functional.

3.4 Renormalization

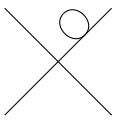
The one-loop corrections to the two- and four-point functions read

$$\langle\phi(q_1)\phi(q_2)\rangle = G_0(q_1)\delta_{q_1+q_2,0} + G_0^2(q_1)\delta_{q_1+q_2,0}\,\Sigma + \mathcal{O}\left(\lambda^2\right)$$
$$= G_0^2(q_1)\delta_{q_1+q_2,0}\left(\frac{m^2\lambda}{(4\pi)^2\epsilon}\right) + \text{finite} + \mathcal{O}\left(\lambda^2\right) \tag{3.30}$$

and

$$\langle\phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4)\rangle = G_0(q_1)G_0(q_2)G_0(q_3)G_0(q_4)\delta_{q_1+q_2+q_3+q_4,0}\left(\lambda\mu^\epsilon\frac{3\lambda}{(4\pi)^2\epsilon}\right)$$
$$+ \text{finite} + \mathcal{O}\left(\lambda^3\right). \tag{3.31}$$

The factor of three arises since we have three distinct diagrams leading to the same $1/\epsilon$-pole. In principle, corrections of the form



(3.32)

and three similar ones appear as well, but these just dress one of the external propagators and therefore do not renormalize the vertex factor itself. The divergence from the loop will be cancelled by the same mechanism as the one from the two-point function.

The minimal substraction scheme can be used to make the correlation functions UV-finite, which goes as follows. We include counterterms in the energy functional of the same structure as the original $E[\phi]$

$$E_{CT}[\phi] = \int_k \frac{\mathbf{k}^2}{2}\frac{Z_{\phi,1}}{\epsilon}\phi(k)\phi(-k) + \frac{m^2}{2}\int_k \frac{Z_{m,1}}{\epsilon}\phi(k)\phi(-k)$$

$$+\frac{\lambda\mu^\epsilon}{4!}\int_{k_1,k_2,k_3,k_4}\frac{Z_{\lambda,1}}{\epsilon}\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta_{k_1+k_2+k_3+k_4,0}, \tag{3.33}$$

treat them as new interactions and require that the $Z_{i,1}\propto\lambda$ cancel the divergencies in the correlation functions exactly. The two-point corrections then have two additional terms arising from

$$\begin{aligned}&-\int_k\frac{1}{2}\left(\mathbf{k}^2\frac{Z_{\phi,1}}{\epsilon}+m^2\frac{Z_{m,1}}{\epsilon}\right)\langle\phi(q_1)\phi(q_2)\phi(k)\phi(-k)\rangle_0\\&=-G_0^2(q_1)\delta_{q_1+q_2,0}\left(\mathbf{q}_1^2\frac{Z_{\phi,1}}{\epsilon}+m^2\frac{Z_{m,1}}{\epsilon}\right),\end{aligned} \tag{3.34}$$

s.t. we can immediately read off $Z_{\phi,1}=0$ and $Z_{m,1}=\frac{\lambda}{(4\pi)^2}$. Repeating an analogous calculation for the vertex factor, we find $Z_{\lambda,1}=\frac{3\lambda}{(4\pi)^2}$. The renormalized energy functional

$$\begin{aligned}E_{ren}[\phi]=E[\phi]+E_{CT}[\phi]&=\frac{1}{2}\int_k\left(\mathbf{k}^2Z_\phi+m^2Z_m\right)\phi(k)\phi(-k)\\&+\frac{\lambda\mu^\epsilon}{4!}Z_\lambda\int_{k_1,k_2,k_3,k_4}\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta_{k_1+k_2+k_3+k_4,0}\end{aligned} \tag{3.35}$$

then does not contain any divergencies at one-loop order. The renormalization constants are given by $Z_m=1+\frac{Z_{m,1}}{\epsilon}=1+\frac{\lambda}{(4\pi)^2\epsilon}$, $Z_\lambda=1+\frac{3\lambda}{(4\pi)^2\epsilon}$ and $Z_\phi=1$, which we will keep for instructive reasons. This renormalized energy functional can be brought to its original form by defining the bare quantities

$$\phi_B=\phi\ Z_\phi^{\frac{1}{2}},\qquad m_B^2=m^2Z_mZ_\phi^{-1},\qquad \lambda_B=\lambda\mu^\epsilon Z_\lambda Z_\phi^{-2}, \tag{3.36}$$

i.e. the bare energy functional reads

$$\begin{aligned}E_B[\phi]&=\frac{1}{2}\int_k\left(\mathbf{k}^2+m_B^2\right)\phi_B(k)\phi_B(-k)\\&+\frac{\lambda_B}{4!}\int_{k_1,k_2,k_3,k_4}\phi_B(k_1)\phi_B(k_2)\phi_B(k_3)\phi_B(k_4)\delta_{k_1+k_2+k_3+k_4,0}.\end{aligned} \tag{3.37}$$

Finally, we derive an equation first introduced in [18] which will prove to be very powerful later. The bare correlation functions are related to the renormalized ones by

$$\begin{aligned}G_B^{(n)}\left(\{k_i\};m_B,\lambda_B,\epsilon\right)&=\langle\phi_B(k_1)\phi_B(k_2)...\phi_B(k_n)\rangle_0=Z_\phi^{-\frac{n}{2}}\langle\phi(k_1)\phi(k_2)...\phi(k_n)\rangle_0\\&=Z_\phi^{-\frac{n}{2}}G^{(n)}\left(\{k_i\};m,\lambda,\mu,\epsilon\right).\end{aligned} \tag{3.38}$$

Note that the renormalization constant $Z_\phi=Z_\phi\big(\lambda(\mu),\epsilon\big)$ depends on the arbitrary mass scale μ via the interaction constant only, whereas the renormalized correlation function depends explicitly on μ as well as via $g(\mu)$ and $m(\mu)$. Applying the operator $\mu\frac{d}{d\mu}$ on both sides, we get a differential equation for the correlation functions

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)\frac{\partial}{\partial m}-n\gamma(\lambda)\right]G^{(n)}\left(\{k_i\};m,\lambda,\mu,\epsilon\right)=0, \tag{3.39}$$

which is called renormalization group equation. The renormalization group functions are defined by

$$\beta(\lambda) = \mu\frac{d\lambda}{d\mu}, \qquad \gamma_m(\lambda) = \mu\frac{dm}{d\mu}, \qquad \gamma(\lambda) = \frac{\mu}{2}\frac{d\ln Z_\phi}{d\mu} \tag{3.40}$$

and are finite in the limit $\epsilon \to 0$, otherwise the theory would not be renormalizable. Fixed points of the theory, thus indicating phase transitions, are given by $\beta(\lambda) = 0$ and the renormalization group functions evaluated at the fixed point value λ^* then determine the critical exponents of the phase transition. Equation (3.39) can be used to obtain the scaling behavior of the correlation functions at the critical point.

4 CDW Model and Methods

4.1 Quantum Critical CDW Action

Now we begin with the main part of this thesis: The quantum phase transition from an ordinary Fermi liquid metal to a CDW ordered state with incommensurate ordering wave vector $\mathbf{Q} = 2k_F$, which connects two points on the Fermi surface, further also called hot-spots, with parallel tangents. We label the right hot-spot with a plus sign, the left one with a minus sign, corresponding to their positions $\pm\mathbf{Q}/2$, as can be seen in Fig. 4. The action describing the quantum phase transition consists of fermions and bosons coupled to each other via

$$S = \int_k \psi^\dagger(k)\left(-ik_0 + \xi_k\right)\psi(k) + \frac{1}{2}\int_k \phi(k)\chi^{-1}(k)\phi(-k) + \lambda\int_{k,p}\phi(p)\psi^\dagger(k+p)\psi(k). \quad (4.1)$$

Here, the real field $\phi(k)$ describes CDW fluctuations, the Grassmannian field $\psi(k)$ represents electrons with dispersion ξ_k measured from the Fermi surface and $\chi(k)$ is the CDW susceptibility. All calculations are done at zero temperature, s.t. the Matsubara frequencies become continuous and are integrated over. The continuous frequencies are then denoted by k_0 and we use the convention $k = (k_0, \mathbf{k})$ in two dimensions as well as $\int_k = \int \frac{dk_0 d^2\mathbf{k}}{(2\pi^3)}$, where $\mathbf{k}$ represents the momenta k_x and k_y. We assume the susceptibility to be peaked at $\pm\mathbf{Q} = \pm 2k_F$, s.t. mainly electrons in the vicinity of the two hot-spots scatter. Consequently, we can expand fluctuation momenta around $\pm\mathbf{Q}$, obtaining

$$\begin{aligned} S = &\int_k \psi^\dagger(k)\left(-ik_0 + \xi_k\right)\psi(k) + \int_k \phi^+(k)\chi_+^{-1}(k)\phi^-(-k) \\ &+ \lambda\int_{k,p}\left[\phi^+(p)\psi^\dagger(k_0+p_0, \mathbf{k}+\mathbf{Q}+\mathbf{p})\psi(k) + \phi^-(p)\psi^\dagger(k_0+p_0, \mathbf{k}-\mathbf{Q}+\mathbf{p})\psi(k)\right], \quad (4.2) \end{aligned}$$

where we introduced the notations $\phi(k_0, \mathbf{k}\pm\mathbf{Q}) = \phi^\pm(k)$ and $\chi_+(k) = \chi(k_0, \mathbf{k}+\mathbf{Q}) = \chi(k_0, -\mathbf{k}-\mathbf{Q}) = \chi_-(-k)$. Since we are interested in low energy excitations of the electrons near the hot-spots, we can expand the electronic momenta around $\pm\mathbf{Q}/2$. Introducing the fermion fields near the hot-spots $\psi(k_0, \mathbf{k}\pm\mathbf{Q}/2) = \psi_\pm(k)$ and considering N fermion flavors, the action can be written as

$$\begin{aligned} S = &\sum_{s=\pm}\sum_{j=1}^{N}\int_k \psi_{s,j}^\dagger(k)(-ik_0 + sk_x + k_y^2)\psi_{s,j}(k) + \int_k \phi^+(k)\left(k_0^2 + k_x^2 + k_y^2\right)\phi^-(-k) \\ &+ \frac{\lambda}{\sqrt{N}}\sum_{j=1}^{N}\int_{k,p}\left[\phi^+(p)\psi_{+,j}^\dagger(k+p)\psi_{-,j}(k) + \phi^-(p)\psi_{-,j}^\dagger(k+p)\psi_{+,j}(k)\right], \quad (4.3) \end{aligned}$$

where we expanded the fermion dispersion and the CDW susceptibility to second order around the hot-spots and rescaled all momenta and fields s.t. the proportionality constants are equal to one. The mass term in the CDW susceptibility vanishes by tuning the theory to the quantum critical point and the peak at $\pm\mathbf{Q} = \pm 2k_F$ translates to $\chi_+^{-1}(0) = \chi_-^{-1}(0) = 0$.

4.2 Generalization to $d+1$ Dimensions

To exploit the framework of dimensional regularization, we first need to generalize the action to arbitrary dimensions $d+1$. To do so, we follow the procedure first presented by Dalidovich and Lee in [1], where the co-dimension of the Fermi surface is increased such that the Fermi surface is kept one-dimensional. The action can be rewritten in spinor representation via the spinors

$$\Psi_j(k) = \begin{pmatrix} \psi_{+,j}(k) \\ \psi^\dagger_{-,j}(-k) \end{pmatrix} \tag{4.4}$$

and

$$\overline{\Psi}_j(k) = \Psi^\dagger_j(k)\sigma_y = \begin{pmatrix} i\psi_{-,j}(-k) & -i\psi^\dagger_{+,j}(k) \end{pmatrix}. \tag{4.5}$$

Using Einstein's sum convention for the spin indices, the action in the new basis takes the form

$$\begin{aligned} S = &\int_k \overline{\Psi}_j(k)\left(-ik_0\sigma_y + i\delta_k\sigma_x\right)\Psi_j(k) + \int_k \overline{\Phi}(k)\left(k_0^2 + k_x^2 + k_y^2\right)\Phi(k) \\ &- \frac{i\lambda}{2\sqrt{N}}\int_{k,p}\left[\Phi(p)\overline{\Psi}_j(k+p)\sigma_y\overline{\Psi}^T_j(-k) + \overline{\Phi}(p)\Psi^T_j(k+p)\sigma_y\Psi_j(-k)\right] \end{aligned} \tag{4.6}$$

with the fermion dispersion $\delta_k = k_x + k_y^2$ and the Pauli matrices σ_x and σ_y. The detailed derivation of the spinor action is presented in Appendix A. Note that we also introduced a new notation for the boson, namely $\Phi(k) = \phi^+(k)$ and $\overline{\Phi}(k) = \phi^-(-k)$. This is just for simplicity, but one still has to keep in mind that the bosons ϕ^+ and ϕ^- in principle describe different degrees of freedom.

The action can now be promoted to general $d+1$ dimensions via

$$\begin{aligned} S = &\int_k \overline{\Psi}_j(k)\left[-i\mathbf{K}\boldsymbol{\Gamma} + i\delta_k\sigma_x\right]\Psi_j(k) + \int_k \overline{\Phi}(k)\left(\mathbf{K}^2 + k_x^2 + k_y^2\right)\Phi(k) \\ &- \frac{i\lambda}{2\sqrt{N}}\int_{k,p}\left[\Phi(p)\overline{\Psi}_j(k+p)\sigma_y\overline{\Psi}^T_j(-k) + \overline{\Phi}(p)\Psi^T_j(k+p)\sigma_y\Psi_j(-k)\right] \end{aligned} \tag{4.7}$$

with $\mathbf{K} = (k_0, k_1, ..., k_{d-2})$, $\boldsymbol{\Gamma} = (\sigma_y, \sigma_z, ..., \sigma_z)$, $\int_k = \int \frac{d^{d-1}\mathbf{K}dk_{d-1}dk_d}{(2\pi)^{d+1}}$ and the general dispersion $\delta_k = k_{d-1} + k_d$. The momenta $\mathbf{k} = (k_1, ..., k_{d-2})$ represent the added $d-2$ space dimensions and therefore $\boldsymbol{\Gamma}$ has $d-2$ entries σ_z. To recover the original two dimensional action, we just have to set $d=2$ and make the replacement $(k_{d-1}, k_d) \to (k_x, k_y)$. The propagators in general dimensions read

$$\begin{aligned} &\left\langle \Psi^a_i(k)\overline{\Psi}^b_j(p)\right\rangle_0 = \left(-i\frac{-\boldsymbol{\Gamma}\mathbf{K} + \delta_k\sigma_x}{\mathbf{K}^2 + \delta_k^2}\right)_{ab}\delta_{ij}(2\pi)^{d+1}\delta^{(d+1)}(k-p) = G_{ab}(k)\ \delta_{ij}\ \delta_{k,p}, \\ &\left\langle \overline{\Psi}^a_i(k)\Psi^b_j(p)\right\rangle_0 = -G_{ba}(k)\ \delta_{ij}\ \delta_{k,p}, \\ &\left\langle \overline{\Phi}(k)\Phi(p)\right\rangle_0 = \left\langle \Phi(k)\overline{\Phi}(p)\right\rangle_0 = \frac{1}{\mathbf{K}^2 + k_{d-1}^2 + k_d^2}\ (2\pi)^{d+1}\delta^{(d+1)}(k-p) = D_0(k)\ \delta_{k,p}, \end{aligned} \tag{4.8}$$

where the upper indices of the fermion fields label spinor components, whereas the lower indices label flavors. Note that the last line implies $D_0(k) = \chi_+(k) = \chi_-(-k)$.

The energy of the spinors is given by poles in the fermion propagator after replacing $k_0 \to i\epsilon_k$, i.e.

$$\epsilon_k = \pm\sqrt{\mathbf{k}^2 + \delta_k^2}, \tag{4.9}$$

which vanishes for $\mathbf{k} = 0$ and $\delta_k = k_{d-1} + k_d^2 = 0$. Thus, the Fermi surface indeed is the original one-dimensional Fermi surface embedded in a d-dimensional space.

4.3 Scale Transformations

As mentioned in the introductory chapter about renormalization, in dimensional regularization the results of the loop-corrections are expanded in small deviations from the dimension in which the interaction is dimensionless. To find this special dimension, we note that the kinetic terms of the action (4.7) are invariant under the scale transformations

$$\mathbf{K} = \frac{\mathbf{K}'}{b}, \quad k_{d-1} = \frac{k'_{d-1}}{b}, \quad k_d = \frac{k'_d}{\sqrt{b}}, \quad \Psi(k) = \Psi'(k')b^{\frac{d}{2}+\frac{3}{4}}, \quad \Phi(k) = \Phi'(k')b^{\frac{d}{2}+\frac{3}{4}}. \tag{4.10}$$

At tree level, the terms $\propto \mathbf{K}^2, k_{d-1}^2$ in the boson propagator are irrelevant and hence can be neglected in the following. Under this rescaling, the interaction λ transforms as

$$\lambda \int_{k,p} \overline{\Phi}\Psi\Psi = \lambda \int_{k',p'} b^{-2d-1} b^{\frac{3}{2}d+\frac{9}{4}} \overline{\Phi}'\Psi'\Psi' = \lambda b^{-\frac{d}{2}+\frac{5}{4}} \int_{k',p'} \overline{\Phi}'\Psi'\Psi' \tag{4.11}$$

and therefore $\lambda' = \lambda b^{\frac{1}{2}\left(\frac{5}{2}-d\right)}$. Thus, the one-loop results need to be expanded around small $\epsilon = \frac{5}{2} - d$. To make the coupling dimensionless, we introduce an arbitrary mass scale μ and replace $\lambda \to \lambda\mu^{\frac{\epsilon}{2}}$.

All further calculations are then based on the action

$$\begin{aligned} S = &\int_k \overline{\Psi}(k)\left[-i\mathbf{\Gamma}\mathbf{K} + i\delta_k\sigma_x\right]\Psi(k) + \int_k \overline{\Phi}(k)\; k_d^2\; \Phi(k) \\ &- \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}} \int_{k,p} \left[\Phi(p)\overline{\Psi}(k+p)\sigma_y \overline{\Psi}^T(-k) + \overline{\Phi}(p)\Psi^T(k+p)\sigma_y\Psi(-k)\right], \end{aligned} \tag{4.12}$$

which is invariant under the scalings

$$\begin{aligned} &\mathbf{K} = \frac{\mathbf{K}'}{b}, \quad k_{d-1} = \frac{k'_{d-1}}{b}, \quad k_d = \frac{k'_d}{\sqrt{b}}, \quad \mu = \frac{\mu'}{b}, \\ &\Psi(k) = \Psi'(k')b^{\frac{d}{2}+\frac{3}{4}}, \quad \Phi(k) = \Phi'(k')b^{\frac{d}{2}+\frac{3}{4}}. \end{aligned} \tag{4.13}$$

We denote the interaction terms as $S_{\text{int},1} \propto \Phi\overline{\Psi\Psi}$ and $S_{\text{int},2} \propto \overline{\Phi}\Psi\Psi$.

5 Boson Self-Energy

The one-loop considerations start with the leading order contribution to the boson self-energy. To warm up with the kind of integrals we have to deal with and to get to know how the derivation of corrections in general dimensions work, this chapter will be rather detailed and technical. We start by calculating the one-loop boson self-energy in $2+1$ dimensions, before turning to the derivation of the $d+1$ dimensional expression of the one-loop integral and its evaluation.

5.1 Two Dimensional Boson Self-Energy

The solution to the two dimensional boson self-energy can be found in [10], but no explicit calculations are done there. Our result will differ from the one by Sýkora et al., since they haven't rescaled there momenta and fields the way we have done, but the form of the self-energy will be the same.

The two dimensional boson self-energy using our conventions reads

$$\begin{aligned}\Pi(q) &= -\lambda^2 \int_p G_+(p) G_-(p-q) \\ &= -\lambda^2 \int_p \frac{1}{-ip_0 + p_x + p_y^2} \frac{1}{-i(p_0-q_0) - (p_x - q_x) + (p_y - q_y)^2}. \end{aligned} \tag{5.1}$$

Note that $\Pi(q)$ dresses $D_0(q) = \chi_+(q) = \chi_-(-q)$. Defining new integration variables $\xi_p = p_x + p_y^2$ and $y = \frac{1}{\sqrt{2}}(2p_y - q_y)$ as well as shifting $p_0 \to p_0 + \frac{q_0}{2}$, we obtain

$$\Pi(q) = \frac{\lambda^2}{\sqrt{2}} \int_{p_0,\xi_p,y} \frac{1}{\xi_p - i(p_0 + \frac{q_0}{2})} \frac{1}{\xi_p - e_q - y^2 + i(p_0 - \frac{q_0}{2})}, \tag{5.2}$$

where $e_q = q_x + \frac{q_y^2}{2}$. The factor $\frac{1}{\sqrt{2}}$ comes from the Jacobian of changing integration variables $(p_x, p_y) \to (\xi_p, y)$.

The ξ_p-integral can be immediately performed using the Residue theorem with poles $\xi_p^1 = i(p_0 + \frac{q_0}{2})$ and $\xi_p^2 = e_q + y^2 - i(p_0 - \frac{q_0}{2})$. Closing the contour in the complex upper half plane yields

$$\begin{aligned}\Pi(q) &= \frac{i\lambda^2}{\sqrt{2}} \int \frac{dp_0 dy}{(2\pi)^2} \frac{\Theta\left(-p_0 + \frac{q_0}{2}\right) - \Theta\left(p_0 + \frac{q_0}{2}\right)}{y^2 + e_q - 2ip_0} \\ &= \frac{i\lambda^2}{\sqrt{2}} \left(\int_{-\infty}^{\frac{q_0}{2}} - \int_{-\frac{q_0}{2}}^{\infty} \right) \frac{dp_0}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{1}{y^2 + e_q - 2ip_0}. \end{aligned} \tag{5.3}$$

The y-integration can be done in the same fashion, s.t. the self-energy is given by the expression

$$\Pi(q) = -\frac{\lambda^2}{\sqrt{2}\ 4\pi} \left(\int_{-\infty}^{\frac{q_0}{2}} - \int_{-\frac{q_0}{2}}^{\infty} \right) dp_0 \frac{\Theta(p_0) - \Theta(-p_0)}{\sqrt{-e_q + 2ip_0}}$$

$$= \frac{\lambda^2}{\sqrt{2}\ 4\pi} \int_{\frac{|q_0|}{2}}^{\infty} dp_0 \left(\frac{1}{\sqrt{-e_q + 2ip_0}} + \frac{1}{\sqrt{-e_q - 2ip_0}} \right)$$
$$= \frac{\lambda^2}{4\pi} \int_{\frac{|q_0|}{2}}^{\infty} dp_0 \frac{\sqrt{\sqrt{e_q^2 + 4p_0^2} - e_q}}{\sqrt{e_q^2 + 4p_0^2}}. \tag{5.4}$$

Substituting $x = \sqrt{e_q^2 + 4p_0^2}$, the above integral evaluates to

$$\Pi(q) = \frac{\lambda^2}{4\pi} \int_{\sqrt{e_q^2+q_0^2}}^{\infty} dx \frac{1}{2\sqrt{x + e_q}} = -\frac{\lambda^2 N}{4\pi} \sqrt{\sqrt{e_q^2 + q_0^2} + e_q} + \frac{\lambda^2}{4\pi} \sqrt{x}\Big|_{x\to\infty}. \tag{5.5}$$

To cancel the divergence, we substract $\Pi(0, \mathbf{Q})$, which corresponds to setting the momenta $\mathbf{q}$ labelling $\Pi(q) = \Pi(q_0, \mathbf{q})$ equal to zero, since they are measured as deviations from the fluctuation wave vector $\mathbf{Q}$. Defining $\Pi(0, \mathbf{Q}) = \Pi(0)$, we obtain the UV-regularized and thus finite result

$$\Pi(q) - \Pi(0) = -\frac{\lambda^2}{4\pi} \sqrt{\sqrt{e_q^2 + q_0^2} + e_q}. \tag{5.6}$$

Now we turn to the task of calculating the boson self-energy in general dimensions. A simple check, indicating whether the general result is correct, is given by setting $d = 2$ and comparing it to (5.6).

5.2 One-Loop Expression for General Dimensions

The first correction to the boson self-energy occurs in second order perturbation theory in the interaction. As argued in the introductory chapter about ϕ^4-theory, only connected terms

$$\left\langle \Phi(q_1)\overline{\Phi}(q_2) \right\rangle = \left\langle \Phi(q_1)\overline{\Phi}(q_2) \right\rangle_0 + \frac{1}{2} \left\langle \Phi(q_1)\overline{\Phi}(q_2) S_{\text{int}}^2 \right\rangle_0^{con}, \tag{5.7}$$

have to be considered. The second order correction can be simplified to

$$\frac{1}{2} \left\langle \Phi(q_1)\overline{\Phi}(q_2) S_{\text{int}}^2 \right\rangle_0^{con} = \left\langle \Phi(q_1)\overline{\Phi}(q_2) S_{\text{int},1} S_{\text{int},2} \right\rangle_0^{con} = -\frac{\lambda^2 \mu^\epsilon}{4N} \sigma_y^{ab} \sigma_y^{cd}$$
$$\int_{k_1,p_1,k_2,p_2} \left\langle \Phi(q_1)\overline{\Phi}(q_2)\Phi(p_1)\overline{\Psi}_a(k_1 + p_1)\overline{\Psi}_b(-k_1)\overline{\Phi}(p_2)\Psi_c(k_2 + p_2)\Psi_d(-k_2) \right\rangle_0^{con}. \tag{5.8}$$

Note that we have suppressed the spin indices of the fermions. The two possible ways of contracting the fermion fields lead to the same result and we get an additional factor of N, since the suppressed spin indices evaluate to $\sum_{i,j=1}^{N} \delta_{ij}\delta_{ij} = N$. Hence, the correction reads

$$\frac{1}{2} \left\langle \Phi(q_1)\overline{\Phi}(q_2) S_{\text{int}}^2 \right\rangle_0^{con} = -\frac{\lambda^2 \mu^\epsilon}{2N} \sigma_y^{ab} \sigma_y^{cd}$$
$$\int_{k_1,p_1,k_2,p_2} \Big\langle \Phi(q_1)\overline{\Phi}(q_2)\Phi(p_1)\overline{\Psi}_a(k_1 + p_1)\sigma_y^{ab}\overline{\Psi}_b(-k_1)\overline{\Phi}(p_2)\Psi_c(k_2 + p_2)\Psi_d(-k_2) \Big\rangle_0^{con}$$

$$
\begin{aligned}
&= -\frac{\lambda^2\mu^\epsilon}{2}\int_{k_1,p_1,k_2,p_2} D_0(q_1)\delta_{q_1,p_2}D_0(q_2)\delta_{q_2,p_1}G_{da}(-k_2)\delta_{-k_2,k_1+p_1}G_{cb}(-k_1)\delta_{-k_1,k_2+p_2}\sigma_y^{ab}\sigma_y^{cd}\\
&= -\frac{\lambda^2\mu^\epsilon}{2}D_0(q_1)D_0(q_2)\int_{k_1,k_2} G_{da}(-k_2)\delta_{-k_2,k_1+q_2}G_{cb}(-k_1)\delta_{-k_1,k_2+q_1}\sigma_y^{ab}\sigma_y^{cd}\\
&= -\frac{\lambda^2\mu^\epsilon}{2}D_0^2(q_1)\delta_{q_1,q_2}\int_{k_1} G_{da}(k_1+q_2)G_{cb}(-k_1)\sigma_y^{ab}\sigma_y^{cd}\\
&= D_0^2(q_1)\delta_{q_1,q_2}\left[-\frac{\lambda^2\mu^\epsilon}{2}\int_p \mathrm{Tr}\Big(G(p)\sigma_y G^T(q-p)\sigma_y\Big)\right], \qquad (5.9)
\end{aligned}
$$

where we set $p = k_1 + q_2$ in the last step. Amputating the external propagators together with the energy and momentum conserving δ-function, the integral for the boson self-energy is given by

$$\Pi(q) = -\frac{\lambda^2\mu^\epsilon}{2}\int_p \mathrm{Tr}\Big(G(p)\sigma_y G^T(q_1-p)\sigma_y\Big). \qquad (5.10)$$

Note again that this self-energy modifies $D_0(q) = \chi_+(q) = \chi_-(-q)$ in general dimensions.

5.3 Calculation of Integral

The evaluation of

$$
\begin{aligned}
\Pi(q) = &-\frac{\lambda^2\mu^\epsilon}{2}(-i)^2\\
&\int_p \frac{\mathrm{Tr}\left[\Big(-p_0\sigma_y - \mathbf{p}\sigma_\mathbf{z} + \delta_p\sigma_x\Big)\sigma_y\Big((q_0-p_0)\sigma_y - (\mathbf{q}-\mathbf{p})\sigma_\mathbf{z} + \delta_{q-p}\sigma_x\Big)\sigma_y\right]}{\Big(\mathbf{P}^2+\delta_p^2\Big)\Big((\mathbf{Q}-\mathbf{P})^2+\delta_{q-p}^2\Big)} \qquad (5.11)
\end{aligned}
$$

is much more involved than in the two dimensional case. We begin with simplifying the trace over Pauli matrices using their properties $\mathrm{Tr}(\sigma_i) = 0$ and $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, which yields

$$\mathrm{Tr}\left(\sigma_i\sigma_y\sigma_j\sigma_y\right) = \begin{cases} 2 & i=j=y \\ -2 & i=j\neq y \\ 0 & i\neq j \end{cases} \qquad (5.12)$$

and thus

$$\Pi(q) = -\lambda^2\mu^\epsilon\int_p \frac{\delta_p\delta_{q-p} - \mathbf{P}(\mathbf{P}-\mathbf{Q})}{\Big(\mathbf{P}^2+\delta_p^2\Big)\Big((\mathbf{P}-\mathbf{Q})^2+\delta_{q-p}^2\Big)}. \qquad (5.13)$$

We shift $p_{d-1} \to p_{d-1} - p_d^2$ and define a new integration variable $y = \frac{1}{\sqrt{2}}(2p_d - q_d)$

$$\Pi(q) = \frac{\lambda^2\mu^\epsilon}{\sqrt{2}}\int_{\mathbf{P},p_{d-1},y} \frac{p_{d-1}\left(p_{d-1} - e_q - y^2\right) + \mathbf{P}(\mathbf{P}-\mathbf{Q})}{\Big(\mathbf{P}^2+p_{d-1}^2\Big)\left[(\mathbf{P}-\mathbf{Q})^2 + \Big(p_{d-1} - e_q - y^2\Big)^2\right]}, \qquad (5.14)$$

where $e_q = q_{d-1} + \frac{q_d^2}{2}$ is the d-dimensional analog to the two dimensional case. The denominator can be rewritten using the Feynman parametrization (3.27). Similar to the vertex correction in ϕ^4-theory, we can get rid of a cross term $\propto \mathbf{PQ}$ in the new denominator by shifting $\mathbf{P} \to \mathbf{P} + x\mathbf{Q}$. The resulting term $\propto \mathbf{PQ}$ in the numerator then vanishes by antisymmetry under $\mathbf{P} \to -\mathbf{P}$, s.t. the self-energy reads

$$\Pi(q) = \frac{\lambda^2 \mu^\epsilon}{\sqrt{2}} \int_{\mathbf{P}, p_{d-1}, y} \int_0^1 dx \frac{p_{d-1}^2 - \left(e_q + y^2\right) p_{d-1} + \mathbf{P}^2 - x(1-x)\mathbf{Q}^2}{\left[p_{d-1}^2 - 2x\left(e_q + y^2\right) p_{d-1} + x\left(e_q + y^2\right)^2 + \mathbf{P}^2 + x(1-x)\mathbf{Q}^2\right]^2}. \tag{5.15}$$

The p_{d-1}-integral can directly be evaluated to

$$\Pi(q) = \frac{\lambda^2 \mu^\epsilon}{2^{\frac{3}{2}}} \int_{\mathbf{P}, y} \int_0^1 dx \frac{\mathbf{P}^2}{\left[\mathbf{P}^2 + x(1-x)\mathbf{Q}^2 + x(1-x)\left(e_q + y^2\right)^2\right]^{\frac{3}{2}}}. \tag{5.16}$$

By rescaling $\mathbf{P} \to \sqrt{x(1-x)}\mathbf{P}$, the x-integral can be carried out

$$\int_0^1 dx \, \left[x(1-x)\right]^{\frac{d}{2}-1} = \frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma(d)}, \tag{5.17}$$

thus leaving

$$\Pi(q) = \lambda^2 \mu^\epsilon \frac{\Gamma^2\left(\frac{d}{2}\right)}{2^{\frac{3}{2}} \Gamma(d)} \int_{\mathbf{P}, y} \frac{\mathbf{P}^2}{\left[\mathbf{P}^2 + \mathbf{Q}^2 + \left(e_q + y^2\right)^2\right]^{\frac{3}{2}}}. \tag{5.18}$$

The integral over the momenta $\mathbf{P}$ is now of the form (3.24) with $a = 1$, $b = \frac{3}{2}$ and d replaced by $d - 1$, yielding

$$\Pi(q) = \lambda^2 \mu^\epsilon \frac{\Gamma^2\left(\frac{d}{2}\right)}{2^{\frac{3}{2}} \Gamma(d)} \frac{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right) \Gamma\left(\frac{3}{2}\right)} \int_y \left[\mathbf{Q}^2 + \left(e_q + y^2\right)^2\right]^{\frac{d}{2}-1}. \tag{5.19}$$

This integral requires UV-regularization, i.e. we substract $\Pi(0)$ such that

$$\Pi(q) - \Pi(0) = \lambda^2 \mu^\epsilon \frac{\Gamma^2\left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right)(d-1)}{2^{d+\frac{1}{2}} \pi^{\frac{d}{2}+1} \Gamma(d)} \int_0^\infty dy \left\{ \left[\mathbf{Q}^2 + \left(e_q + y^2\right)^2\right]^{\frac{d}{2}-1} - \left(y^4\right)^{\frac{d}{2}-1} \right\}, \tag{5.20}$$

where we used $\Gamma\left(\frac{d+1}{2}\right) = \left(\frac{d-1}{2}\right) \Gamma\left(\frac{d-1}{2}\right)$. After the substitution $z = y^2 + e_q$, we have to distinguish two different cases, namely $e_q > 0$ and $e_q < 0$.

For $e_q > 0$, we get the final result

$$\Pi(q) - \Pi(0) = \lambda^2 \mu^\epsilon \frac{\Gamma^2\left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right)(d-1)}{2^{d+\frac{3}{2}} \pi^{\frac{d}{2}+1} \Gamma(d)} \frac{\Gamma\left(\frac{3}{2} - d\right) \sqrt{\pi}}{\Gamma(2-d)} e_q^{d-\frac{3}{2}}$$

$$\times\, {}_2F_1\left[\frac{3-2d}{4}, \frac{5-2d}{4}, \frac{3-d}{2}, -\frac{\mathbf{Q}^2}{e_q^2}\right], \tag{5.21}$$

where ${}_2F_1$ is the hypergeometric function. Note that the pole for $d = 2$ in $\Gamma\left(1-\frac{d}{2}\right)$ is cancelled by $\Gamma(2-d)$. Thus the finite expression for $d = 2$ reads

$$\Pi(q) - \Pi(0) = -\frac{\lambda^2}{4\sqrt{2}\pi}\left(\sqrt{e_q + i|q_0|} + \sqrt{e_q - i|q_0|}\right) = -\frac{\lambda^2}{4\pi}\sqrt{\sqrt{e_q^2 + q_0^2} + e_q}, \tag{5.22}$$

which indeed coincides with (5.6). As mentioned in Section 4.3, the result (5.21) needs to be expanded in $\epsilon = \frac{5}{2} - d$ and we find a pole

$$\Pi(q) - \Pi(0) = -\lambda^2 \frac{\Gamma\left(\frac{5}{4}\right)}{8\sqrt{2}\pi^{\frac{7}{4}}}\frac{e_q}{\epsilon} + \text{finite}. \tag{5.23}$$

The expression

$$\begin{aligned}\Pi(q) - \Pi(0) = \lambda^2\mu^\epsilon \frac{\Gamma^2\left(\frac{5}{4}-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}-\frac{1}{4}\right)\left(\frac{3}{2}-\epsilon\right)}{2^{4-\epsilon}\pi^{\frac{9}{4}-\frac{\epsilon}{2}}\Gamma\left(\frac{5}{2}-\epsilon\right)} \frac{|\mathbf{Q}|^{-\epsilon}}{\Gamma\left(\frac{\epsilon}{2}-\frac{1}{4}\right)}\Bigg(2|\mathbf{Q}|\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{\epsilon-1}{2}\right) \\ \times\, {}_2F_1\left[\frac{1}{4}, \frac{\epsilon-1}{2}, \frac{1}{2}, -\frac{e_q^2}{\mathbf{Q}^2}\right] - e_q\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{\epsilon}{2}\right){}_2F_1\left[\frac{3}{4}, \frac{\epsilon}{2}, \frac{3}{2}, -\frac{e_q^2}{\mathbf{Q}^2}\right]\Bigg),\end{aligned} \tag{5.24}$$

is the solution to (5.20) for the case $e_q < 0$, where we already set $\epsilon = \frac{5}{2} - d$. The two dimensional result (5.6) is again recovered for $\epsilon = \frac{1}{2}$ (see Appendix B) and by expanding around $\epsilon = 0$, we obtain the same pole as for $e_q > 0$. Thus, the final result for the one-loop boson self-energy in dimensional regularization is of the form

$$\Pi(q) - \Pi(0) = -u_1\lambda^2\frac{e_q}{\epsilon} + \text{finite}, \tag{5.25}$$

where $u_1 = \frac{\Gamma\left(\frac{5}{4}\right)}{8\sqrt{2}\pi^{\frac{7}{4}}}$.

The $1/\epsilon$-pole is $\propto e_k = k_{d-1} + \frac{k_d^2}{2}$, hence renormalizes the boson propagator $D_0^{-1}(k) = k_d^2$, but also a term not present in the boson kinetic action. Thus, we need to include an additional term $\propto k_{d-1}$ in the boson propagator, then given by the expression $D^{-1} = k_d^2 + ak_{d-1}$, where a is a real parameter which will be determined by the RG flow later.

From $D^{-1}(k) = k_d^2 + ak_{d-1} = \chi_+^{-1}(k) = \chi_-^{-1}(-k)$, the new expression for the CDW susceptibility near the two momenta $\pm 2k_F$ can be derived to be $\chi_\pm^{-1}(k) = k_d^2 \pm ak_{d-1}$. Note that k_{d-1} has the same scaling transformation as k_d^2, i.e. the new term $\propto k_{d-1}$ is relevant and the parameter a is dimensionless.

The boson self-energy is in contrast to other works using dimensional regularization, such as [19] or the original work [1]. There, no poles occurred in the boson self-energy, which was of the form of a standard Landau damping term $\propto |k_0|$ in two dimensions. In our case, such a term is generated by expanding (5.24) in small $|\mathbf{K}|$ and afterwards in small ϵ, where the leading order contribution to the finite part is $\propto |\mathbf{K}|^{3/2}$, the $d = \frac{5}{2}$ dimensional analog to the Landau damping term in two dimensions.

In [1] and [19], the Landau damping term had to be included in the boson propagator, since otherwise the one-loop integral for the fermion self-energy would be IR-divergent due to the factor $1/k_d^2$ coming from the boson propagator. In our case, however, the same divergence is not present when including the term $\propto k_{d-1}$, as seen in the next chapter. Note that by neglecting frequency depending terms of the finite part of the boson self-energy, the fermion self-energy cannot depend on external frequencies either.

6 Fermion Self-Energy

Neglecting frequency-depending, non-analytic terms in the boson self-energy, our approach to the fermion self-energy differs from the two dimensional considerations in [9] and [10]. In the first part of this chapter, we therefore compute the fermion self-energy in two dimensions without non-analytic terms and check whether additional divergencies occur. If yes, it indicates that the non-analytic terms are essential for the treatment of the transition to CDW order. If not, further calculations in dimensional regularization can be done with $D^{-1} = k_d^2 + ak_{d-1}$ and should lead to reliable physical quantities when continuing the dimension to $d = 2$.

6.1 Two Dimensional Fermion Self-Energy

Using our conventions, the two dimensional fermion self-energy, modifying the propagator $G_+(q) = (-ik_0 + k_x + k_y^2)^{-1}$, is given by the integral

$$\begin{aligned}\Sigma_+(q) &= \frac{\lambda^2}{N}\int_p D(p)G_-(q-p) = \frac{\lambda^2}{N}\int_p \frac{1}{p_y^2 + ap_x}\frac{1}{-i(q_0-p_0)-(q_x-p_x)+(q_y-p_y)^2} \\ &= \frac{\lambda^2}{Na}\int_p \frac{1}{p_x + \frac{p_y^2}{a}}\frac{1}{p_x - q_x + (q_y-p_y)^2 - ip_0}. \end{aligned} \tag{6.1}$$

The p_x-integral can be evaluated using the principal value (see Appendix C.1), yielding

$$\begin{aligned}\Sigma_+(q) &= \frac{i\lambda^2}{2Na}\int_{p_0,p_y} \frac{\operatorname{sgn}(p_0)}{\frac{p_y^2}{a} + q_x - (q_y-p_y)^2 + ip_0} \\ &= \frac{i\lambda^2\tilde{a}}{2Na}\int_{p_0,p_y} \frac{\operatorname{sgn}(p_0)}{p_y^2 + 2\tilde{a}q_yp_y - \tilde{a}\delta_{-q} + i\tilde{a}p_0} \end{aligned} \tag{6.2}$$

with $\tilde{a} = \frac{a}{1-a}$ and $\delta_{-q} = -q_x + q_y^2$. The p_y integral is derived in Appendix C.2, s.t. we obtain

$$\begin{aligned}\Sigma_+(q) &= \frac{\lambda^2\tilde{a}}{2Na}\int_{p_0} \operatorname{sgn}(p_0)\frac{\operatorname{sgn}(\tilde{a}p_0)}{\sqrt{4\tilde{a}^2q_y^2 + 4\tilde{a}\delta_{-q} - 4i\tilde{a}p_0}} \\ &= \frac{\lambda^2|\tilde{a}|}{2Na}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{1}{2\sqrt{|\tilde{a}|}}\frac{1}{\sqrt{|\tilde{a}|q_y^2 + \operatorname{sgn}(\tilde{a})\,\delta_{-q} - i\operatorname{sgn}(\tilde{a})\,p_0}} \\ &= \frac{\lambda^2\sqrt{|\tilde{a}|}}{8\pi Na}\int_{-\infty}^{\infty} dp_0 \frac{1}{\sqrt{c(q) - i\operatorname{sgn}(\tilde{a})\,p_0}}, \end{aligned} \tag{6.3}$$

where we defined $c(q) = |\tilde{a}|\,q_y^2 + \operatorname{sgn}(\tilde{a})\,\delta_{-q}$. The remaining integral diverges like $\sqrt{p_0}$, thus requiring UV-regularization by substracting $\Sigma(0)$. The calculation of the resulting integral is done in Appendix C.3, leading to the final expression

$$\Sigma_+(q) - \Sigma_+(0) = -\frac{\lambda^2\sqrt{|\tilde{a}|}}{2\pi Na}\sqrt{\left|\tilde{a}q_y^2 + \delta_{-q}\right|}\;\Theta\left(-|\tilde{a}|\,q_y^2 - \operatorname{sgn}(\tilde{a})\,\delta_{-q}\right). \tag{6.4}$$

The only additional divergence occurred in the p_x-integral, but could be made finite by using the principle value. Thus it should be reasonable to calculate the fermion self-energy in general dimensions without a non-analytic term and still getting physically sensible results for $d = 2$.

6.2 One-Loop Expression for General Dimensions

Similar to the one-loop correction to the boson propagator, the correction to the fermion propagator in second order perturbation theory can by calculated via

$$\frac{1}{2}\left\langle \Psi_a(q_1)\overline{\Psi}_b(q_2)S_{\text{int}}^2\right\rangle_0^{con} = \left\langle \Psi_a(q_1)\overline{\Psi}_b(q_2)S_{\text{int},1}S_{\text{int},2}\right\rangle_0^{con} = -\frac{\lambda^2\mu^\epsilon}{4N}\sigma_y^{cd}\sigma_y^{ef}$$
$$\int_{k_1,p_1,k_2,p_2}\left\langle \Psi_a(q_1)\overline{\Psi}_b(q_2)\Phi(p_1)\overline{\Psi}_c(k_1+p_1)\overline{\Psi}_d(-k_1)\overline{\Phi}(p_2)\Psi_e(k_2+p_2)\Psi_f(-k_2)\right\rangle_0^{con}. \quad (6.5)$$

We have four possibilities connecting the external Fermions to the interaction terms giving rise to a factor of four and hence

$$\begin{aligned}
&\frac{1}{2}\left\langle \Psi_a(q_1)\overline{\Psi}_b(q_2)S_{\text{int}}^2\right\rangle_0^{con} = -\frac{\lambda^2\mu^\epsilon}{N}\sigma_y^{cd}\sigma_y^{ef}\\
&\quad\int_{k_1,p_1,k_2,p_2}\left\langle \underbrace{\Psi_a(q_1)\overline{\Psi}_b(q_2)\overbrace{\Phi(p_1)\overline{\Psi}_c(k_1+p_1)\overline{\Psi}_d(-k_1)\overbrace{\Phi}(p_2)}\Psi_e(k_2+p_2)\Psi_f(-k_2)}\right\rangle_0^{con}\\
&= \frac{\lambda^2\mu^\epsilon}{N}\int_{k_1,p_1,k_2,p_2} G_{ac}(q_1)\delta_{q_1,k_1+p_1}G_{fb}(q_2)\delta_{q_2,-k_2}D(p_1)\delta_{p_1,p_2}G_{ed}(-k_1)\delta_{-k_1,k_2+p_2}\sigma_y^{cd}\sigma_y^{ef}\\
&= \frac{\lambda^2\mu^\epsilon}{N}G_{ac}(q_1)G_{fb}(q_1)\delta_{q_1,q_2}\int_{p_1} D(p_1)G_{ed}(p_1-q_1)\sigma_y^{cd}\sigma_y^{ef}\\
&= G_{ac}(q_1)\left[\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y^{cd}G_{de}^T(p-q_1)\sigma_y^{ef}D(p)\right]G_{fb}(q_1)\delta_{p_1,q_1}.
\end{aligned} \quad (6.6)$$

Ignoring the external propagators and the energy and momentum conserving δ-function, we get the amputated expression for the fermion self-energy

$$\Sigma(q) = \frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y G^T(p-q)\sigma_y D(p). \quad (6.7)$$

6.3 Calculation of Integral

As usual, the first step to evaluate $\Sigma(q)$ is the simplification of the product of Pauli matrices

$$\begin{aligned}
\sigma_y G^T(k)\sigma_y &= -i\sigma_y\frac{k_0\sigma_y - \mathbf{k}\sigma_z + \delta_k\sigma_x}{\mathbf{K}^2+\delta_k^2}\sigma_y = -i\sigma_y\frac{k_0\sigma_y\sigma_y + \mathbf{k}\sigma_y\sigma_z - \delta_k\sigma_y\sigma_x}{\mathbf{K}^2+\delta_k^2}\\
&= -G(k),
\end{aligned} \quad (6.8)$$

leading to

$$\Sigma(q) = -\frac{\lambda^2\mu^\epsilon}{N}\int_p G(p-q)D(p) = i\lambda^2\mu^\epsilon\int_p \frac{-\mathbf{\Gamma}(\mathbf{P}-\mathbf{Q})+\sigma_x\delta_{p-q}}{(\mathbf{P}-\mathbf{Q})^2+\delta_{p-q}^2}\frac{1}{p_d^2+ap_{d-1}}$$

$$= \frac{i\lambda^2\mu^\epsilon\sigma_x}{N}\int_p \frac{p_{d-1}-q_{d-1}+(p_d-q_d)^2}{\mathbf{P}^2+(p_{d-1}-q_{d-1}+(p_d-q_d)^2)^2}\frac{1}{p_d^2+ap_{d-1}}, \tag{6.9}$$

where we shifted $\mathbf{P}\to\mathbf{P}+\mathbf{Q}$ and used the antisymmetry of the term $\propto\mathbf{P}$ under $\mathbf{P}\to-\mathbf{P}$. $\Sigma(q)$ therefore is a matrix $\propto\sigma_x$ and thus can renormalize the shape of the Fermi surface in the action only.

Shifting $p_{d-1}\to p_{d-1}-p_d^2+2q_dp_d$ yields

$$\Sigma(q) = \frac{i\lambda^2\mu^\epsilon\sigma_x}{N(1-a)}\int_p \frac{p_{d-1}+\delta_{-q}}{\mathbf{P}^2+(p_{d-1}+\delta_{-q})^2}\ \frac{1}{p_d^2+2\tilde{a}q_dp_d+\tilde{a}p_{d-1}}, \tag{6.10}$$

where we again defined $\tilde{a}=\frac{a}{1-a}$. The p_d-integral can be evaluated using the principal value (see Appendix C.4), s.t. we obtain together with the substitution $y=\operatorname{sgn}(\tilde{a})\,p_{d-1}-|\tilde{a}|\,q_d^2$

$$\begin{aligned}\Sigma(q) &= \frac{i\lambda^2\mu^\epsilon\sigma_x}{N(1-a)}\int_{\mathbf{P},p_{d-1}}\frac{p_{d-1}+\delta_{-q}}{\mathbf{P}^2+(p_{d-1}+\delta_{-q})^2}\frac{\Theta\left(\operatorname{sgn}(\tilde{a})\,p_{d-1}-|\tilde{a}|\,q_d^2\right)}{\sqrt{4\,|\tilde{a}|\left(\operatorname{sgn}(\tilde{a})\,p_{d-1}-|\tilde{a}|\,q_d^2\right)}}\\ &= \frac{i\lambda^2\mu^\epsilon\sigma_x}{2N(1-a)\sqrt{|\tilde{a}|}}\int_{\mathbf{P}}\int_{-\operatorname{sgn}(\tilde{a})\infty}^{\operatorname{sgn}(\tilde{a})\infty}\frac{dy}{2\pi}\operatorname{sgn}(\tilde{a})\frac{dy}{2\pi}\ \frac{\operatorname{sgn}(\tilde{a})\,y+\tilde{a}q_d^2+\delta_{-q}}{\mathbf{P}^2+(\operatorname{sgn}(\tilde{a})\,y+\tilde{a}q_d^2+\delta_{-q})^2}\frac{\Theta(y)}{\sqrt{y}}\\ &= \frac{i\lambda^2\mu^\epsilon\sigma_x\,\operatorname{sgn}(\tilde{a})}{2N(1-a)\sqrt{|\tilde{a}|}}\int_{\mathbf{P}}\int_0^\infty\frac{dy}{2\pi}\frac{y+c(q)}{\mathbf{P}^2+\big(y+c(q)\big)^2}\frac{1}{\sqrt{y}},\end{aligned} \tag{6.11}$$

where we defined similar to the two dimensional case $c(q)=|\tilde{a}|\,q_d^2+\operatorname{sgn}(\tilde{a})\,\delta_{-q}$. The y-integral leads to

$$\Sigma(q) = \frac{i\lambda^2\mu^\epsilon\sigma_x\,\operatorname{sgn}(\tilde{a})}{N(1-a)\sqrt{|\tilde{a}|}}\frac{1}{2^{d+1}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\ 2\ \mathrm{Re}\left\{\int_0^\infty dr\frac{r^{d-2}}{\sqrt{c(q)+ir}}\right\}. \tag{6.12}$$

The above integral converges for $d<\frac{3}{2}$, but since we are interested in $2<d<\frac{5}{2}$, we need to substract $\Sigma(0)$. The integral can then be evaluated to

$$\begin{aligned}\Sigma(q)-\Sigma(0) = \frac{i\lambda^2\mu^\epsilon\sigma_x\,\operatorname{sgn}(\tilde{a})}{N(1-a)\sqrt{|\tilde{a}|}}&\frac{1}{2^d\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\frac{\Gamma\left(\frac{3}{2}-d\right)\Gamma(d-1)}{\sqrt{\pi}}\\ &\times\mathrm{Re}\left\{i^{1-d}\big(c(q)\big)^{d-\frac{3}{2}}\right\}.\end{aligned} \tag{6.13}$$

Setting $d=\frac{5}{2}-\epsilon$ and expanding around $\epsilon=0$ yields the poles

$$\begin{aligned}\Sigma(q)-\Sigma(0) &= \frac{i\lambda^2\sigma_x\,\operatorname{sgn}(\tilde{a})}{N(1-a)\sqrt{|\tilde{a}|}}\frac{1}{16\pi^{\frac{3}{4}}\Gamma\left(\frac{3}{4}\right)}\frac{c(q)}{\epsilon}+\text{finite}\\ &= -\frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}\frac{i\sigma_xq_{d-1}}{\epsilon}+\frac{2u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}\frac{i\sigma_xq_d^2}{\epsilon}+\text{finite}.\end{aligned} \tag{6.14}$$

The original IR-divergence, which we avoided by including the term $\propto k_{d-1}$ in the boson propagator, is encoded in the above expression since it diverges for $a\to0$ as $|a|^{-1/2}$.

We can check if (6.13) reproduces the correct two dimensional expression (6.4), but since the generalized self-energy is a matrix, we first have to identify the right component

dressing the two-dimensional scalar propagator $G_+(k)$ via $G_+^{-1}(k) - \Sigma_+(k)$. Writing out the inverse two dimensional fermion propagator in spinor representation explicitly

$$\begin{aligned} G^{-1}(k) &= -ik_0\sigma_y + i\delta_k\sigma_x = \begin{pmatrix} 0 & i\,(ik_0+\delta_k) \\ i\,(-ik_0+\delta_k) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & iG_-^{-1}(-k) \\ iG_+^{-1}(k) & 0 \end{pmatrix}, \end{aligned} \tag{6.15}$$

we find that the lower left component should describe the modification of G_+ by the self-energy Σ_+. Indeed, using

$$G(k) = \begin{pmatrix} 0 & -iG_+(k) \\ -iG_-(-k) & 0 \end{pmatrix}, \tag{6.16}$$

the expression of the self-energy in spinor representation reads

$$\Sigma(q) = -\frac{\lambda^2}{N}\int_p G(p-q)D(p) = -\frac{\lambda^2}{N}\int_p \begin{pmatrix} 0 & -iG_+(p-q) \\ -iG_-(q-p) & 0 \end{pmatrix} D(p), \tag{6.17}$$

where the lower off-diagonal element corresponds to $i\Sigma_+$.

Setting $d = 2$ in (6.13) leads to

$$\Sigma(q) - \Sigma(0) = -\frac{i\lambda^2\sigma_x\ \mathrm{sgn}\,(\tilde{a})}{2\pi N(1-a)\sqrt{|\tilde{a}|}}\mathrm{Re}\left\{i^{-1}\sqrt{c(q)}\right\}. \tag{6.18}$$

For the real part to be not zero, $c(q)$ has to be negative, s.t. we obtain the expression

$$\begin{aligned} \Sigma(q) - \Sigma(0) &= -\frac{i\lambda^2\sigma_x\sqrt{|\tilde{a}|}}{2\pi N a}\mathrm{Re}\left\{-i\sqrt{-\,|c(q)|}\right\}\Theta\left(-c(q)\right) \\ &= -\frac{i\lambda^2\sigma_x\sqrt{|\tilde{a}|}}{2\pi N a}\sqrt{|\tilde{a}q_y^2+\delta_{-q}|}\,\Theta\left(-\,|\tilde{a}|\,q_y^2 - \mathrm{sgn}\,(\tilde{a})\,\delta_{-q}\right) \\ &= i\sigma_x\big(\Sigma_+(q) - \Sigma_+(0)\big). \end{aligned} \tag{6.19}$$

Hence the lower left component of the self-energy calculated in general dimensions reproduces the correct result (6.4) in two dimensions.

7 Vertex Correction and Cancellation of Divergencies

As was the case in ϕ^4-theory, vertices can in principle have one-loop corrections as well. In this chapter we introduce the Feynman rules and use them to argue that there are no vertex corrections in our theory. Afterwards, we introduce two different approaches to cancelling the divergencies of the one-loop fermion and boson self-energies, both using the minimal substraction scheme already introduced in Section 3.4.

7.1 Feynman Rules

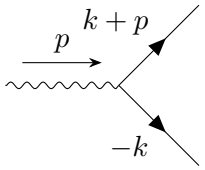
First we recall the action

$$\begin{aligned} S = &\int_k \overline{\Psi}(k) \left[-i\mathbf{\Gamma K} + i\delta_k \sigma_x\right] \Psi(k) + \int_k \overline{\Phi}(k) \left(k_d^2 + a k_{k_{d-1}}\right) \Phi(k) \\ &- \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}} \int_{k,p} \left[\Phi(p)\overline{\Psi}(k+p)\sigma_y \overline{\Psi}^T(-k) + \overline{\Phi}(p)\Psi^T(k+p)\sigma_y \Psi(-k)\right] \end{aligned} \tag{7.1}$$

and the propagators

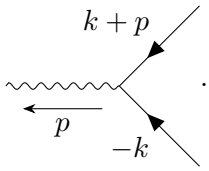
$$\begin{aligned} \left\langle \Psi_a(k_1)\overline{\Psi}_b(k_2)\right\rangle_0 &= G_{ab}(k_1)\delta_{k_1,k_2}, \\ \left\langle \overline{\Psi}_a(k_1)\Psi_b(k_2)\right\rangle_0 &= -G_{ba}(k_1)\delta_{k_1,k_2}, \\ \left\langle \Phi(k)\overline{\Phi}(k)\right\rangle_0 &= \left\langle \overline{\Phi}(k)\Phi(k)\right\rangle_0 = D(k)\delta_{k_1,k_2}, \end{aligned} \tag{7.2}$$

where $\delta_{k_1,k_2} = (2\pi)^{d+1}\delta^{(d+1)}(k_1 - k_2)$. The interaction terms can be represented by the following Feynman diagrams. The first vertex has the form



, (7.3)

whereas the second vertex is given by



. (7.4)

The wavy lines represent bosons, the straight lines fermions. The vertex factors corresponding to the above diagrams can be obtained by calculating the first order contribution

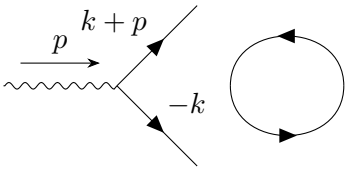
$$\begin{aligned} \left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)\right\rangle &= -\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{int}\right\rangle_0 + \mathcal{O}\left(\lambda^3\right) \\ &= D(p)\left(-G^T\right)_{ad}(k+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)^{dc} G_{cb}(-k) + \mathcal{O}\left(\lambda^3\right). \end{aligned} \tag{7.5}$$

Note that the momenta of the fields in the expectation value were already chosen properly, s.t. there is no energy and momentum conserving δ-function in the final expression. The explicit calculation of the contractions is done in Appendix D.1. We attributed the minus sign to the transposed fermion propagator to get the correct signs in the following diagrams. Thus, the vertex factor is a matrix $\propto \sigma_y$ given by the expression in the brackets.

For the second vertex we find

$$\begin{aligned}\left\langle \overline{\Phi}(p)\Psi_a(k+p)\Psi_b(-k)\right\rangle &= -\left\langle \overline{\Phi}(p)\Psi_a(k+p)\Psi_b(-k)S_{int}\right\rangle_0 + \mathcal{O}\left(\lambda^3\right) \\ &= D(p)G_{ad}(k+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)^{dc}\left(-G^T\right)_{cb}(-k) + \mathcal{O}\left(\lambda^3\right), \end{aligned} \tag{7.6}$$

s.t. the vertex factor is the same for both vertices. An interesting thing to notice is the placement of the transposed fermion propagators, which seems to come from the direction of the fermion lines in the diagrams. Imagine a circle with counterclockwise direction



. (7.7)

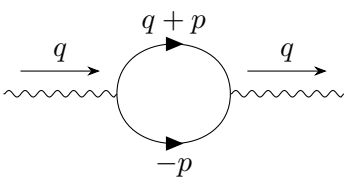
Following the fermion lines in the direction of this circle, we get a regular fermion propagator if the directions coincide and a negative transposed propagator when they differ. At appropriate places, we have to include the vertex factors of course. For the diagram in (7.7) this implies a negative transposed fermion propagator, followed by a vertex factor and then a regular propagator, just like in (7.5).

We further assume that we have to take the trace over fermion loops with an additional factor $-N$, as usual for fermionic theories. Integration over internal momenta and symmetry factors have to be included as well.

A first check for these Feynman rules is the one-loop correction to the boson propagator with the mathematical expression

$$\Pi^{n.a.} = D^2(q)\left[-\frac{\lambda^2\mu^\epsilon}{2}\int_p \mathrm{Tr}\left(G(p)\sigma_y G^T(q-p)\sigma_y\right)\right], \tag{7.8}$$

where the superscript denotes that the external legs are not amputated yet. Given the diagrammatic vertices in (7.3) and (7.4), the only diagram with two external bosonic legs we can draw is



. (7.9)

Symmetry factors of diagrams at n^{th} order perturbation theory, denoted by $S(n)$ in the following, can be calculated as follows. Expanding the interaction exponential to n^{th} order

always yields a factor $\frac{1}{n!\times 2^n}$, where $\frac{1}{n!}$ comes from expanding the exponential function itself and $\frac{1}{2^n}$ comes from the one-half in the interaction S_{int} which was not included in the vertex factor. Counteracting these factors is the number of possibilities of drawing a certain diagram in n^{th} order perturbation theory, denoted as $p(n)$. Note that $p(n)$ does not have to be the same for all diagrams at a certain order. Thus, the symmetry factor is obtained by the formula $S(n) = \frac{p(n)}{n!\times 2^n}$.

For the one-loop correction to the boson propagator, we have a factor of two for interchanging the vertices and two possibilities of connecting the fermion lines, hence $p(2) = 4$ and consequently $S(2) = \frac{4}{2!\times 2^2} = \frac{1}{2}$. Using our Feynman rules, the diagram in (7.9) is translated into the mathematical expression

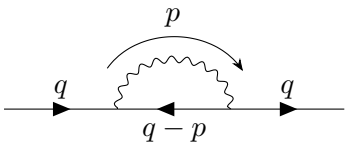
$$\begin{aligned} &D^2(q)\int_p\left(-\frac{N}{2}\right)\mathrm{Tr}\left[G(-p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)\left(-G^T\right)(q+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)\right] \\ &= D^2(q)\left[-\frac{\lambda^2\mu^\epsilon}{2}\int_p \mathrm{Tr}\left(G(p)\sigma_y G^T(q-p)\sigma_y\right)\right], \end{aligned} \tag{7.10}$$

indeed coinciding with (7.8).

A second check is provided by the fermion self-energy

$$\Sigma^{n.a.} = G(q)\left[\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y G^T(p-q)\sigma_y D(p)\right]G(q). \tag{7.11}$$

At second order in the interaction, the only diagram we can draw is given by



. (7.12)

The symmetry factor can be calculated to be one, since we have two possibilities of ordering the vertices, two possibilities to fix the first external leg at the first vertex and two possibilities to fix the second external leg at the second vertex, hence $p(2) = 2^3$ and $S(2) = \frac{2^3}{2!\times 2^2} = 1$. Translating (7.12) into a mathematical expression yields

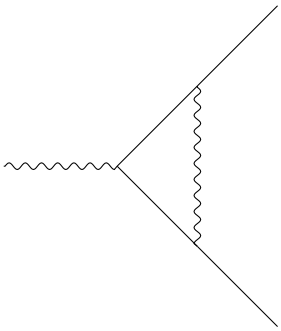
$$\begin{aligned} &G(q)\left[\int_p\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)D(p)\left(-G^T\right)(q-p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)\right]G(q) \\ &= G(q)\left[\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y G^T(q-p)\sigma_y D(p)\right]G(q), \end{aligned} \tag{7.13}$$

thus again coinciding with the result obtained by calculating the contractions explicitly.

7.2 Application of Feynman Rules to Vertex Corrections

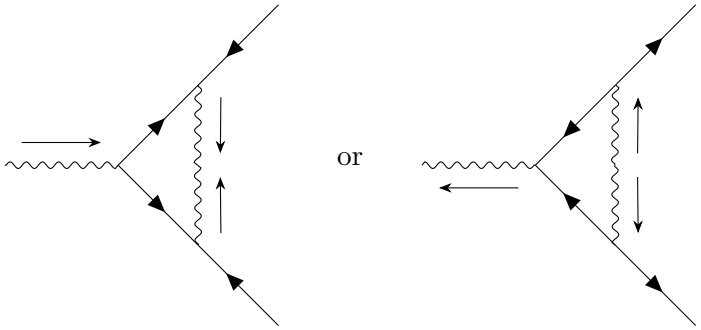
Now we are in the position to argue that vertex corrections don't appear at one-loop order in our theory using the Feynman rules derived in the previous section. The typical one-loop

vertex correction is of the form



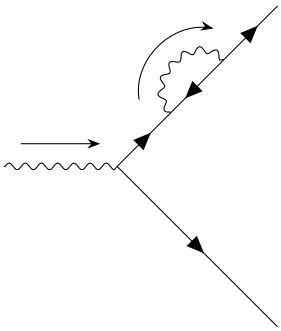
, (7.14)

but with the available vertices we can only draw diagrams like



or (7.15)

which contain objects $\langle \Phi\Phi \rangle$ and $\langle \overline{\Phi}\overline{\Phi} \rangle$ and would renormalize vertices $\propto \Phi\Psi\Psi$ and $\propto \overline{\Phi}\overline{\Psi}\overline{\Psi}$ not present in our theory. The only vertex diagrams existing at one-loop order are the ones that dress one of the external fermion legs like



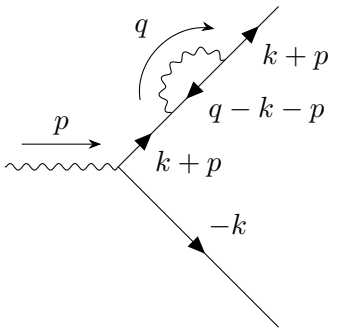
. (7.16)

These types of diagrams don't renormalize the interaction, the divergencies occuring due to the dressed propagators will be cancelled by the counterterms cancelling the divergencies of the two-point functions.

As a last check that our Feynman rules indeed reproduce the correct formulas, we can compare the mathematical and diagrammatical expressions of the diagram (7.16). The calculation of the contractions is done in Appendix D.2, the result reads

$$
\begin{aligned}
&-\frac{1}{3!}\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{int}^3 \right\rangle_0 \\
&= D(p)G_{ae}^T(k+p)\left[-\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\int_q \sigma_y G(q-k-p)D(q)\sigma_y G^T(k+p)\sigma_y\right]^{eh} G_{gb}(-k). \qquad (7.17)
\end{aligned}
$$

The term obtained by using the Feynman rules follows from the labelled diagram



. (7.18)

The possibilities of contractions of this diagram is given by $p(3) = 3 \times 2^4$, since we have three ways of interchanging the vertices, two ways of choosing the first external fermion line, two ways of choosing the second external fermion line and two ways of choosing the external boson line. The symmetry factor then evaluates to $S(3) = \frac{3\times 2^4}{3!\times 2^3} = 1$ and the diagram (7.18) translates to

$$\begin{aligned} &D(p)\left(-G^T\right)(k+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\right)^3 \sigma_y \int_q G(q-k-p)D(q)\sigma_y\left(-G^T\right)(k+p)\sigma_y G(-k) \\ &= D(p)G^T(k+p)\left[-\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\int_q \sigma_y G(q-k-p)D(q)\sigma_y G^T(k+p)\sigma_y\right] G(-k). \end{aligned} \tag{7.19}$$

Therefore, the right expression is reproduced, indicating that our Feynman rules indeed are correct.

7.3 Cancellation of Divergencies

In the previous chapters, we derived the $1/\epsilon$-poles of the one-loop corrections to the boson and fermion propagators in dimensional regularization. In Section 3.4, we've already seen how the divergencies can be cancelled by introducing counterterms and treating them as interactions. Since the counterterms to the kinetic parts of the action are quadratic in the fields, one could include them as modifications to the propagators as well. In this section we compare the two ways of cancelling the divergencies by treating the counterterms either as interactions or as modifications to the propagators. We show that both ways are essentially the same since the original divergencies have to be added to the action, but the mechanism generating the correct sign for cancellation is different. The following calculations are done for corrections to the fermion propagator at one-loop level, the same results apply to the boson as well.

Recall that the full fermion propagator is given by the formula

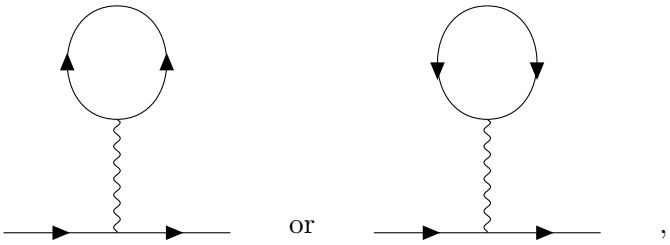
$$\begin{aligned} \left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle &= \left\langle \Psi_a(k)\overline{\Psi}_b(k)e^{-S_{\text{int}}}\right\rangle_0^{con} \\ &= \left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle_0 + \frac{1}{2!}\left\langle \Psi_a(k)\overline{\Psi}_b(k)S_{\text{int}}^2\right\rangle_0^{con} + \mathcal{O}\left(\lambda^4\right) \end{aligned} \tag{7.20}$$

where the superscript denotes that only connected diagrams contribute. Diagrammatically

this can be represented by

$$= G(k) + G(k)\Sigma(k)G(k) + \mathcal{O}\left(\lambda^4\right), \tag{7.21}$$

where the thick fermion line stands for the full fermion propagator. Here we quickly note that no tadpole diagrams are present in our theory, since we cannot even draw reasonable tadpole-like diagrams



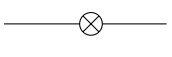
, (7.22)

which would require quadratic terms $\propto \Psi\Psi$ and $\propto \overline{\Psi}\overline{\Psi}$ and interactions of the form $\overline{\Phi}\overline{\Psi}\Psi$ in the action. The absence of tadpole diagrams holds true in two dimensions as well.

The expression in (7.21) is divergent since the self-energy is of the form

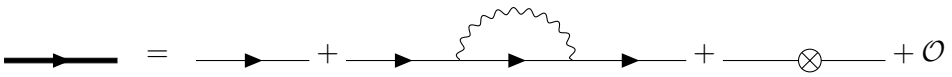
$$\Sigma(k) = \frac{\lambda^2 Z(k)}{\epsilon} + \Sigma_{\text{finite}}(k), \tag{7.23}$$

where $Z(k)$ is a matrix and $\Sigma_{\text{finite}}(k) \propto \lambda^2$ is the finite part of the self-energy. To make the theory finite and therefore a physical sensible theory, we need to include a term in the action that cancels the diverging term exactly, i.e. a term that produces a diagram



$$= G(k)\left(-\frac{\lambda^2 Z(k)}{\epsilon}\right)G(k), \tag{7.24}$$

s.t. we get the finite result



$$\begin{aligned} &= G(k) + G(k)\left(\frac{\lambda^2 Z(k)}{\epsilon} + \Sigma_{\text{finite}}(k)\right)G(k) - G(k)\frac{\lambda^2 Z(k)}{\epsilon}G(k) + \mathcal{O}\left(\lambda^4\right) \\ &= G(k) + G(k)\Sigma_{\text{finite}}(k)G(k) + \mathcal{O}\left(\lambda^4\right). \end{aligned} \tag{7.25}$$

This can be achieved by including

$$S_{\text{ct}} = \int_k \overline{\Psi}(k)\frac{\lambda^2 Z(k)}{\epsilon}\Psi(k), \tag{7.26}$$

hence adding the diverging term in the action.

We start by treating S_{ct} as an interaction, noting that our new action, which we call

renormalized action, reads

$$S_{\text{ren}} = S + S_{\text{ct}}. \tag{7.27}$$

Then the full propagator to one-loop level is given by

$$\begin{aligned}\left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle &= \left\langle \Psi_a(k)\overline{\Psi}_b(k)e^{-S_{\text{ct}}}e^{-S_{\text{int}}}\right\rangle_0 \\ &= \left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle_0 - \left\langle \Psi_a(k)\overline{\Psi}_b(k)S_{\text{ct}}\right\rangle_0^{con} + \frac{1}{2!}\left\langle \Psi_a(k)\overline{\Psi}_b(k)S_{\text{int}}^2\right\rangle_0^{con} + \mathcal{O}\left(\lambda^4\right),\end{aligned} \tag{7.28}$$

because $S_{\text{ct}}^2 \propto \lambda^4$. We can calculate

$$\begin{aligned}-\left\langle \Psi_a(k)\overline{\Psi}_b(k)S_{\text{ct}}\right\rangle_0^{con} &= -\int_p \left\langle \overbrace{\Psi_a(k)\overline{\Psi}_b(k)}\overline{\Psi}_c(p)\frac{\lambda^2 Z_{cd}(p)}{\epsilon}\Psi_d(p)\right\rangle_0^{con} \\ &= G_{ac}(k)\left(-\frac{\lambda^2 Z(k)}{\epsilon}\right)_{cd} G_{db}(k),\end{aligned} \tag{7.29}$$

which indeed has the correct form to cancel the divergence coming from the term proportional to S_{int}^2. We emphasize that the needed minus sign is generated by the expansion of the exponential function to first order.

In the second method, the minus sign is generated in a different way. Including the counterterm into the fermion propagator, we get the action

$$\begin{aligned}S_{\text{ren}} = S_0 + S_{\text{int}} + S_{\text{ct}} &= \int_k \overline{\Psi}(k)\underbrace{\left(G^{-1}(k) + \frac{\lambda^2 Z(k)}{\epsilon}\right)}_{:=\Gamma^{-1}(k)}\Psi(k) + S_\Phi + S_{\text{int}} \\ &= S_0' + S_{\text{int}}.\end{aligned} \tag{7.30}$$

The expectation values are then calculated w.r.t. the new quadratic action S_0' according to

$$\left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle = \left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle_{0'} + \frac{1}{2!}\left\langle \Psi_a(k)\overline{\Psi}_b(k)S_{\text{int}}^2\right\rangle_{0'}^{con}, \tag{7.31}$$

where the fermion propagator in the new free theory is

$$\left\langle \Psi_a(k)\overline{\Psi}_b(k)\right\rangle_{0'} = \Gamma_{ab}(k). \tag{7.32}$$

To find the right expression for $\Gamma(k)$, we note that multiplied with $\Gamma^{-1}(k)$ it has to give the unit matrix up to order $\mathcal{O}\left(\lambda^4\right)$. The expression

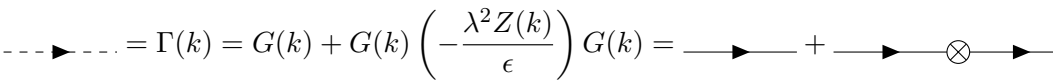
$$\Gamma(k) = G(k)\left(\mathbb{1} - \frac{\lambda^2 Z(k)}{\epsilon}G(k)\right) \tag{7.33}$$

yields the correct result, since

$$\Gamma^{-1}(k)\Gamma(k) = \left(G^{-1}(k) + \frac{\lambda^2 Z(k)}{\epsilon}\right)G(k)\left(\mathbb{1} - \frac{\lambda^2 Z(k)}{\epsilon}G(k)\right)$$

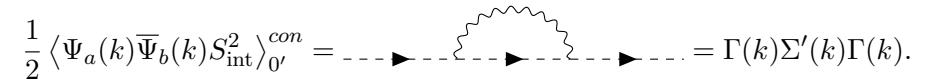
$$= \mathbb{1} + \frac{\lambda^2 Z(k)}{\epsilon} G(k) - \frac{\lambda^2 Z(k)}{\epsilon} G(k) + \mathcal{O}\left(\lambda^4\right) = \mathbb{1} + \mathcal{O}\left(\lambda^4\right). \tag{7.34}$$

$\Gamma(k)$, represented by a dashed line, produces the original bare fermion propagator plus the correct counterterm



(7.35)

The one-loop correction takes the same form as before, but the original fermion propagator $G(k)$ gets replaced by $\Gamma(k)$, such that

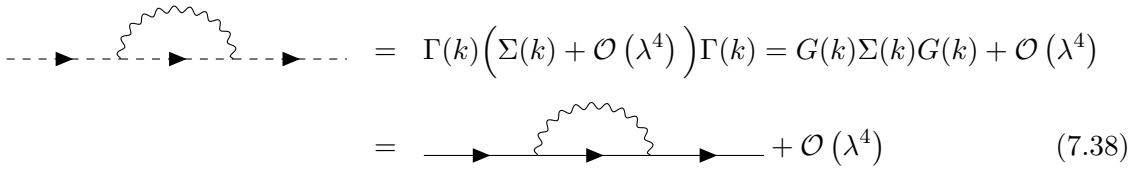


(7.36)

The self-energy is given by

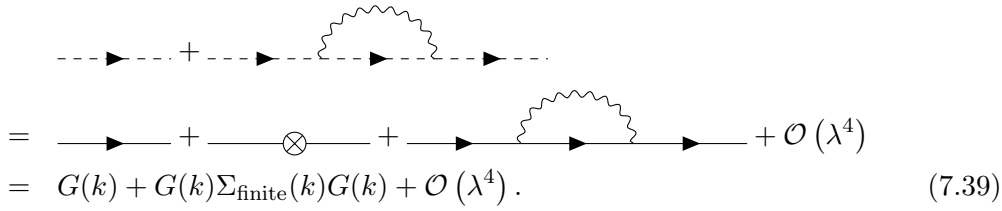
$$\begin{aligned} \Sigma'(k) &= \frac{\lambda^2 \mu^\epsilon}{N} \int_p \sigma_y \Gamma^T(p-q)\sigma_y D(p) = \frac{\lambda^2 \mu^\epsilon}{N} \int_p \sigma_y G^T(p-k)\sigma_y D(p) + \mathcal{O}\left(\lambda^4\right) \\ &= \Sigma(k) + \mathcal{O}\left(\lambda^4\right), \end{aligned} \tag{7.37}$$

thus the whole diagram reads



(7.38)

and hence



$$= G(k) + G(k)\Sigma_{\text{finite}}(k)G(k) + \mathcal{O}\left(\lambda^4\right). \tag{7.39}$$

We note that in the first method - treating the counterterm as an interaction - the required minus sign of the counterterm gets generated by expanding the exponential function $e^{-S_{\text{ct}}}$ to first order, but in the second method the correct sign comes from inverting the new propagator to appropriate order in λ.

Either way, we conclude that we need to add the diverging terms of the self-energies to the action to cancel the divergencies and get finite results.

8 RG Analysis

Now we are in the position to begin with the one-loop renormalization of our theory. We already mentioned that the value of the parameter a in the boson propagator $D^{-1}(k) = k_d^2 + ak_{d-1}$ will be determined by the RG flow. This is achieved by treating both λ and a as interactions renormalized by the $1/\epsilon$-poles in the one-loop corrections to the boson and fermion propagator. Consequently, we won't have just one β-function determining the fixed points and therefore the quantum critical point as is the case in ϕ^4-theory, but two.

The outline of this chapter is as follows. First we determine the two β-functions for general one-loop renormalization constants. This will be done in detail to get familiar with the procedure of obtaining the explicit expressions of β-functions. The derivation of the other renormalization group functions is presented in Appendix E.1. Afterwards, we identify the fixed points of our theory using the explicit form of the counterterms. The renormalization group equation and its solution at the fixed points for the two-point boson and fermion correlation functions will be derived in the third section.

8.1 Renormalization and β-Functions

The $1/\epsilon$-poles in (5.25) and (6.14) renormalize the boson propagator and the dispersion in the fermion propagator, s.t. we add

$$S_{\text{ct}} = \int_k \overline{\Psi}(k) \left[i\sigma_x k_{d-1} \frac{Z_{2,1}}{\epsilon} + i\sigma_x k_d^2 \frac{Z_{3,1}}{\epsilon} \right] \Psi(k) + \int_k \overline{\Phi}(k) \left(k_d^2 \frac{Z_{4,1}}{\epsilon} + ak_{d-1} \frac{Z_{5,1}}{\epsilon} \right) \Phi(k) \tag{8.1}$$

to the original action, obtaining the renormalized action

$$\begin{aligned} S_{\text{ren}} = &\int_k \overline{\Psi}(k) \left[-i\boldsymbol{\Gamma}\mathbf{K} + i\sigma_x k_{d-1} Z_2 + i\sigma_x k_d^2 Z_3 \right] \Psi(k) + \int_k \overline{\Phi}(k) \left[k_d^2 Z_4 + ak_{d-1} Z_5 \right] \Phi(k) \\ &- \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}} \int_{k,p} \left[\Phi(p) \overline{\Psi}(k+p) \sigma_y \overline{\Psi}^T(-k) + \overline{\Phi}(p) \Psi^T(k+p) \sigma_y \Psi(-k) \right], \end{aligned} \tag{8.2}$$

where $Z_n = Z_n(\lambda, a) = 1 + \frac{Z_{n,1}}{\epsilon}$ are functions of λ and a only. The renormalized action can be brought into its initial, bare form by multiplicative renormalization

$$\begin{aligned} &\mathbf{K} = \mathbf{K}_B, \quad k_{d-1} = Z_2^{-1} k_{B,d-1}, \quad k_d = Z_3^{-\frac{1}{2}} k_{B,d}, \quad \Psi(k) = Z_2^{\frac{1}{2}} Z_3^{\frac{1}{4}} \Psi_B(k_B), \\ &\Phi(k) = Z_2^{\frac{1}{2}} Z_3^{\frac{3}{4}} Z_4^{-\frac{1}{2}} \Phi_B(k_B), \quad \lambda_B = \lambda\mu^{\frac{\epsilon}{2}} Z_2^{-\frac{1}{2}} Z_3^{\frac{1}{4}} Z_4^{-\frac{1}{2}}, \quad a = Z_2 Z_3^{-1} Z_4 Z_5^{-1} a_B. \end{aligned} \tag{8.3}$$

In the renormalization procedure of this theory, there is a certain freedom of keeping one momentum fixed, while the others renormalize. We chose $\mathbf{K} = \mathbf{K}_B$, but in principle it would be possible to keep $k_{d-1} = k_{B,d-1}$ or $k_d = k_{B,d}$ without effecting the final, physical expressions.

The β-functions of λ and a are defined similar to the one in ϕ^4-theory, namely

$$\begin{aligned}\beta_\lambda(\lambda,a) &= \frac{d\lambda}{d\ln\mu} = \mu\frac{d\lambda}{d\mu},\\ \beta_a(\lambda,a) &= \frac{da}{d\ln\mu} = \mu\frac{da}{d\mu}.\end{aligned} \tag{8.4}$$

We begin with rewriting β_λ

$$\begin{aligned}\beta_\lambda &= \mu\frac{d}{d\mu}\left(\lambda_B\mu^{-\frac{\epsilon}{2}}Z_2^{\frac{1}{2}}Z_3^{-\frac{1}{4}}Z_4^{\frac{1}{2}}\right)\\ &= \mu\lambda_B\mu^{-\frac{\epsilon}{2}}Z_2^{\frac{1}{2}}Z_3^{-\frac{1}{4}}Z_4^{\frac{1}{2}}\left(-\frac{\epsilon}{2}\mu^{-1}+\frac{1}{2}Z_2^{-1}\frac{dZ_2}{d\mu}-\frac{1}{4}Z_3^{-1}\frac{dZ_3}{d\mu}+\frac{1}{2}Z_4^{-1}\frac{dZ_4}{d\mu}\right)\\ &= \lambda\left[-\frac{\epsilon}{2}+\frac{1}{2}\frac{1}{Z_2}\left(\beta_\lambda Z_2'+\beta_a\dot{Z}_2\right)-\frac{1}{4}\frac{1}{Z_2}\left(\beta_\lambda Z_3'+\beta_a\dot{Z}_3\right)+\frac{1}{2}\frac{1}{Z_4}\left(\beta_\lambda Z_4'+\beta_a\dot{Z}_4\right)\right]\\ &= -\frac{\epsilon}{2}\lambda+\beta_\lambda\left(\frac{\lambda}{2}\frac{Z_2'}{Z_2}-\frac{\lambda}{4}\frac{Z_3'}{Z_3}+\frac{\lambda}{2}\frac{Z_4'}{Z_4}\right)+\frac{\lambda}{2}\beta_a\left(\frac{\dot{Z}_2}{Z_2}-\frac{1}{2}\frac{\dot{Z}_3}{Z_3}+\frac{\dot{Z}_4}{Z_4}\right),\end{aligned} \tag{8.5}$$

where primes denote derivatives w.r.t. λ and dots w.r.t. a. Multiplying both sides by $Z_2Z_3Z_4$ yields

$$\begin{aligned}&\beta_\lambda\left(Z_2Z_3Z_4-\frac{\lambda}{2}Z_2'Z_3Z_4+\frac{\lambda}{4}Z_2Z_3'Z_4-\frac{\lambda}{2}Z_2Z_3Z_4'\right)\\ &\quad=-\frac{\lambda}{2}\epsilon Z_2Z_3Z_4+\frac{\lambda}{2}\beta_a\left(\dot{Z}_2Z_3Z_4-\frac{1}{2}Z_2\dot{Z}_3Z_4+Z_2Z_3\dot{Z}_4\right).\end{aligned} \tag{8.6}$$

The same procedure can be done for β_a, i.e. calculating

$$\begin{aligned}\beta_a &= \mu\frac{d}{d\mu}\left(a_BZ_2Z_3^{-1}Z_4Z_5^{-1}\right) = a\left[\frac{\mu}{Z_2}\frac{dZ_2}{d\mu}-\frac{\mu}{Z_3}\frac{dZ_3}{d\mu}+\frac{\mu}{Z_4}\frac{dZ_4}{d\mu}-\frac{\mu}{Z_5}\frac{dZ_5}{d\mu}\right]\\ &= a\beta_\lambda\left(\frac{Z_2'}{Z_2}-\frac{Z_3'}{Z_3}+\frac{Z_4'}{Z_4}-\frac{Z_5'}{Z_5}\right)+a\beta_a\left(\frac{\dot{Z}_2}{Z_2}-\frac{\dot{Z}_3}{Z_3}+\frac{\dot{Z}_4}{Z_4}-\frac{\dot{Z}_5}{Z_5}\right),\end{aligned} \tag{8.7}$$

and then multiplying with $Z_2Z_3Z_4Z_5$

$$\begin{aligned}&\beta_a\left(Z_2Z_3Z_4Z_5-a\dot{Z}_2Z_3Z_4Z_5+aZ_2\dot{Z}_3Z_4Z_5-aZ_2Z_3\dot{Z}_4Z_5+aZ_2Z_3Z_4\dot{Z}_5\right)\\ &=a\beta_\lambda\left(Z_2'Z_3Z_4Z_5-Z_2Z_3'Z_4Z_5+Z_2Z_3Z_4'Z_5-Z_2Z_3Z_4Z_5'\right).\end{aligned} \tag{8.8}$$

Taking only the regular parts of (8.6) and (8.8) in the limit $\epsilon\to 0$ into account, we can determine the β-functions to one-loop level.

First note that in general the renormalization constants and their derivatives are given by expansions in negative powers of ϵ

$$Z_n(\lambda,a) = 1+\sum_{k=1}^{\infty}\frac{Z_{n,k}(\lambda,a)}{\epsilon^k},\qquad Z_n'(\lambda,a)=\sum_{k=1}^{\infty}\frac{Z_{n,k}'(\lambda,a)}{\epsilon^k}. \tag{8.9}$$

The expansion of the derivative of course holds for derivates w.r.t. to λ as well as a. The

products of renormalization constants and their first order derivatives evaluate to

$$Z_l Z_m Z_n = 1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right), \quad Z_l' Z_m Z_n = \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right), \tag{8.10}$$

where the terms in brackets represent a series in negative powers of ϵ with coefficients which don't have to be specified exactly for now. These product relations also hold for four or more renormalization constants as long as just one derivative is included. Equation (8.6) can then be written in the general form

$$\beta_\lambda \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] = -\frac{\lambda}{2}\epsilon \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] + \frac{\lambda}{2}\beta_a \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \tag{8.11}$$

and (8.8) yields

$$\beta_a \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] = a\beta_\lambda \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right). \tag{8.12}$$

These are just two equations with two wanted parameters β_λ and β_a, i.e. we naively solve (8.12) for β_a, insert it into (8.11) and solve for β_λ. The resulting equation

$$\begin{aligned} &\beta_\lambda \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] \left[1 + \left(\propto \sum_{m=1}^{\infty} \epsilon^{-m} \right) \right] \\ &= -\frac{\lambda}{2}\epsilon \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] + \frac{\lambda a}{2}\beta_\lambda \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \left(\propto \sum_{m=1}^{\infty} \epsilon^{-m} \right) \end{aligned} \tag{8.13}$$

then simplifies to

$$\beta_\lambda \left[1 + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right) \right] = -\frac{\lambda}{2}\epsilon + \text{constant} + \left(\propto \sum_{k=1}^{\infty} \epsilon^{-k} \right), \tag{8.14}$$

where the constant term is independent of ϵ. Since renormalizability of the theory requires β_λ to stay finite in the limit $\epsilon \to 0$, we can extract the general form of β_λ via the regular parts of the above equation, i.e.

$$\beta_\lambda = \beta_\lambda^{(0)} + \beta_\lambda^{(1)}\epsilon \tag{8.15}$$

with $\beta_\lambda^{(0)} = \text{constant}$ and $\beta_\lambda^{(1)} = -\frac{\lambda}{2}$. Inserting this back into (8.12), we obtain

$$\beta_a = \beta_a^{(0)} = \text{constant}. \tag{8.16}$$

The explicit expression of $\beta_\lambda^{(0)}$ can be derived via (8.6). Using that we have determined the renormalization constants only to one-loop order, i.e. $Z_n = 1 + \frac{Z_{n,1}}{\epsilon}$, as well as $\beta_\lambda =$

$\beta_\lambda^{(0)} + \beta_\lambda^{(1)}\epsilon$ and $\beta_a =$ constant and keeping only the regular terms, we obtain

$$\beta_\lambda^{(0)} = \frac{\lambda^2}{4}\left(\frac{1}{2}Z'_{3,1} - Z'_{2,1} - Z'_{4,1}\right) \tag{8.17}$$

and thus

$$\beta_\lambda = \frac{\lambda^2}{4}\left(\frac{1}{2}Z'_{3,1} - Z'_{2,1} - Z'_{4,1}\right) - \frac{\lambda}{2}\epsilon. \tag{8.18}$$

The same can be done for β_a via (8.8) with the help of the explicit expression of β_λ, yielding

$$\beta_a = -\frac{\lambda a}{2}\left(Z'_{2,1} + Z'_{4,1} - Z'_{3,1} - Z'_{5,1}\right). \tag{8.19}$$

8.2 Fixed Points

The fixed points of the theory are given by the values of λ and a for which both β-functions are zero. Hence we need the explicit form of the renormalization constants obtainable from the poles in (5.25) and (6.14). As was derived in Section 7.3, the divergencies have to be added to the original action, such that

$$\begin{aligned} Z_{2,1} &= -\frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}, \quad Z_{3,1} = \frac{2u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}, \\ Z_{4,1} &= -\frac{u_1\lambda^2}{2}, \quad Z_{5,1} = -\frac{u_1\lambda^2}{a}, \end{aligned} \tag{8.20}$$

where $\tilde{a} = \frac{a}{1-a}$ and $u_1 = \frac{\Gamma\left(\frac{5}{4}\right)}{8\sqrt{2}\pi^{\frac{7}{4}}} \approx 0.010807$. Note that the $1/\epsilon$-pole in the boson self-energy $\propto k_{d-1}$ specifies $aZ_{5,1}$, not $Z_{5,1}$ itself. The β-functions are therefore given by

$$\begin{aligned} \beta_\lambda &= \frac{\lambda^2}{4}\left(\frac{2u_1\lambda}{N(1-a)^2\sqrt{|\tilde{a}|}} + \frac{4u_1\lambda}{N(1-a)\sqrt{|\tilde{a}|}} + u_1\lambda\right) - \frac{\lambda}{2}\epsilon \\ &= \frac{u_1\lambda^3}{4}\left(\frac{2(3-2a)}{N(1-a)^2\sqrt{|\tilde{a}|}} + 1\right) - \frac{\lambda}{2}\epsilon \end{aligned} \tag{8.21}$$

and

$$\begin{aligned} \beta_a &= -\frac{\lambda a}{2}\left(-\frac{4u_1\lambda}{N(1-a)\sqrt{|\tilde{a}|}} - u_1\lambda - \frac{4u_1\lambda}{N(1-a)^2\sqrt{|\tilde{a}|}} + \frac{2u_1\lambda}{a}\right) \\ &= \frac{u_1\lambda^2}{2}\left(\frac{4a(2-a)}{N(1-a)^2\sqrt{|\tilde{a}|}} + a - 2\right). \end{aligned} \tag{8.22}$$

Rewriting the arbitrary mass scale by introducing a logarithmic scale l via $\mu = \mu_0 e^{-l}$, we get coupled non-linear differential equations for λ and a

$$\frac{d\lambda}{dl} = -\frac{u_1\lambda^3}{4}\left(\frac{2(3-2a)}{N(1-a)^2\sqrt{|\tilde{a}|}} + 1\right) + \frac{\lambda}{2}\epsilon = -\beta_\lambda,$$

$$\frac{da}{dl} = -\frac{u_1\lambda^2}{2}\left(\frac{4a(2-a)}{N(1-a)^2\sqrt{|\tilde{a}|}} + a - 2\right) = -\beta_a \tag{8.23}$$

as a function of l. The flow diagram of these differential equations is shown in Fig. 5. The

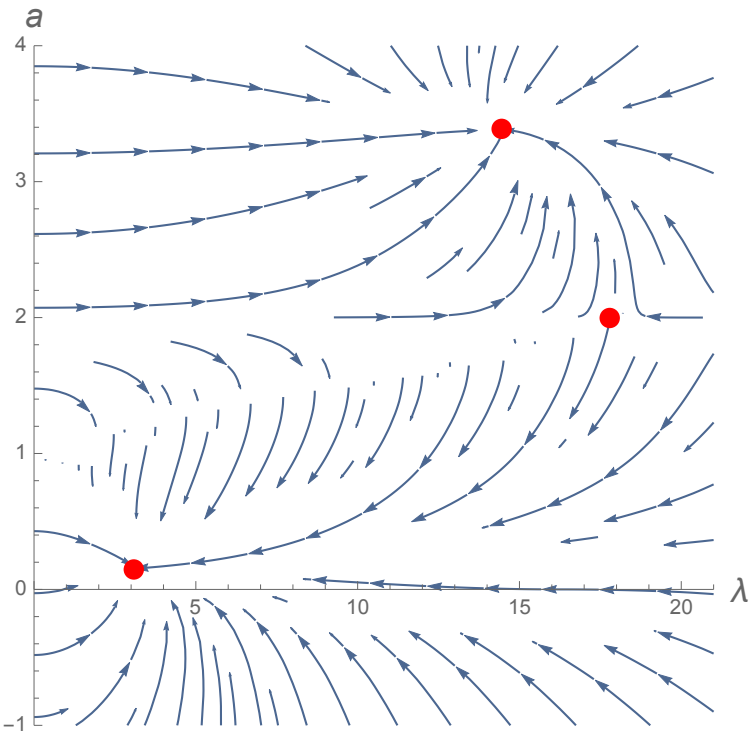


Figure 5: Flow diagram to the differential equations in (8.23) in two dimensions, i.e. $\epsilon = \frac{1}{2}$, and $N = 2$. The red dots denote the fixed points.

fixed points follow from solving both $\beta_\lambda = 0$ and $\beta_a = 0$. For the physical case $N = 2$ and restricting ourselves to positive λ, we hence obtain

$$\begin{aligned}(\lambda_1^*, a_1^*) &= (4.335\sqrt{\epsilon}, 0.153),\\ (\lambda_2^*, a_2^*) &= (20.43\sqrt{\epsilon}, 3.383),\\ (\lambda_3^*, a_3^*) &= (25.137\sqrt{\epsilon}, 2.0).\end{aligned} \tag{8.24}$$

The stability of the fixed points is determined by the sign of the eigenvalues of the Jacobian

$$J = \begin{pmatrix} -\beta'_\lambda & -\dot{\beta}_\lambda \\ -\beta'_a & -\dot{\beta}_a \end{pmatrix} \tag{8.25}$$

evaluated at the fixed point values. We find that the first and second fixed point are stable, whereas the fixed point at $a = 2$ has an unstable component.

As already mentioned, the fixed points of an RG flow indicate (quantum) phase transitions to new states of matter. The question is whether all fixed points correspond to physically sensible quantum phase transitions. Our original assumption about the CDW susceptibility was a peak at the wave vectors $\mathbf{Q} = \pm 2k_F$, which corresponds to $D^{-1}(0) = 0$.

Although this condition is fulfilled for any value of a, a finite a introduces an additional peak at $k_{d-1} = -k_d^2/a$, contradicting our initial assumptions. Thus, a sensible initial value for the RG flow is a infinitesimal parameter a and hence we identify the first fixed point as the one describing a quantum phase transition to incommensurate $2k_F$ CDW order.

The case for $a = 2$ is a little bit different, because for this specific value the CDW susceptibility is peaked along the entire $2k_F$ line and we get infinitely many pairs of hot-spots independent of each other. This can be seen as follows: Consider an electron on the left hot-spot with momentum (k_x, k_y), which gets scattered via a CDW fluctuation with momentum (p_x, p_y) to the right hot-spot with resulting momentum (k_x+p_x, k_y+p_y). For the original fermion to be on the Fermi surface, we have $k_x = k_y^2$ and the fluctuation propagator is peaked for $p_x = -p_y^2/2$. Plugging these two conditions into the Fermi surface equation of the electron on the right hot-spot $k_x + p_x + (k_y + p_y)^2 = 0$, we find $p_y = -2k_y$ and hence $p_x = -2k_x$. Therefore, not just electrons at the original hot-spots, but any electron with arbitrary momentum (k_x, k_y) can be scattered to one specific electron on the opposite site of the Fermi surface with $(-k_x, -k_y)$. These scattering processes, however, are still local in momentum space, thus creating infinitely many decoupled hot-spot pairs consistent with our original assumptions of the hot-spot theory itself. Note that these considerations for the third fixed point are only possible because the value of a doesn't flow for the initial condition $a = 2$ and hence the shape of the Fermi surface is fixed to its bare form (see later).

8.3 RG Equation and Scaling Forms

At this point, we have two finite theories related to each other by multiplicative renormalization, the bare action which is independent of the artificially introduced mass scale μ and the renormalized action depending on μ. Thus, renormalized correlation functions change when varying μ, whereas bare correlation functions are not effected. This can be used to derive a differential equation for the renormalized correlation functions, yielding the scaling behavior of two-point functions at the quantum critical point, as we will see in this section.

The connection between renormalized and bare correlation functions can be extracted from

$$\begin{aligned}
&G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)\delta^{(d+1)}\left(\{k_i\}\right)\\
&=\left\langle\Psi(k_1)...\Psi(k_m)\overline{\Psi}(k_{m+1})...\overline{\Psi}(k_{2m})\Phi(k_{2m+1})...\Phi(k_{2m+n})\overline{\Phi}(k_{2m+n+1})...\overline{\Phi}(k_{2m+2n})\right\rangle\\
&=Z_\Psi^{-m}Z_\Phi^{-n}\left\langle\Psi_B(k_{B,1})...\overline{\Phi}_B(k_{B,2m+2n})\right\rangle\\
&=Z_\Psi^{-m}Z_\Phi^{-n}Z_2^{-1}Z_3^{-\frac{1}{2}}G^{(m,m,n,n)}\left(\{k_{B,i}\},\lambda_B,a_B\right)\delta^{(d+1)}\left(\{k_i\}\right),
\end{aligned}\tag{8.26}$$

where the δ-functions ensure energy and momentum conservation, factors of $(2\pi)^{d+1}$ have been suppressed and we used $\delta^{(d+1)}\left(\{k_{B,i}\}\right) = Z_2^{-1}Z_3^{-1/2}\delta^{(d+1)}\left(\{k_i\}\right)$, resulting in the key relation

$$G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right) = Z_\Psi^{-m}Z_\Phi^{-n}Z_2^{-1}Z_3^{-\frac{1}{2}}G^{(m,m,n,n)}\left(\{k_{B,i}\},\lambda_B,a_B\right).\tag{8.27}$$

Applying the operator $\mu\frac{d}{d\mu}$ on both sides and using the chain rule on the left hand side leads to

$$\left[\sum_{i=1}^{2m+2n}\left(\mu\frac{dk_{d-1,i}}{d\mu}\frac{\partial}{\partial k_{d-1,i}}+\mu\frac{dk_{d,i}}{d\mu}\frac{\partial}{\partial k_{d,i}}\right)+\mu\frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda}+\mu\frac{da}{d\mu}\frac{\partial}{\partial a}+\mu\frac{\partial}{\partial\mu}\right] \tag{8.28}$$

acting on $G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)$, whereas the right hand side depends on μ only via the renormalization constants

$$\begin{aligned}
&\mu\frac{d}{d\mu}\left(Z_\Psi^{-m}Z_\Phi^{-n}Z_2^{-1}Z_3^{-\frac{1}{2}}\right)\\
&\quad=Z_\Psi^{-m}Z_\Phi^{-n}Z_2^{-1}Z_3^{-\frac{1}{2}}\left(-m\frac{d\ln Z_\Psi}{d\ln\mu}-n\frac{d\ln Z_\Phi}{d\ln\mu}-\frac{d\ln Z_2}{d\ln\mu}-\frac{1}{2}\frac{d\ln Z_3}{d\ln\mu}\right)\\
&\quad=Z_\Psi^{-m}Z_\Phi^{-n}Z_2^{-1}Z_3^{-\frac{1}{2}}\left(-2m\eta_\Psi-2n\eta_\Phi+\left(1-z_{d-1}^{-1}\right)+\frac{1}{2}\left(1-z_d^{-1}\right)\right).
\end{aligned} \tag{8.29}$$

Here we have defined the anomalous dimensions of the fields

$$\eta_{\Psi/\Phi}=\frac{1}{2}\frac{d\ln Z_{\Psi/\Phi}}{d\ln\mu}. \tag{8.30}$$

and the dynamical critical exponents

$$z_{d-1}^{-1}(\lambda,a)=1+\frac{d\ln Z_2}{d\ln\mu},\qquad z_d^{-1}(\lambda,a)=1+\frac{d\ln Z_3}{d\ln\mu}, \tag{8.31}$$

which also appear in (8.28) since

$$\begin{aligned}
\mu\frac{dk_{d-1,i}}{d\mu}&=\mu k_{B,d-1,i}\frac{dZ_2^{-1}}{d\mu}=-k_{d-1,i}\frac{d\ln Z_2}{d\ln\mu}=k_{d-1,i}\left(1-z_{d-1}^{-1}\right),\\
\mu\frac{dk_{d,i}}{d\mu}&=\mu k_{B,d,i}\frac{dZ_3^{-\frac{1}{2}}}{d\mu}=-\frac{1}{2}k_{d,i}\frac{d\ln Z_3}{d\ln\mu}=\frac{1}{2}k_{d,i}\left(1-z_d^{-1}\right).
\end{aligned} \tag{8.32}$$

Thus, the differential equation for the correlation functions obtained by the operator $\mu\frac{d}{d\mu}$ acting on (8.27) reads

$$\begin{aligned}
&\left[\sum_{i=1}^{2m+2n}\left(\left(z_{d-1}^{-1}-1\right)k_{d-1,i}\frac{\partial}{\partial k_{d-1,i}}+\left(z_d^{-1}-1\right)\frac{k_{d,i}}{2}\frac{\partial}{\partial k_{d,i}}\right)-\beta_\lambda\frac{\partial}{\partial\lambda}-\beta_a\frac{\partial}{\partial a}-\mu\frac{\partial}{\partial\mu}\right.\\
&\left.-2m\eta_\Psi-2n\eta_\Phi+\left(1-z_{d-1}^{-1}\right)+\frac{1}{2}\left(1-z_d^{-1}\right)\right]G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)=0.
\end{aligned} \tag{8.33}$$

In the same fashion, we can derive another differential equation for the renormalized correlation functions by using the scale transformations (4.13). Under these transformations, the correlation functions rescale as

$$G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)=b^{\frac{4-\epsilon}{2}(2m+2n)+\epsilon-3}G^{(m,m,n,n)}\left(\{k_i'\},\mu',\lambda,a\right) \tag{8.34}$$

and applying the differential operator $\frac{d}{db}|_{b=1}$ on both sides yields

$$\left[\sum_{i=1}^{2m+2n}\left(\mathbf{K}_i\nabla_{\mathbf{K}_i}+k_{d-1,i}\frac{\partial}{\partial k_{d-1,i}}+\frac{1}{2}k_{d,i}\frac{\partial}{\partial k_{d,i}}\right)+\mu\frac{\partial}{\partial\mu}+\frac{4-\epsilon}{2}(2m+2n)+\epsilon-3\right] G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)=0. \tag{8.35}$$

Equations (8.33) and (8.35) can be combined to one single differential equation, the final renormalization group equation independent of derivatives w.r.t. μ, namely

$$\left[\sum_{i=1}^{2m+2n}\left(\mathbf{K}_i\nabla_{\mathbf{K}_i}+\frac{k_{d-1,i}}{z_{d-1}}\frac{\partial}{\partial k_{d-1,i}}+\frac{k_{d,i}}{2z_d}\frac{\partial}{\partial k_{d,i}}\right)-\beta_\lambda\frac{\partial}{\partial\lambda}-\beta_a\frac{\partial}{\partial a}-2m\left(\eta_\Psi-\frac{4-\epsilon}{2}\right)\right.$$
$$\left.-2n\left(\eta_\Phi-\frac{4-\epsilon}{2}\right)+\left(\epsilon-\frac{3}{2}-z_{d-1}^{-1}-\frac{1}{2}z_d^{-1}\right)\right]G^{(m,m,n,n)}\left(\{k_i\},\mu,\lambda,a\right)=0. \tag{8.36}$$

For further computations, we need the explicit forms of the anomalous dimensions and dynamical critical exponents derived in Appendix E.1. For completeness, we state the results

$$\eta_\Psi=\frac{u_1\lambda^2}{2}\frac{2a-1}{N(1-a)^2\sqrt{|\tilde{a}|}},\qquad \eta_\Phi=\frac{u_1\lambda^2}{2}\left(\frac{2a+1}{N(1-a)^2\sqrt{|\tilde{a}|}}+\frac{1}{2}\right),$$
$$z_{d-1}^{-1}=1+\frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}},\qquad z_d^{-1}=1-\frac{2u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}. \tag{8.37}$$

Note that the dynamical critical exponents are the same for $a=2$.

Equation (8.36) contains all important information about quantum critical points when inserting the right fixed point values. The renormalized shape of the Fermi surface at such a critical point, for example, can be obtained by solving the renormalization group equation for the fermion two-point function while setting $\beta_{\lambda/a}=0$. In (8.36) we have $2m+2n$ momenta, but due to energy and momentum conservation only $2m+2n-1$ of them are independent. This implies for the fermion two-point function, where $m=1$ and $n=0$, that the RG equation only depends on one momentum, such that

$$\left[\mathbf{K}\nabla_{\mathbf{K}}+\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}+\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}+1\right]G(k)=0. \tag{8.38}$$

Note that the constant terms sum up to one and that the dynamical critical exponents are evaluated at one of the fixed points (λ_i^*,a_i^*). This equation is solved by any function of the form

$$G(k)=\frac{1}{|k_d|^{2z_d}}\,f\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}},\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right) \tag{8.39}$$

where f is a universal scaling function (see Appendix E.2).

In the bare case, the shape of the Fermi surface at the two hot-spots is given by the

equation

$$\frac{k_{d-1}}{k_d^2} = \pm 1, \tag{8.40}$$

which corresponds to simple poles of the bare fermion propagators at zero frequency. Additionaly, the dynamical critical exponents in the bare case are equal to one, implying the scaling of the fermion propagator

$$G(k) = \frac{1}{k_d^2}\, f\left(0, \frac{k_{d-1}}{k_d^2}\right) \tag{8.41}$$

at zero frequency. Thus, the simple pole for (8.40) has to be encoded in the scaling function f via

$$f^{-1}(0, \pm 1) = 0. \tag{8.42}$$

Transferring condition (8.42) to the renormalized case, we find the renormalized shape of the Fermi surface at the hot-spots

$$\operatorname{sgn}(k_{d-1})\, |k_{d-1}|^{z_{d-1}} = \pm\, |k_d|^{2 z_d}\,. \tag{8.43}$$

As seen in Fig. 6, the Fermi surface is flattened at the hot-spots for the first fixed point. Thus we conclude that the quantum phase transition from an ordinary Fermi liquid metal

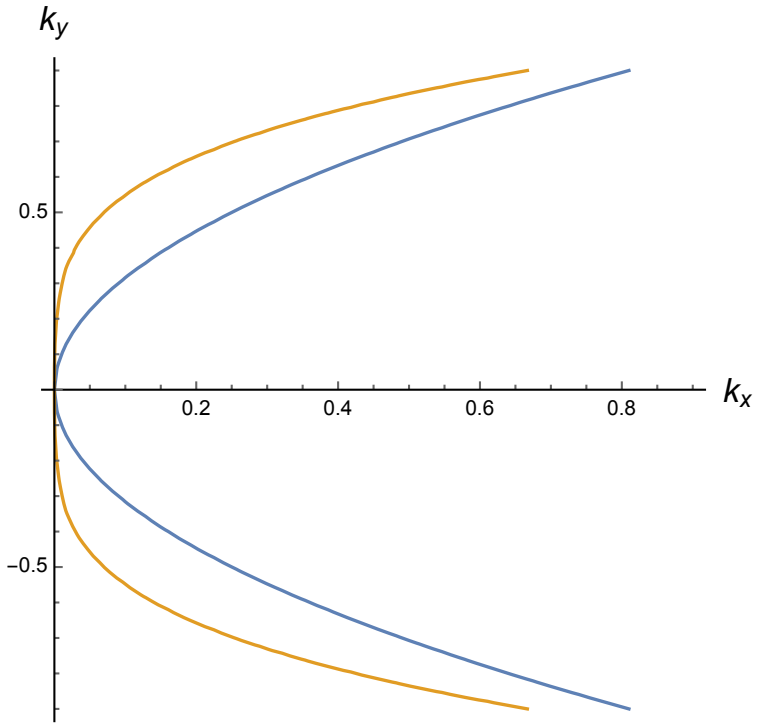


Figure 6: Blue: Bare Fermi surface at the left hot-spot. Orange: Renormalized, flattened Fermi surface at the left hot-spot for the first fixed point. The dynamical critical exponents are evaluated for $N = 2$ and $\epsilon = \frac{1}{2}$.

to the incommensurate $2k_F$ CDW ordered phase is of second order with a flattened Fermi surface at the hot-spots.

Note that the Fermi surface shape doesn't change for $a = 2$ since the dynamical critical exponents are the same, reducing (8.43) to the bare form.

Experimentally observable predictions about the quantum phase transition driven by Fermi surface nesting can be extracted from the scaling behavior of the renormalized boson two-point function, which is obtained from the RG equation (8.36) by setting $m = 0$, $n = 1$ and $\beta_{\lambda/a} = 0$

$$\left[\mathbf{K}\nabla_{\mathbf{K}} + \frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}} + \frac{k_d}{2z_d}\frac{\partial}{\partial k_d} + p\right] D(k) = 0 \tag{8.44}$$

with $p = z_d^{-1} - \frac{u_1\lambda^2}{2}$. This differential equation is solved by any function of the form

$$D(k) = \frac{1}{|k_d|^{2z_d p}}\, g\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}}, \frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right), \tag{8.45}$$

where g is an universal scaling function (see Appendix E.3).

One of such experimental detectable signatures is the power-law frequency dependence of $D(k)$ when approaching the quantum critical point with wave vector $\mathbf{Q} = 2k_F$. One can easily set $k_{d-1} = 0$ in (8.45), but when approaching $k_d = 0$, the scaling from should reduce to a power-law behavior dependent on $|\mathbf{K}|$ only, i.e.

$$\lim_{k_d\to 0} D(\mathbf{K}, 0, k_d) \propto \lim_{k_d\to 0} \frac{1}{|k_d|^{2z_d p}} \left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}}\right)^{\alpha} = |\mathbf{K}|^{-p}, \tag{8.46}$$

where the exponent α is determined by the cancellation of the k_d-terms. In two dimensional metals with spin degeneracy $N = 2$, one therefore should find $D(\omega) \propto |\omega|^{-p}$ with $p \approx 0.616$.

A very important point still to discuss is the mass term in the boson propagator. Usually, the theory is tuned to the quantum critical point by setting the mass term to zero, s.t. the susceptibility is peaked at $\mathbf{Q} = 2k_F$, i.e. $k = 0$. Adding a mass term in the boson propagator, the linear term $\propto k_{d-1}$, however, turns the boson massless at momenta different from $k = 0$, thus contradicting our original assumptions.

The flattening of the Fermi surface during the RG flow resolves this problem, which can be concluded from the boson self-energy. On the one hand, the forms in (5.21) and (5.24) don't show a peak for $k = 0$ in dimensions $d \geq 2$. On the other hand, calculating the boson self-energy in two dimensions with a nested Fermi surface of the form $\pm k_{d-1} + |k_d|^{\alpha}$ with $\alpha > 2$, the density susceptibility is indeed peaked for $\mathbf{Q} = 2k_F$, thus gapping out the boson at all momenta when adding a mass term.

We now have finished our RG analysis of the quantum phase transition to incommensurable $2k_F$ CDW order in two dimensional metals using dimensional regularization and the minimal substraction scheme. This was done by treating the low-energy excitations of the electrons and the CDW order parameter fluctuations on equal footing.

Another approach to symmetry breaking quantum phase transitions, already mentioned

in the introduction, was introduced by John A. Hertz in [6], where he integrates out the fermions of the underlying theory, obtaining an effective ϕ^4-like action of the order parameter fluctuations alone. Here we quickly argue why this method fails in two dimensional materials, following the treatment given in Chapter 18 of [4].

The underlying assumption is that the electronic quasi-particles have Fermi-liquid behavior, i.e. that the width of the quasi-particle peak vanishes faster than the quasi-particle energy sufficiently close to the Fermi surface. By integrating out the fermions, the main modification to the order parameter propagator comes from the fermion polarizability, which has a typical Landau damping form $|k_0| / \gamma(k)$ [20]. The function $\gamma(k)$ depends on the problem under consideration and in our case is given by $\gamma(k) = \sqrt{|e_k|}$. This is the leading order term when expanding the $2d$ boson self-energy in small frequencies, coming from negative e_k.

In the usual RG analysis, the quadratic term is required to be invariant under certain scale transformations. For this to be possible, we need to allow for a dynamic critical exponent in the rescaling of the frequency $k_0 = k_0' \, b^{-z}$, s.t. all terms in the new fluctuation propagator transform uniformly. This can be seen by the typical form of the modified propagator $\propto k^2 + |k_0| / \gamma(k)$, where we get the condition that the frequency transforms in the same fashion as $k_0 \sim k^2 \gamma(k)$. In our case, when ignoring terms $\propto k_{d-1}$ which have the same scaling dimension as k_d^2, we get $k_0 \sim k_d^3$ and therefore $z = 3$.

Having introduced the dynamical critical exponent for the frequency leading to an effective dimensionality $d + z$ of the system, we can derive the scale transformations for the order parameter field and afterwards the dimension of the interaction strength of the quartic term, just like in the introductory chapter about ϕ^4-theory. In the Hertz theory, we arrive at the conclusion that the interaction is irrelevant in dimensions $d > 1$ when $z = 3$, s.t. the critical properties of the phase transition are described by the stable Gaussian fixed point of the new effective theory.

However, this conclusion is wrong for two dimensional systems. This can be seen by calculating the imaginary part of the self-energy of the fermionic excitations within the Hertz approach for the fluctuation propagator, which in our case is similar to the calculation done in [10] with the result $\mathrm{Im}(\Sigma) \propto \omega^{2/3}$ at the hot-spots. Hence, we don't have a well-defined quasi-particle peak since the width of the peak vanishes sublinearly with energy, undermining the original assumption of Fermi-liquid behavior and leading to a breakdown of the Hertz approach.

This is the reason why it is actual necessary to treat the order parameter fluctuations and the low-energy excitations of the fermions equally to obtain consistent, physical results.

9 Large N Limit

The expansions in this thesis are arranged in powers of the interaction λ. Another, often used, expansion parameter is $1/N$, i.e. large numbers of fermion species N are considered to obtain a power series in $1/N$, as for example in [9]. In this short chapter we expand the fixed point values and their RG eigenvalues to first order in large N and derive the implications.

As was seen in the previous chapter, the fixed point values are obtained by solving the equations

$$\begin{aligned} &-\frac{u_1\lambda^3}{4}\left(\frac{2(3-2a)}{N(1-a)^2\sqrt{|\tilde{a}|}}+1\right)+\frac{\lambda}{2}\epsilon=0,\\ &-\frac{u_1\lambda^2}{2}\left(\frac{4a(2-a)}{N(1-a)^2\sqrt{|\tilde{a}|}}+a-2\right)=0 \end{aligned} \tag{9.1}$$

for λ and a. The values of a can be determined by the second equation and the corresponding values of λ follow via the first equation, which can be solved for

$$\lambda=\sqrt{\frac{2\epsilon}{u_1}}\frac{1}{\sqrt{\frac{2(3-2a)}{N(1-a)^2\sqrt{|\tilde{a}|}}+1}}. \tag{9.2}$$

Note that we restricted ourselves to positive λ as before.

A N-independent solution is given by $a=2$, since the second equation in (9.1) is $\propto(2-a)$. The other fixed point values of a then follow from the equation

$$\frac{\sqrt{|a|}}{|1-a|^{\frac{3}{2}}}-\frac{N}{4}=0. \tag{9.3}$$

The ansatz $a=1\pm\frac{c}{N^b}$ leads to $c=2^{4/3}$ and $b=2/3$ in the limit of large N. Thus, we in total obtain three values for a

$$a_1^*=1-\frac{2^{\frac{4}{3}}}{N^{\frac{2}{3}}},\qquad a_2^*=1+\frac{2^{\frac{4}{3}}}{N^{\frac{2}{3}}},\qquad a_3^*=2, \tag{9.4}$$

where we labelled the values such that they represent the corresponding fixed points in (8.24), only for large N. Inserting these values into (9.2) and expanding around $1/N=0$, the three pairs of fixed points are then given by

$$\begin{aligned} (\lambda_1^*,a_1^*)&=\left(\frac{2}{\sqrt{3u_1}}\sqrt{\epsilon}-\frac{5\times2^{\frac{1}{3}}}{3\sqrt{3u_1}}\frac{\sqrt{\epsilon}}{N^{\frac{2}{3}}},1-\frac{2^{\frac{4}{3}}}{N^{\frac{2}{3}}}\right),\\ (\lambda_2^*,a_2^*)&=\left(\frac{2}{\sqrt{3u_1}}\sqrt{\epsilon}+\frac{5\times2^{\frac{1}{3}}}{3\sqrt{3u_1}}\frac{\sqrt{\epsilon}}{N^{\frac{2}{3}}},1+\frac{2^{\frac{4}{3}}}{N^{\frac{2}{3}}}\right),\\ (\lambda_3^*,a_3^*)&=\left(\sqrt{\frac{2}{u_1}}\sqrt{\epsilon}+\frac{1}{\sqrt{u_1}}\frac{\sqrt{\epsilon}}{N},2\right) \end{aligned} \tag{9.5}$$

to leading order in $1/N$. Note that for $N \to \infty$ the first two fixed points merge to a single one.

The stability can be studied via the eigenvalues of the Jacobian

$$J = \begin{pmatrix} -\beta'_\lambda & -\dot{\beta}_\lambda \\ -\beta'_a & -\dot{\beta}_a \end{pmatrix} \tag{9.6}$$

when evaluated at the fixed point values. In leading order, we find for the first two fixed points

$$\begin{aligned} \nu_1 &= \mp \left(\frac{1}{2^{\frac{4}{3}}} \epsilon N^{\frac{2}{3}} - \frac{23}{12 \times 2^{\frac{2}{3}}} \frac{\epsilon}{N^{\frac{2}{3}}} \right), \\ \nu_2 &= -\epsilon \mp \frac{2^{\frac{1}{3}}}{3} \frac{\epsilon}{N^{\frac{2}{3}}}, \end{aligned} \tag{9.7}$$

where the upper sign represents (λ_1^*, a_1^*) and the lower sign the (λ_2^*, a_2^*), and for the third fixed point

$$\begin{aligned} \nu_1 &= -\epsilon + 3\sqrt{2} \frac{\epsilon}{N}, \\ \nu_2 &= -\epsilon + \frac{9}{4} \frac{\epsilon}{N^2}. \end{aligned} \tag{9.8}$$

We note that the first fixed point, stable for $N = 2$ as seen in Fig. 5, stays stable for any large value of N. The second fixed point, however, changes from stable to unstable when passing through the line $a = 2$ on its way to merging with the first fixed point, whereas the third fixed point gets stable at this crossing point.

The UV initial conditions of positive λ and infinitesimal small a terminate in the first fixed point for any N and the dynamical critical exponents indicate a flattening of the Fermi surface for $N = 2$ as well as for large N, since $z_{d-1} = 1 + \mathcal{O}\left(N^{-2/3}\right)$ and $z_d^{-1} = 1 - \frac{2}{3}\epsilon + \mathcal{O}\left(N^{-2/3}\right)$. We might therefore conclude that our previous conclusion, namely that the second fixed point does not represent a physical sensible one, is strengthened in the large N sense since it merges with the fixed point identified with the quantum phase transition to incommensurate $2k_F$ CDW order.

10 Superconducting Instabilities

Some materials with CDW order become superconducting in the vicinity of the potential CDW quantum critical point, for example in Pd-intercalated rare earth poly-telluride $RETe_n$ CDW systems [21] or in Cu_xTaS_2 [22].

Whether superconductivity is favored or suppressed near quantum critical points can be studied theoretically as well by including terms corresponding to the creation and annihilation of Cooper pairs. In our case, we consider an interaction term coupling two electrons of the opposite hot-spots with zero total momentum in a spin-singlett form and check via one-loop vertex renormalization if the anomalous dimension of the coupling constant is increased or suppressed at the fixed point describing the quantum phase transition.

10.1 One-Loop Correction to Superconducting Vertex

The interaction term described above has the form

$$\begin{aligned} S_{\text{cp}} &= g \int_k \left[\psi_{+,\uparrow}(k)\psi_{-,\downarrow}(-k) + \overline{\psi}_{-,\downarrow}(-k)\overline{\psi}_{+,\uparrow}(k) - \psi_{+,\downarrow}(k)\psi_{-,\uparrow}(-k) - \overline{\psi}_{-,\uparrow}(-k)\overline{\psi}_{+,\downarrow}(k)\right] \\ &= g \int_k \sigma_y^{\alpha\beta}\overline{\Psi}_\alpha(k)\Psi_\beta(k), \end{aligned} \tag{10.1}$$

where g is the coupling constant. Greek indices act in spin space, s.t. $\sigma_y^{\uparrow\downarrow} = -i$, $\sigma_y^{\downarrow\uparrow} = i$. Thus our whole action in general dimensions now reads

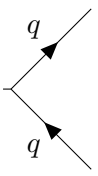
$$\begin{aligned} S = &\int_k \Psi_\mu(k)\left[-i\mathbf{\Gamma K} + i\delta_k\sigma_x\right]\Psi_\mu(k) + \int_k \overline{\Phi}(k)\left[k_d^2 + ak_{d-1}\right]\Phi(k) + g\int_k \sigma_y^{\alpha\beta}\overline{\Psi}_\alpha(k)\Psi_\beta(k) \\ &- \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}}\int_{k,p}\left[\Phi(p)\overline{\Psi}_\mu(k+p)\sigma_y\overline{\Psi}_\mu^T(-k) + \overline{\Phi}(p)\Psi_\mu^T(k+p)\sigma_y\Psi_\mu(-k)\right], \end{aligned} \tag{10.2}$$

where summation over repeated indices is implied. The full vertex to one-loop order follows from

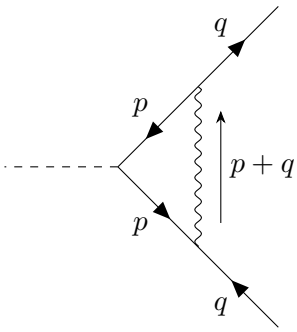
$$\left\langle\overline{\Psi}_\alpha^a(q)\Psi_\beta^b(q)\right\rangle = -\left\langle\overline{\Psi}_\alpha^a(q)\Psi_\beta^b(q)S_{int}\right\rangle_0 - \frac{1}{3!}\left\langle\overline{\Psi}_\alpha^a(q)\Psi_\beta^b(q)S_{int}^3\right\rangle_0^{con}. \tag{10.3}$$

Explicit calculations of the contractions are done in Appendix F, here we state the results. The vertex factor is given by

$$-\left\langle\overline{\Psi}_\alpha^a(q)\Psi_\beta^b(q)S_{int}\right\rangle_0 = G_{ad}^T(q)\left(-g\sigma_y^{\alpha\beta}\mathbb{1}\right)_{dc}G_{cb}^T(q) = \tag{10.4}$$



and the only one-loop diagram we can draw is of the from



$$= g\sigma_y^{\alpha\beta} G^T(q) \underbrace{\left[\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y G^2(p)\sigma_y D(p+q)\right]}_{:=V(q)} G^T(q). \tag{10.5}$$

Thus, (10.3) can be written as

$$\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)\right\rangle = G^T_{ad}(q)\left[-g\sigma_y^{\alpha\beta}\mathbb{1} + g\sigma_y^{\alpha\beta}V(q)\right]_{dc} G^T_{cb}(q), \tag{10.6}$$

where $V(q)$ is a matrix in spinor space.

To calculate $V(q)$, we need to simplify the product of Pauli matrices

$$\sigma_y G^2(p)\sigma_y \propto -\sigma_y\left(-\mathbf{\Gamma P} + \sigma_x\delta_p\right)\left(-\mathbf{\Gamma P} + \sigma_x\delta_p\right)\sigma_y = \left(-p_0^2 - p_i p_j - \delta_p^2\right)\mathbb{1}, \tag{10.7}$$

where we used the notation $p_i \widehat{=} \sum_{i=1}^{d-2} p_i$. With

$$p_i p_j \widehat{=} \sum_{i=1}^{d-2}\sum_{j=1}^{d-2} p_i p_j = \mathbf{p}^2 + 2\sum_{i=1}^{d-3}\sum_{j=i+1}^{d-2} p_i p_j, \tag{10.8}$$

we get

$$V(q) = -\frac{\lambda^2\mu^\epsilon}{N}\int_p \frac{p_0^2 + \mathbf{p}^2 + \delta_p^2 + 2\sum_{i=1}^{d-3}\sum_{j=i+1}^{d-2} p_i p_j}{\left(\mathbf{P}^2 + \delta_p^2\right)^2}\frac{1}{(p_d+q_d)^2 + a(p_{d-1}+q_{d-1})}. \tag{10.9}$$

Note that $V(q) \propto \mathbb{1}$, but we suppressed the unity matrix. Changing $\mathbf{p} \to -\mathbf{p}$, all the non-quadratic terms in the first numerator vanish by antisymmetry, leading to

$$\begin{aligned} V(q) &= -\frac{\lambda^2\mu^\epsilon}{N}\int_p \frac{p_0^2 + \mathbf{p}^2 + \delta_p^2}{\left(\mathbf{P}^2 + \delta_p^2\right)^2}\frac{1}{(p_d+q_d)^2 + a(p_{d-1}+q_{d-1})} \\ &= -\frac{\lambda^2\mu^\epsilon}{N(1-a)}\int_p \frac{1}{\mathbf{P}^2 + p_{d-1}^2}\frac{1}{p_d^2 + \frac{2q_d}{1-a}p_d + \frac{1}{1-a}q_d^2 + \tilde{a}p_{d-1} + \tilde{a}q_{d-1}}, \end{aligned} \tag{10.10}$$

where we shifted $p_{d-1} \to p_{d-1} - p_d^2$. The p_d-integral is of the form I_4^Σ evaluated in Appendix C.4, thus simplifying $V(q)$

$$V(q) = -\frac{\lambda^2\mu^\epsilon}{N(1-a)}\int_{\mathbf{P},p_{d-1}} \frac{1}{\mathbf{P}^2 + p_{d-1}^2}\frac{\Theta\left(|\tilde{a}|\,p_{d-1} - b(q)\right)}{\sqrt{4\,|\tilde{a}|\left(|\tilde{a}|\,p_{d-1} - b(q)\right)}}, \tag{10.11}$$

where $b(q) = -|\tilde{a}|\,q_{d-1} + \frac{|\tilde{a}|}{1-a}q_d^2$. Substituting $y = \operatorname{sgn}(\tilde{a})\,p_{d-1} - b(q)$, the remaining integrals

can be computed straightforward

$$\begin{aligned} V(q) &= -\frac{\lambda^2\mu^\epsilon}{2N(1-a)\sqrt{|\tilde{a}|}}\frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty dr\ r^{d-2}\int_0^\infty\frac{dy}{2\pi}\frac{1}{r^2+\big(y+b(q)\big)^2}\frac{1}{\sqrt{y}} \\ &= -\frac{i\lambda^2\mu^\epsilon}{8N(1-a)\sqrt{|\tilde{a}|}}\frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty dr\ r^{d-3}\left(\frac{1}{\sqrt{b(q)+ir}}-\frac{1}{\sqrt{b(q)-ir}}\right) \\ &= \frac{i\lambda^2\mu^\epsilon}{N(1-a)\sqrt{|\tilde{a}|}}\frac{\Gamma\left(\frac{5}{2}-d\right)\Gamma\left(d-2\right)}{2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d-1}{2}\right)}\big(b(q)\big)^{d-\frac{5}{2}}\left[i^{-d}-(-i)^{-d}\right]. \end{aligned} \tag{10.12}$$

Setting $d=\frac{5}{2}-\epsilon$ and expanding around $\epsilon=0$, we find

$$V(q) = -\frac{\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}\frac{1}{8\pi^{\frac{3}{4}}\Gamma\left(\frac{3}{4}\right)\epsilon}+\text{finite} = -\frac{4u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}\epsilon}+\text{finite}. \tag{10.13}$$

Hence the superconducting vertex to one-loop order in (10.3) takes the form

$$\left\langle\overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)\right\rangle = G^T_{ad}(q)\left[-g\sigma_y^{\alpha\beta}\mathbb{1}-g\sigma_y^{\alpha\beta}\mathbb{1}\frac{4u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}\epsilon}+\text{finite}\right]_{dc}G^T_{cb}(q). \tag{10.14}$$

10.2 Renormalization

The diverging $1/\epsilon$-pole is cancelled by including the counterterm

$$S^{\text{ct}}_{\text{cp}} = g\int_k\sigma_y^{\alpha\beta}\overline{\Psi}_\alpha(k)\frac{Z_{g,1}}{\epsilon}\Psi_\beta(k) \tag{10.15}$$

with $Z_{g,1}=-\frac{4u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}$ and treating it as a new interaction. Since the interaction g has mass dimension one, we again introduce the arbitrary mass scale μ, s.t. the renormalized Cooper pair term reads

$$S^{\text{ren}}_{\text{cp}} = S_{\text{cp}}+S^{\text{ct}}_{\text{cp}} = g\mu Z_g\int_k\sigma_y^{\alpha\beta}\overline{\Psi}_\alpha(k)\Psi_\beta(k) \tag{10.16}$$

and can be brought to its bare, initial form using the multiplicative renormalizations in (8.3) together with $g_B=g\mu Z_g$. The β-function for the interaction then reads

$$\beta_g = \mu\frac{dg}{d\mu} = g_B\mu\frac{d}{d\mu}\left(\mu^{-1}Z_g^{-1}\right) = g\left(-1-\frac{\mu}{Z_g}\frac{dZ_g}{d\mu}\right) = g\left(-1-\eta_g\right), \tag{10.17}$$

where the anomalous dimension η_g can be obtain as usual by comparing the regular parts of

$$Z_g\eta_g = \mu\frac{dZ_g}{\partial\mu} = \beta_\lambda Z_g'+\beta\dot{Z}_g, \tag{10.18}$$

thus leading to

$$\eta_g = \beta_\lambda^{(1)} Z'_{g,1} = -\frac{\lambda}{2}\left(-\frac{8u_1\lambda}{N(1-a)\sqrt{|\tilde{a}|}}\right) = \frac{4u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}. \tag{10.19}$$

For $\eta_g > 0$, the scaling dimension of g is increased and therefore the superconducting instabilities are enhanced, whereas for $\eta_g < 0$ the superconducting instabilities are suppressed. The first fixed point describing the quantum phase transition to incommensurate $2k_F$ CDW order leads to the numerical value $\eta_g \approx 0.564$ in two dimensions and for the physical case $N = 2$. Consequently superconducting instabilities are enhanced in the vicinity of our quantum critical point.

11 Conclusion

In this thesis, we studied the quantum phase transition from a normal Fermi liquid metal to a charge density wave ordered phase with incommensurate wave vector $\mathbf{Q} = 2k_F$. Based on the epsilon expansion by Dalidovich and Lee [1], we calculated one-loop contributions to the bosonic CDW fluctuation and the fermion self-energy at one-loop order in dimensional regularization and found $1/\epsilon$-poles for both.

A crucial difference to other works using the dimensional regularization procedure by Dalidovich and Lee [1, 19] is given by the fact that we did not need to include the Landau damping term in the boson propagator to handle with the IR-divergence in the fermion self-energy. This divergence was avoided by another term $\propto k_{d-1}$ in the CDW fluctuation propagator, which we became aware of by a pole in the boson self-energy. Since no frequency depending terms were renormalized by the boson self-energy, we kept the boson propagator frequency independent and consequently only the Fermi surface form was renormalized by the fermion self-energy. One-loop corrections to the vertex factor were not present due to the structure of the interaction vertices in our theory.

We derived the fixed points by renormalizing our theory via the minimal substraction scheme and identified a stable fixed point corresponding to a continuous, second order phase transition to the incommensurate $2k_F$ CDW ordered phase with a flattening of the Fermi surface at the hot-spots. Thus, based on a controlled, perturbative RG analysis, we came to the same conclusion as Sýkora at al. in [10].

After showing that the physically important fixed point is stable with a Fermi surface flattening at the hot-spots in the large N limit, we further concluded that superconductivity is favored in the vicinity of our quantum critical point.

All calculations were done at one-loop order, since already two-loop diagrams are very hard to evaluate. It would be very interesting, however, to include higher order diagrams and take also frequency dependent terms into account. Unfortunately, this has to be left for future work.

A CDW Model and Methods

Here we present how the action in (4.3) can be rewritten in the spinor representation given the spinor components

$$\Psi_j^1(k) = \psi_{+,j}(k), \quad \Psi_j^2(k) = \psi_{-,j}^\dagger(-k), \quad \overline{\Psi}_j^1(k) = i\psi_{-,j}(-k), \quad \overline{\Psi}_j^2(k) = -i\psi_{+,j}^\dagger(k). \tag{A.1}$$

We use the notation $S = S_\Psi + S_\Phi + S_{\text{int},1} + S_{\text{int},2}$. The kinetic part of the fermions can be written as

$$\begin{aligned}
S_\Psi &= \int_k \left\{ \psi_{+,j}^\dagger(k) \left(-ik_0 + k_x + k_y^2\right) \psi_{+,j}(k) + \psi_{-,j}^\dagger(-k) \left(ik_0 + k_x + k_y^2\right) \psi_{-,j}(-k) \right\} \\
&= \int_k \left\{ i\overline{\Psi}_j^2(k) \left(-ik_0 + k_x + k_y^2\right) \Psi_j^1(k) + \Psi_j^2(k) \left(ik_0 + k_x + k_y^2\right) (-i)\overline{\Psi}_j^1(k) \right\} \\
&= \int_k \left\{ (-ik_0) \left[i\overline{\Psi}_j^2(k)\Psi_j^1(k) - i\overline{\Psi}_j^1(k)\Psi_j^2(k) \right] + i\left(k_x + k_y^2\right) \left[\overline{\Psi}_j^2(k)\Psi_j^1(k) + \overline{\Psi}_j^1(k)\Psi_j^2(k) \right] \right\} \\
&= \int_k \overline{\Psi}_j(k) \left[-ik_0\sigma_y + i\left(k_x + k_y^2\right)\sigma_x \right] \Psi_j(k),
\end{aligned} \tag{A.2}$$

whereas the first interaction term takes the form

$$\begin{aligned}
S_{\text{int},1} &= \lambda \int_{k,p} \Phi(p) \psi_{+,j}^\dagger(p+k) \psi_{-,j}(k) \\
&= \frac{\lambda}{2} \int_{k,p} \Phi(p) \big[\psi_{+,j}^\dagger(p+k)\psi_{-,j}(k) - \underbrace{\psi_{-,j}(k)\psi_{+,j}^\dagger(p+k)}_{k\to -k-p} \big] \\
&= \frac{\lambda}{2} \int_{k,p} \Phi(p) \left[\psi_{+,j}^\dagger(p+k)\psi_{-,j}(k) - \psi_{-,j}(-k-p)\psi_{+,j}^\dagger(-k) \right] \\
&= \frac{\lambda}{2} \int_{k,p} \Phi(p) \left[i\overline{\Psi}_j^2(k+p)(-i)\overline{\Psi}_j^1(-k) - (-i)\overline{\Psi}_j^1(k+p) i\overline{\Psi}_j^2(-k) \right] \\
&= -\frac{i}{2}\lambda \int_{k,p} \Phi(p) \overline{\Psi}_j(k+p) \sigma_y \overline{\Psi}_j^T(-k).
\end{aligned} \tag{A.3}$$

Similarly, the second interaction term reads

$$\begin{aligned}
S_{\text{int},2} &= \lambda \int_{k,p} \overline{\Phi}(p) \psi_{-,j}^\dagger(k-p) \psi_{+,j}(k) \\
&= \frac{\lambda}{2} \int_{k,p} \overline{\Phi}(p) \big[\underbrace{\psi_{-,j}^\dagger(k-p)\psi_{+,j}(k)}_{k\to -k} - \underbrace{\psi_{+,j}(k)\psi_{-,j}^\dagger(k-p)}_{k\to k+p} \big] \\
&= \frac{\lambda}{2} \int_{k,p} \overline{\Phi}(p) \left[\psi_{-,j}^\dagger(-k-p)\psi_{+,j}(-k) - \psi_{+,j}(k+p)\psi_{-,j}^\dagger(k) \right] \\
&= \frac{\lambda}{2} \int_{k,p} \overline{\Phi}(p) \left[\Psi_j^2(k+p)\Psi_j^1(-k) - \Psi_j^1(k+p)\Psi_j^2(-k) \right] \\
&= -\frac{i}{2}\lambda \int_{k,p} \overline{\Phi}(p) \Psi_j^T(k+p) \sigma_y \Psi_j(-k),
\end{aligned} \tag{A.4}$$

s.t. in total we get the action in (4.6).

B Boson Self-Energy

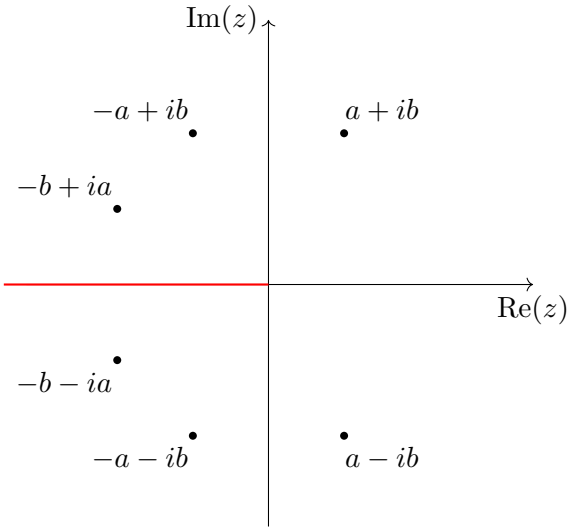
When setting $\epsilon = \frac{1}{2}$ in (5.24), we get the expression

$$\begin{aligned}\Pi(q) - \Pi(0) &= -\frac{\lambda^2}{8\pi}\left(\sqrt{|q_0| - ie_q} + \sqrt{|q_0| + ie_q} + \frac{\sqrt{2}e_q}{\sqrt{\sqrt{q_0^2 + e_q^2} + |q_0|}}\right) \\ &= -\frac{\lambda^2}{8\pi}\left(\sqrt{|q_0| - ie_q} + \sqrt{|q_0| + ie_q} + \frac{e_q}{|e_q|}\sqrt{2}\sqrt{\sqrt{q_0^2 + e_q^2} - |q_0|}\right) \\ &= -\frac{\lambda^2}{8\pi}\left(\sqrt{|q_0| + i|e_q|} + \sqrt{|q_0| - i|e_q|} - \sqrt{-|q_0| + i|e_q|} - \sqrt{-|q_0| - i|e_q|}\right). \end{aligned} \tag{B.1}$$

We now need to show that

$$\sqrt{a + ib} + \sqrt{a - ib} - \sqrt{-a + ib} - \sqrt{-a - ib} = \sqrt{2}\left(\sqrt{-b + ia} + \sqrt{-b - ia}\right), \tag{B.2}$$

where $a, b > 0$. Naively factoring out $\sqrt{i}$ or $\sqrt{-i}$ on the left hand side leads to a wrong result, since the complex square-root function has a branch cut in the complex plane, which we have to avoid. We choose the branch cut to be on the negative real axis, s.t. there are



two trivial cases

$$\begin{aligned} a + ib &= e^{-i\frac{\pi}{2}}(-b + ia) = -i(-b + ia), \\ a - ib &= e^{i\frac{\pi}{2}}(-b - ia) = i(-b - ia) \end{aligned} \tag{B.3}$$

and therefore

$$\begin{aligned} \sqrt{a + ib} &= \sqrt{-i}\sqrt{-b + ia}, \\ \sqrt{a - ib} &= \sqrt{i}\sqrt{-b - ia}. \end{aligned} \tag{B.4}$$

When going from $-a+ib$ to $-b-ia$ and from $-a-ib$ to $-b+ia$ however, we need to avoid the negative real axis, i.e.

$$\begin{aligned}-a+ib &= e^{i\frac{3\pi}{2}}(-b-ia) = e^{2\pi i}e^{-i\frac{\pi}{2}}(-b-ia) = e^{2\pi i}(-i)(-b-ia),\\ -a-ib &= e^{-i\frac{3\pi}{2}}(-b+ia) = e^{-2\pi i}e^{i\frac{\pi}{2}}(-b+ia) = e^{-2\pi i}i(-b+ia)\end{aligned} \tag{B.5}$$

and therefore, by raising both sides to the power $\frac{1}{2}$,

$$\begin{aligned}\sqrt{-a+ib} &= e^{\pi i}\sqrt{-i}\sqrt{-b-ia} = -\sqrt{-i}\sqrt{-b-ia},\\ \sqrt{-a-ib} &= e^{-\pi i}\sqrt{i}\sqrt{-b+ia} = -\sqrt{i}\sqrt{-b+ia}.\end{aligned} \tag{B.6}$$

Thus, we can derive

$$\begin{aligned}&\sqrt{a+ib}+\sqrt{a-ib}-\sqrt{-a+ib}-\sqrt{-a-ib}\\ &= \sqrt{-b+ia}\left(\sqrt{i}+\sqrt{-i}\right)+\sqrt{-b-ia}\left(\sqrt{i}+\sqrt{-i}\right) = \sqrt{2}\left(\sqrt{-b+ia}+\sqrt{-b-ia}\right)\end{aligned} \tag{B.7}$$

and hence

$$\begin{aligned}\Pi(q)-\Pi(0) &= -\frac{\lambda^2}{8\pi}\left(\sqrt{|q_0|+i|e_q|}+\sqrt{|q_0|-i|e_q|}-\sqrt{-|q_0|+i|e_q|}-\sqrt{-|q_0|-i|e_q|}\right)\\ &= -\frac{\lambda^2}{8\pi}\sqrt{2}\left(\sqrt{-|e_q|+i\,|q_0|}+\sqrt{-|e_q|-i\,|q_0|}\right) = -\frac{\lambda^2}{4\pi}\sqrt{\sqrt{q_0^2+e_q^2}-|e_q|}\\ &= -\frac{\lambda^2}{4\pi}\sqrt{\sqrt{q_0^2+e_q^2}+e_q},\end{aligned} \tag{B.8}$$

which again coincides with (5.6).

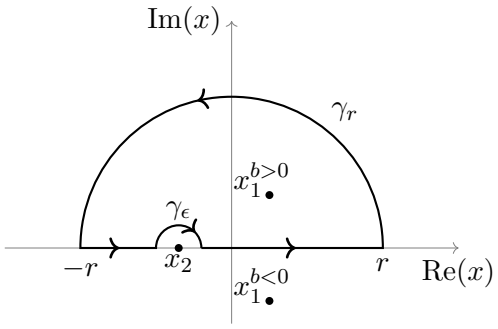
C Fermion Self-Energy

C.1 Derivation of I_1^Σ

The integral

$$I_1^\Sigma = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \underbrace{\frac{1}{x+a-ib}\frac{1}{x+c}}_{:=f(x)} \tag{C.1}$$

is solved as follows: It has two poles, one at $x_1 = -a + ib$ and one at $x_2 = -c$. Since x_2 lies on the real axis, we can't just close the contour in the upper half plane in the standard fashion, but have to avoid the pole on the real axis. Hence we choose the contour as shown in the graph below.



The whole contour is set together by four different paths

$$\oint_C \frac{dx}{2\pi} = \left(\int_{-r}^{x_2-\epsilon} + \int_{\gamma_\epsilon} + \int_{x_2+\epsilon}^{r} + \int_{\gamma_r}\right)\frac{dx}{2\pi}, \tag{C.2}$$

where γ_ϵ is a clockwise half-circle around x_2 with radius ϵ.

For the case $b > 0$, for which x_1 lies in the closed contour, the left hand side of (C.2) evaluates to

$$\oint_C \frac{dx}{2\pi} f(x) = i\ \mathrm{Res}(x = x_1) = \frac{i}{c-a+ib}. \tag{C.3}$$

The right hand side is a little more involved. First we can take the limit $r \to \infty$, for which the contribution of the path γ_r vanishes since $f(x)$ decays fast enough. After that, we can take the limit $\epsilon \to 0$, for which we get

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{dx}{2\pi}f(x)f(x) = -\frac{i}{2}\mathrm{Res}(x = x_2) = -\frac{i}{2}\frac{1}{-c+a-ib} = \frac{i}{2}\frac{1}{c-a+ib}. \tag{C.4}$$

Hence in the limits of our interest the right hand side of (C.2) yields

$$\begin{aligned}\lim_{r\to\infty}\lim_{\epsilon\to 0}\left(\int_{-r}^{x_2-\epsilon} + \int_{\gamma_\epsilon} + \int_{x_2+\epsilon}^{r} + \int_{\gamma_r}\right)\frac{dx}{2\pi}f(x) &= \left(\int_{-\infty}^{x_2} + \int_{x_2}^{\infty}\right)\frac{dx}{2\pi}f(x) + \frac{i}{2}\frac{1}{c-a+ib} \\ &= \int_{-\infty}^{\infty}\frac{dx}{2\pi}f(x) + \frac{i}{2}\frac{1}{c-a+ib}.\end{aligned} \tag{C.5}$$

In total (C.2) then reads

$$\frac{i}{c-a+ib} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) + \frac{i}{2} \frac{1}{c-a+ib} \tag{C.6}$$

and therefore we get for $b > 0$

$$I_1^{\Sigma}\Big|_{b>0} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) = \frac{i}{2} \frac{1}{c-a+ib}. \tag{C.7}$$

For $b < 0$, the pole at x_1 doesn't lie in the contour C and therefore the right hand side of (C.2) is zero. This leads to

$$0 = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) + \frac{i}{2} \frac{1}{c-a+ib} \tag{C.8}$$

and hence

$$I_1^{\Sigma}\Big|_{b<0} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) = -\frac{i}{2} \frac{1}{c-a+ib}. \tag{C.9}$$

The complete solution to (C.1) then is given by

$$I_1^{\Sigma} = \frac{i}{2} \frac{\Theta(b) - \Theta(-b)}{c-a+ib} = \frac{i}{2} \frac{\operatorname{sgn}(b)}{c-a+ib}. \tag{C.10}$$

C.2 Derivation of I_2^{Σ}

Let's evaluate the integral

$$I_2^{\Sigma} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \underbrace{\frac{1}{x^2+bx+c+id}}_{:=g(x)}, \tag{C.11}$$

where the integrand has two complex poles at $x_{\pm} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2-4c-4id}$. For $d > 0$, the pole at x_+ lies in the lower half plane and x_- in the upper half plane, whereas for $d < 0$ the opposite is the case. By closing the contour in the upper half plane, we therefore get

$$\begin{aligned} I_2^{\Sigma} &= \Theta(d)\; i\; \operatorname{Res}(x = x_-) + \Theta(-d)\; i\; \operatorname{Res}(x = x_+) = \frac{i\;\Theta(d)}{x_- - x_+} + \frac{i\;\Theta(-d)}{x_+ - x_-} \\ &= i\frac{\Theta(d) - \Theta(-d)}{x_- - x_+} = -i\frac{\operatorname{sgn}(d)}{\sqrt{b^2-4c-4id}}. \end{aligned} \tag{C.12}$$

C.3 Derivation of I_3^{Σ}

The integral

$$I_3^{\Sigma}(q) = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{c(q) - i\operatorname{sgn}(b)\, x}} \tag{C.13}$$

with $c(0) = 0$ and real b can be calculated as follows. First we split the integration region into the intervals $]-\infty, 0]$ and $[0, \infty[$ and then substitute $x \to -x$ in the negative integration region, which yields

$$\begin{aligned} I_3^\Sigma(q) &= \int_0^\infty dx \left(\frac{1}{\sqrt{c(q) - i\mathrm{sgn}\,(b)\, x}} + \frac{1}{\sqrt{c(q) + i\mathrm{sgn}\,(b)\, x}} \right) \\ &= \int_0^\infty dx \left(\frac{1}{\sqrt{c(q) - ix}} + \frac{1}{\sqrt{c(q) + ix}} \right) = 2 \int_0^\infty dx\ \mathrm{Re} \left\{ \frac{1}{\sqrt{c(q) + ix}} \right\} \\ &= 2\ \mathrm{Re} \left\{ \int_0^\infty dx \frac{1}{\sqrt{c(q) + ix}} \right\} = 2\ \mathrm{Re} \left\{ -2i \sqrt{c(q) + ix} \Big|_0^\infty \right\}. \end{aligned} \tag{C.14}$$

Substracting $I_3^\Sigma(0)$ to take care of the diverging term, we get

$$I_3^\Sigma(q) - I_3^\Sigma(0) = 4\ \mathrm{Re} \left\{ i \sqrt{c(q)} \right\}. \tag{C.15}$$

To get a non-zero real part, the square root has to have an imaginary component, but since $c(q) \in \mathbb{R}$ we get that $c(q) < 0$ and hence

$$I_3^\Sigma(q) - I_3^\Sigma(0) = 4\ \mathrm{Re} \left\{ i \sqrt{-|c(q)|} \right\} \Theta\big(-c(q)\big) = -4\ \sqrt{|c(q)|}\ \Theta\big(-c(q)\big). \tag{C.16}$$

C.4 Derivation of I_4^Σ

In the derivation of the Fermi self-energy there was an integral of the form

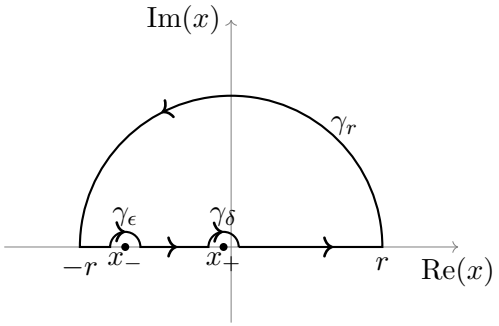
$$I_4^\Sigma = \int_{-\infty}^\infty \frac{dx}{2\pi} \frac{1}{x^2 + bx + c} = \int_{-\infty}^\infty \frac{dx}{2\pi} \frac{1}{(x - x_+)(x - x_-)} \tag{C.17}$$

where the integrand has poles at $x_\pm = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$. Assuming that $4c > b^2$, the poles $x_\pm = -\frac{b}{2} \pm \frac{i}{2}\sqrt{4c - b^2}$ get imaginary, where x_+ lies in the upper half plane, whereas x_- lies in the lower half plane. Closing the contour in the upper half plane, the integral evaluates to

$$I_4^\Sigma = i\ \mathrm{Res}(x = x_+) = \frac{i}{x_+ - x_-} = \frac{1}{\sqrt{4c - b^2}} \qquad \text{for } 4c > b^2. \tag{C.18}$$

On the other hand, when $b^2 > 4c$, the poles are real and thus we need to use the principal value to get a finite result. We choose the contour shown in the figure below, s.t.

$$\oint_C \frac{dx}{2\pi} = \left(\int_{-r}^{x_- - \epsilon} + \int_{\gamma_\epsilon} + \int_{x_- + \epsilon}^{x_+ - \delta} + \int_{\gamma_\delta} + \int_{x_+ + \delta}^{r} + \int_{\gamma_r} \right) \frac{dx}{2\pi}. \tag{C.19}$$



Since there aren't any poles enclosed in our contour, the integral over the closed contour C is equal to zero. In the limit $r \to \infty$, the contribution of the path γ_r vanishes since the integrand decays fast enough. Further taking the limit $\epsilon \to 0$ we get

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{dx}{2\pi}\frac{1}{(x-x_+)(x-x_-)} = -\frac{i}{2}\mathrm{Res}(x=x_-) = -\frac{i}{2}\frac{1}{x_- - x_+} = \frac{i}{2}\frac{1}{\sqrt{b^2-4c}} \tag{C.20}$$

as well as for $\delta \to 0$

$$\lim_{\delta\to 0}\int_{\gamma_\delta}\frac{dx}{2\pi}\frac{1}{(x-x_+)(x-x_-)} = -\frac{i}{2}\mathrm{Res}(x=x_+) = -\frac{i}{2}\frac{1}{x_+ - x_-} = -\frac{i}{2}\frac{1}{\sqrt{b^2-4c}}. \tag{C.21}$$

Hence the right hand side of (C.19) reads

$$\begin{aligned}
&\lim_{r\to\infty}\lim_{\epsilon\to 0}\lim_{\delta\to 0}\left(\int_{-r}^{x_- -\epsilon} + \int_{\gamma_\epsilon} + \int_{x_-+\epsilon}^{x_+-\delta} + \int_{\gamma_\delta} + \int_{x_++\delta}^{r} + \int_{\gamma_r}\right)\frac{dx}{2\pi}\frac{1}{(x-x_+)(x-x_-)} \\
&= \left(\int_{-\infty}^{x_-} + \int_{x_-}^{x_+} + \int_{x_+}^{\infty}\right)\frac{dx}{2\pi}\frac{1}{(x-x_+)(x-x_-)} + \frac{i}{2}\frac{1}{\sqrt{b^2-4c}} - \frac{i}{2}\frac{1}{\sqrt{b^2-4c}} \\
&= \int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{1}{(x-x_+)(x-x_-)}.
\end{aligned} \tag{C.22}$$

In total (C.19) then leads to

$$I_4^\Sigma = 0 \qquad \text{for } b^2 > 4c. \tag{C.23}$$

Combining both cases, we finally get

$$I_4^\Sigma = \frac{\Theta\left(4c-b^2\right)}{\sqrt{4c-b^2}}. \tag{C.24}$$

D Vertex correction and Cancellation of Divergencies

D.1 Feynman Rules

The lowest order contribution to the first vertex

$$\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)\right\rangle = -\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{\text{int}}\right\rangle_0 + \mathcal{O}\left(\lambda^3\right) \tag{D.1}$$

can be calculated using the usual procedure of contracting the fields

$$\begin{aligned}
&-\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{\text{int}}\right\rangle_0 \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}}\int_{q,l} 2\times\left\langle \underbrace{\Phi(p)\overline{\Psi}_a(k+p)}\overbrace{\overline{\Psi}_b(-k)\underbrace{\overline{\Phi}(q)}\Psi_c(l+q)}\sigma_y^{cd}\overbrace{\Psi_d(-l)}\right\rangle_0 \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\int_{q,l} D(p)\delta_{p,q}G_{da}(k+p)\delta_{k+p,-l}G_{cb}(-k)\delta_{-k,l+q}\sigma_y^{cd} \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}D(p)G^T_{ad}(k+p)\left(\sigma_y^T\right)^{dc}G_{cb}(-k) \\
&\qquad = D(p)\left(-G^T\right)_{ad}(k+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)^{dc}G_{cb}(-k).
\end{aligned} \tag{D.2}$$

Note that we chose the momenta of the external fields such that when evaluating the last integral, a δ-function of the form $\delta_{k,k} = (2\pi)^{(d+1)}\delta^{(d+1)}(0)$ remains, which in principal is infinite. However, the appearance of $\delta(0)$ just indicates that the initial momenta are chosen in such a way that energy and momentum are conserved. When using arbitrary initial momenta, $\delta(0)$ just turns into the usual δ-function ensuring energy and momentum conservation.

Similarly, the lowest order contribution to the second vertex

$$\left\langle \overline{\Phi}(p)\Psi_a(k+p)\Psi_b(-k)\right\rangle = -\left\langle \overline{\Phi}(p)\Psi_a(k+p)\Psi_b(-k)S_{\text{int}}\right\rangle_0 + \mathcal{O}\left(\lambda^3\right) \tag{D.3}$$

is evaluated to

$$\begin{aligned}
&-\left\langle \overline{\Phi}(p)\Psi_a(k+p)\Psi_b(-k)S_{\text{int}}\right\rangle_0 \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}}\int_{q,l} 2\times\left\langle \underbrace{\overline{\Phi}(p)}\overbrace{\Psi_a(k+p)\underbrace{\Psi_b(-k)\Phi(q)}\overline{\Psi}_c(l+q)}\sigma_y^{cd}\overbrace{\overline{\Psi}_d(-l)}\right\rangle_0 \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\int_{q,l} D(p)\delta_{p,q}G_{ad}(k+p)\delta_{k+p,-l}G_{bc}(-k)\delta_{-k,l+q}\sigma_y^{cd} \\
&\qquad = \frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}D(p)G_{ad}(k+p)\left(\sigma_y^T\right)^{dc}G^T_{cb}(-k) \\
&\qquad = D(p)G_{ad}(k+p)\left(\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{\sqrt{N}}\sigma_y\right)^{dc}\left(-G^T\right)_{cb}(-k).
\end{aligned} \tag{D.4}$$

D.2 Vertex Corrections

The one-loop vertex diagram shown in (7.16) appears at third order in λ, s.t.

$$\begin{aligned}
&-\frac{1}{3!}\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{\text{int}}^3\right\rangle_0 = -\frac{1}{3!}\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)(S_{\text{int},1}+S_{\text{int},2})^3\right\rangle_0 \\
&= -\frac{1}{2}\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)S_{\text{int},1}S_{\text{int},2}^2\right\rangle_0 . \qquad \text{(D.5)}
\end{aligned}$$

and therefore

$$\begin{aligned}
-\frac{1}{2}\left(-\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}}\right)^3 \int_{l_1,l_2,l_3,q_1,q_2,q_3} &\left\langle \Phi(p)\overline{\Psi}_a(k+p)\overline{\Psi}_b(-k)\Phi(q_1)\overline{\Psi}_c(l_1+q_1)\sigma_y^{cd}\overline{\Psi}_d(-l_1)\overline{\Phi}(q_2)\right. \\
&\left.\Psi_e(l_2+q_2)\sigma_y^{ef}\Psi_f(-l_2)\overline{\Phi}(q_3)\Psi_g(l_3+q_3)\sigma_y^{gh}\Psi_h(-l_3)\right\rangle_0 . \qquad \text{(D.6)}
\end{aligned}$$

To get the mathematical expression that corresponds to the diagram (7.18), we have to contract the external boson and the second external fermion to the same vertex. This leaves two possibilities of contracting the external boson, two possibilities of contracting the second external fermion to the same vertex as the boson and another two possibilities to contract the first external fermion to a different vertex. Together with another factor of two for contracting the remaining fields we have

$$\begin{aligned}
&-\frac{1}{2^4}\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\int_{l_1,l_2,l_3,q_1,q_2,q_3} 2^4\sigma_y^{cd}\sigma_y^{ef}\sigma_y^{gh} \\
&\quad\left\langle \Phi_p\overline{\Psi}_{a,k+p}\overline{\Psi}_{b,-k}\Phi_{q_1}\overline{\Psi}_{c,l_1+q_1}\overline{\Psi}_{d,-l_1}\overline{\Phi}_{q_2}\Psi_{e,l_2+q_2}\Psi_{f,-l_2}\overline{\Phi}_{q_3}\Psi_{g,l_3+q_3}\Psi_{h,-l_3}\right\rangle_0 \\
&= -\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\sigma_y^{cd}\sigma_y^{ef}\sigma_y^{gh}\int_{l_1,l_2,l_3,q_1,q_2,q_3} D(p)\delta_{p,q_3}D(q_1)\delta_{q_1,q_2}G_{ea}(k+p)\delta_{k+p,l_2+q_2}G_{gb}(-k) \\
&\qquad\qquad \delta_{-k,l_3+q_3}G_{hc}(-l_3)\delta_{-l_3,l_1+q_1}G_{fd}(-l_1)\delta_{l_1,l_2} \\
&= -\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\sigma_y^{cd}\sigma_y^{ef}\sigma_y^{gh}\int_{q_1,l_1,l_3} D(p)D(q_1)G_{ea}(k+p)\delta_{k+p,l_1+q_1}G_{gb}(-k)\delta_{-k,l_3+p} \\
&\qquad\qquad G_{hc}(-l_3)\delta_{-l_3,l_1+q_1}G_{fd}(-l_1) \\
&= -\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\sigma_y^{cd}\sigma_y^{ef}\sigma_y^{gh}\int_q D(p)D(q)G_{ea}(k+p)G_{gb}(-k)G_{hc}(k+p)G_{fd}(q-k-p) \\
&= D(p)G_{ae}^T(k+p)\left[-\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\int_q \sigma_y^{ef}G_{fd}(q-k-p)\left(\sigma_y^T\right)^{dc}G_{ch}^T(k+p)D(q)\left(\sigma_y^T\right)^{hg}\right]G_{gb}(-k) \\
&= D(p)G_{ae}^T(k+p)\left[-\frac{i\lambda^3\mu^{\frac{3}{2}\epsilon}}{N^{\frac{3}{2}}}\int_q \sigma_y G(q-k-p)D(q)\sigma_y G^T(k+p)\sigma_y\right]^{eh}G_{gb}(-k). \qquad \text{(D.7)}
\end{aligned}$$

In the line with the contractions, we wrote the momentum arguments as indices to keep the expression short enough.

E RG Analysis

E.1 Anomalous Dimensions and Dynamical Critical Exponents

The anomalous dimensions of the fields are given by

$$\eta_{\Psi/\Phi} = \frac{1}{2}\frac{d\ln Z_{\Psi/\Phi}}{d\ln\mu}. \tag{E.1}$$

and the dynamical critical exponents by

$$z_{d-1}^{-1}(\lambda,a) = 1 + \frac{d\ln Z_2}{d\ln\mu}, \qquad z_d^{-1}(\lambda,a) = 1 + \frac{d\ln Z_3}{d\ln\mu}. \tag{E.2}$$

Here we derive their explicit expressions stated in (8.37).

The anomalous dimension of the fermion fields can be rewritten as

$$\begin{aligned}\eta_\Psi &= \frac{1}{2}\mu Z_2 Z_3^{\frac{1}{2}}\frac{d}{d\mu}\left(Z_2^{-1}Z_3^{-\frac{1}{2}}\right) = \frac{1}{2}\left(-\frac{\mu}{Z_2}\frac{dZ_2}{d\mu} - \frac{1}{2}\frac{\mu}{Z_3}\frac{dZ_3}{d\mu}\right)\\ &= -\frac{1}{2}\beta_\lambda\left(\frac{Z_2'}{Z_2} + \frac{1}{2}\frac{Z_3'}{Z_3}\right) - \frac{1}{2}\beta_a\left(\frac{\dot Z_2}{Z_2} + \frac{1}{2}\frac{\dot Z_3}{Z_3}\right)\end{aligned} \tag{E.3}$$

and after multiplication with Z_2Z_3 as

$$Z_2Z_3\eta_\Psi = -\frac{1}{2}\beta_\lambda\left(Z_2'Z_3 + \frac{1}{2}Z_2Z_3'\right) - \frac{1}{2}\beta_a\left(\dot Z_2 Z_3 + \frac{1}{2}Z_2\dot Z_3\right). \tag{E.4}$$

Similar to the derivation of the β-functions, we compare the regular parts to obtain

$$\begin{aligned}\eta_\Psi &= -\frac{1}{2}\beta_\lambda^{(1)}\left(Z_{2,1}' + \frac{1}{2}Z_{3,1}'\right) = \frac{\lambda}{4}\left(-\frac{4u_1\lambda}{N(1-a)\sqrt{|\tilde a|}} + \frac{2u_1\lambda}{N(1-a)^2\sqrt{|\tilde a|}}\right)\\ &= \frac{u_1\lambda^2}{2}\frac{2a-1}{N(1-a)^2\sqrt{|\tilde a|}}.\end{aligned} \tag{E.5}$$

The anomalous dimension of the boson fields is derived by multiplying

$$\begin{aligned}\eta_\Phi &= \frac{1}{2}\mu Z_2 Z_3^{\frac{3}{2}} Z_4^{-1}\frac{d}{d\mu}\left(Z_2^{-1}Z_3^{-\frac{3}{2}}Z_4\right) = \frac{1}{2}\left(-\frac{\mu}{Z_2}\frac{dZ_2}{d\mu} - \frac{3}{2}\frac{\mu}{Z_3}\frac{dZ_3}{d\mu} + \frac{\mu}{Z_4}\frac{dZ_4}{d\mu}\right)\\ &= -\frac{1}{2}\beta_\lambda\left(\frac{Z_2'}{Z_2} + \frac{3}{2}\frac{Z_3'}{Z_3} - \frac{Z_4'}{Z_4}\right) - \frac{1}{2}\beta_a\left(\frac{\dot Z_2}{Z_2} + \frac{3}{2}\frac{\dot Z_3}{Z_3} - \frac{\dot Z_4}{Z_4}\right)\end{aligned} \tag{E.6}$$

with $Z_2Z_3Z_4$ and comparing the regular parts

$$\begin{aligned}\eta_\Phi &= -\frac{1}{2}\beta_\lambda^{(1)}\left(Z_{2,1}' + \frac{3}{2}Z_{3,1}' - Z_{4,1}'\right) = \frac{\lambda}{4}\left(-\frac{4u_1\lambda}{N(1-a)\sqrt{|\tilde a|}} + \frac{6u_1\lambda}{N(1-a)^2\sqrt{|\tilde a|}} + u_1\lambda\right)\\ &= \frac{u_1\lambda^2}{2}\left(\frac{2a+1}{N(1-a)^2\sqrt{|\tilde a|}} + \frac{1}{2}\right).\end{aligned} \tag{E.7}$$

Comparing the finite parts in the limit $\epsilon \to 0$ of the equation

$$Z_2 z_{d-1}^{-1} = Z_2 + \mu \frac{dZ_2}{d\mu} = Z_2 + \beta_\lambda Z_2' + \beta_a \dot{Z}_2 \tag{E.8}$$

gives the expression of the first anomalous dimension

$$z_{d-1}^{-1} = 1 + \beta_\lambda^{(1)} Z_{2,1}' = 1 + \frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}. \tag{E.9}$$

The regular parts of the equation

$$Z_3 z_d^{-1} = Z_3 + \mu \frac{dZ_3}{d\mu} = Z_3 + \beta_\lambda Z_3' + \beta_a \dot{Z}_3 \tag{E.10}$$

yield the second anomalous dimension

$$z_d^{-1} = 1 + \beta_\lambda^{(1)} Z_{3,1}' = 1 - \frac{2u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}. \tag{E.11}$$

E.2 Renormalized Fermion Two-Point Function

First we show that the constant terms in (8.36) indeed sum up to one when setting $m = 1$, $n = 0$ and $\beta_{\lambda,a} = 0$. Given the expressions in (8.37), we can calculate

$$\begin{aligned}
&- 2\eta_\Psi - z_{d-1}^{-1} - \frac{1}{2} z_d^{-1} + \frac{5}{2} \\
&= -\frac{(2a-1)u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}} - 1 - \frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}} - \frac{1}{2} + \frac{u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}} + \frac{5}{2} \\
&= \frac{u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}\big(-2a + 1 - 2 + 2a + 1\big) + 1 = 1.
\end{aligned} \tag{E.12}$$

Furthermore, we stated that the differential equation

$$\left[\mathbf{K}\nabla_{\mathbf{K}} + \frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}} + \frac{k_d}{2z_d}\frac{\partial}{\partial k_d} + 1\right] G(k) = 0. \tag{E.13}$$

is solved by any function of the form

$$G(k) = \frac{1}{|k_d|^{2z_d}}\, f\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}}, \frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right). \tag{E.14}$$

This can be seen by using

$$\begin{aligned}
\mathbf{K}\nabla_{\mathbf{K}} |\mathbf{K}| &= \mathbf{K}\frac{\mathbf{K}}{|\mathbf{K}|} = |\mathbf{K}|\,, \\
\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}\Big(\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}\Big) &= \frac{k_{d-1}}{z_{d-1}} z_{d-1} k_{d-1} \operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}-2} \\
&= \operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}\,,
\end{aligned}$$

$$\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}\left(\frac{1}{|k_d|^{2z_d}}\right)=\frac{k_d}{2z_d}\left(-\frac{2z_dk_d}{|k_d|^{2z_d+2}}\right)=-\frac{1}{|k_d|^{2z_d}} \tag{E.15}$$

and calculating the derivatives of $G(k)$ explicitly

$$\begin{aligned}
\mathbf{K}\nabla_{\mathbf{K}}G(k)&=\frac{\mathbf{K}}{|k_d|^{2z_d}}\nabla_{\mathbf{K}}f\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}},\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right)\\
&=\frac{1}{|k_d|^{2z_d}}\mathbf{K}\nabla_{\mathbf{K}}\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}}\right)f_1'=\frac{|\mathbf{K}|}{|k_d|^{4z_d}}f_1',\\
\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}G(k)&=\frac{1}{|k_d|^{2z_d}}\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}\left(\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right)f_2'=\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{4z_d}}f_2',\\
\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}G(k)&=\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}\left(\frac{1}{|k_d|^{2z_d}}\right)f+\frac{1}{|k_d|^{2z_d}}\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}}\right)f_1'\\
&\quad+\frac{1}{|k_d|^{2z_d}}\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}\left(\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right)f_2'\\
&=-\frac{1}{|k_d|^{2z_d}}f-\frac{|\mathbf{K}|}{|k_d|^{4z_d}}f_1'-\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{4z_d}}f_2',
\end{aligned} \tag{E.16}$$

which in total leads to

$$\begin{aligned}
&\left[\mathbf{K}\nabla_{\mathbf{K}}+\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}+\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}+1\right]G(k)=\frac{|\mathbf{K}|}{|k_d|^{4z_d}}f_1'+\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{4z_d}}f_2'\\
&-\frac{1}{|k_d|^{2z_d}}f-\frac{|\mathbf{K}|}{|k_d|^{4z_d}}f_1'-\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{4z_d}}f_2'+\frac{1}{|k_d|^{2z_d}}f=0.
\end{aligned} \tag{E.17}$$

E.3 Renormalized Boson Two-Point Function

For $m=0$, $n=1$ and $\beta_{\lambda,a}=0$ the constant terms in (8.36) sum up

$$\begin{aligned}
&-u_1\lambda^2\left(\frac{2a+1}{N(1-a)^2\sqrt{|\tilde{a}|}}+\frac{1}{2}\right)+\frac{5}{2}-1-\frac{2u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}}-\frac{1}{2}+\frac{u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}\\
&-\frac{u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}(2a+1+2-2a-1)-\frac{u_1\lambda^2}{2}+1=1-\frac{2u_1\lambda^2}{N(1-a)^2\sqrt{|\tilde{a}|}}-\frac{u_1\lambda^2}{2}\\
&=z_d^{-1}-\frac{u_1\lambda^2}{2}=p,
\end{aligned} \tag{E.18}$$

yielding the differential equation

$$\left[\mathbf{K}\nabla_{\mathbf{K}}+\frac{k_{d-1}}{z_{d-1}}\frac{\partial}{\partial k_{d-1}}+\frac{k_d}{2z_d}\frac{\partial}{\partial k_d}+p\right]D(k)=0. \tag{E.19}$$

This is solved by any function of the form

$$D(k)=\frac{1}{|k_d|^{2z_dp}}\;g\left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}},\frac{\operatorname{sgn}(k_{d-1})\,|k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}}\right), \tag{E.20}$$

because differentiating this function gives

$$\begin{aligned}
\mathbf{K}\nabla_{\mathbf{K}} D(k) &= \frac{1}{|k_d|^{2z_d p}} \mathbf{K}\nabla_{\mathbf{K}} \left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}} \right) g_1' = \frac{|\mathbf{K}|}{|k_d|^{2z_d p + 2z_d}} g_1' \\
\frac{k_{d-1}}{z_{d-1}} \frac{\partial}{\partial k_{d-1}} D(k) &= \frac{1}{|k_d|^{2z_d p}} \frac{k_{d-1}}{z_{d-1}} \frac{\partial}{\partial k_{d-1}} \left(\frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}} \right) g_2' \\
&= \frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d p + 2z_d}} g_2' \\
\frac{k_d}{2z_d} \frac{\partial}{\partial k_d} D(k) &= \frac{k_d}{2z_d} \frac{\partial}{\partial k_d} \left(\frac{1}{|k_d|^{2z_d p}} \right) g + \frac{1}{|k_d|^{2z_d p}} \frac{k_d}{2z_d} \frac{\partial}{\partial k_d} \left(\frac{|\mathbf{K}|}{|k_d|^{2z_d}} \right) g_1' \\
&\quad + \frac{1}{|k_d|^{2z_d p}} \frac{k_d}{2z_d} \frac{\partial}{\partial k_d} \left(\frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d}} \right) g_2' \\
&= -p \frac{1}{|k_d|^{2z_d p}} g - \frac{|\mathbf{K}|}{|k_d|^{2z_d p + 2z_d}} g_1' - \frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d p + 2z_d}} g_2', \qquad \text{(E.21)}
\end{aligned}$$

which in total leads to

$$\begin{aligned}
&\left[\mathbf{K}\nabla_{\mathbf{K}} + \frac{k_{d-1}}{z_{d-1}} \frac{\partial}{\partial k_{d-1}} + \frac{k_d}{2z_d} \frac{\partial}{\partial k_d} + p \right] D(k) = \frac{|\mathbf{K}|}{|k_d|^{2z_d p + 2z_d}} g_1' + \frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d p + 2z_d}} g_2' \\
&- p \frac{1}{|k_d|^{2z_d p}} g - \frac{|\mathbf{K}|}{|k_d|^{2z_d p + 2z_d}} g_1' - \frac{\operatorname{sgn}(k_{d-1}) |k_{d-1}|^{z_{d-1}}}{|k_d|^{2z_d p + 2z_d}} g_2' + p \frac{1}{|k_d|^{2z_d p}} g = 0. \qquad \text{(E.22)}
\end{aligned}$$

F Superconducting Instabilities

The generalization of the Cooper pair action $S_{\rm cp}$ to arbitrary dimensions goes along the same lines as for the CDW action. Recalling the definition of the spinors

$$\Psi_j(k) = \begin{pmatrix} \psi_{+,j}(k) \\ \psi^\dagger_{-,j}(-k) \end{pmatrix}, \qquad \overline{\Psi}_j(k) = \Psi^\dagger_j(k)\sigma_y = \begin{pmatrix} i\psi_{-,j}(-k) & -i\psi^\dagger_{+,j}(k) \end{pmatrix}, \tag{F.1}$$

the action $S_{\rm cp}$ can be rewritten as

$$\begin{aligned} S_{cp} &= g\int_k \left[\psi_{+,\uparrow}(k)\psi_{-,\downarrow}(-k) + \overline{\psi}_{-,\downarrow}(-k)\overline{\psi}_{+,\uparrow}(k) - \psi_{+,\downarrow}(k)\psi_{-,\uparrow}(-k) - \overline{\psi}_{-,\uparrow}(-k)\overline{\psi}_{+,\downarrow}(k)\right] \\ &= g\int_k \left[\Psi^1_\uparrow(k)(-i)\overline{\Psi}^1_\downarrow(k) + \Psi^2_\downarrow(k)i\overline{\Psi}^2_\uparrow(k) - \Psi^1_\downarrow(k)(-i)\overline{\Psi}^1_\uparrow(k) - \Psi^2_\uparrow(k)i\overline{\Psi}^2_\downarrow(k)\right] \\ &= g\int_k i\left[\overline{\Psi}^1_\downarrow(k)\Psi^1_\uparrow(k) + \overline{\Psi}^2_\downarrow(k)\Psi^2_\uparrow(k) - \overline{\Psi}^1_\uparrow(k)\Psi^1_\downarrow(k) - \overline{\Psi}^2_\uparrow(k)\Psi^2_\downarrow(k)\right] \\ &= g\int_k \left[i\overline{\Psi}_\downarrow(k)\Psi_\uparrow(k) - i\overline{\Psi}_\uparrow(k)\Psi_\downarrow(k)\right] = g\int_k \sigma_y^{\alpha\beta}\overline{\Psi}_\alpha(k)\Psi_\beta(k) \end{aligned} \tag{F.2}$$

and the generalization to arbitrary dimensions follows directly from $\int \frac{dk_0 d^2\mathbf{k}}{(2\pi)^3} \to \int \frac{d^{d-1}\mathbf{K} dk_{d-1} dk_d}{(2\pi)^{d+1}}$.

Next we calculate the contractions of the vertex factor and the lowest order vertex correction in

$$\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)\right\rangle = -\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)S_{int}\right\rangle_0 - \frac{1}{3!}\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)S^3_{int}\right\rangle^{con}_0 \tag{F.3}$$

explicitly. For the vertex factor we obtain

$$\begin{aligned} -\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)S_{int}\right\rangle_0 &= -g\int_k \left\langle \overline{\Psi}^a_\alpha(q)\overline{\Psi^b_\beta(q)\sigma_y^{\mu\nu}\overline{\Psi}^c_\mu(k)}\delta_{cd}\Psi^d_\nu(k)\right\rangle_0 \\ &= g\int_k G_{da}(q)\delta_{q,k}\delta_{\alpha\nu}G_{bc}(q)\delta_{q,k}\delta_{\beta\mu}\sigma_y^{\mu\nu}\delta_{cd} \\ &= gG_{da}(q)G_{bc}(q)\sigma_y^{\beta\alpha}\delta_{cd} = G^T_{ad}(q)\left(-g\sigma_y^{\alpha\beta}\mathbb{1}\right)_{dc}G^T_{cb}(q). \end{aligned} \tag{F.4}$$

The third order term - the one-loop vertex correction - can be evaluated to be

$$\begin{aligned} &-\frac{1}{3!}\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)S^3_{int}\right\rangle^{con}_0 = -g\left(-\frac{i\lambda\mu^{\frac{\epsilon}{2}}}{2\sqrt{N}}\right)^2\int_{k_1,p_1,k_2,p_2,k_3} 2^2\sigma_y^{cd}\sigma_y^{ef}\delta_{gh}\sigma_y^{\gamma\delta}\times \\ &\left\langle \overline{\Psi}^a_\alpha(q)\overline{\Psi^b_\beta(q)\underbrace{\Phi(p_1)\overline{\Psi}^c_\mu(k_1+p_1)}\overline{\Psi}^d_\mu(-k_1)\overline{\Phi}(p_2)}\Psi^e_\nu(k_2+p_2)\overline{\Psi^f_\nu(-k_2)\overline{\Psi}^g_\gamma(k_3)}\Psi^h_\delta(k_3)\right\rangle^{con}_0 \\ &= -\frac{g\lambda^2\mu^\epsilon}{N}\int_{k_1,p_1,k_2,p_2,k_3} G_{ea}(q)\delta_{q,k_2+p_2}\delta_{\alpha\nu}G_{bc}(q)\delta_{q,k_1+p_1}\delta_{\beta\mu}D(p_1)\delta_{p_1,p_2}G_{hd}(k_3)\delta_{k_3,-k_1}\delta_{\mu\delta} \\ &\qquad G_{fg}(k_3)\delta_{k_3,-k_2}\delta_{\nu\gamma}\sigma_y^{cd}\sigma_y^{ef}\delta_{gh}\sigma_y^{\gamma\delta} \\ &= -\frac{g\lambda^2\mu^\epsilon}{N}\int_{p_1,k_3} G_{ea}(q)\delta_{q,-k_3+p_1}G_{bc}(q)\delta_{q,-k_3+p_1}D(p_1)G_{hd}(k_3)G_{fg}(k_3)\sigma_y^{cd}\sigma_y^{ef}\delta_{gh}\sigma_y^{\gamma\delta}\delta_{\alpha\gamma}\delta_{\beta\delta} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{g\lambda^2\mu^\epsilon}{N}\int_{k_3} G_{ea}(q)G_{bc}(q)D(k_3+q)G_{hd}(k_3)G_{fg}(k_3)\sigma_y^{cd}\sigma_y^{ef}\delta_{gh}\sigma_y^{\alpha\beta}\\
&= g\sigma_y^{\alpha\beta}G^T_{ae}(q)\left[-\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y^{ef}G_{fg}(p)\delta_{gh}G_{hd}(p)\left(\sigma_y^T\right)^{dc}D(p+q)\right]G^T_{cb}(q)\\
&= g\sigma_y^{\alpha\beta}G^T_{ae}(q)\underbrace{\left[\frac{\lambda^2\mu^\epsilon}{N}\int_p \sigma_y G^2(p)\sigma_y D(p+q)\right]^{ec}}_{:=V_{ec}(q)}G^T_{cb}(q). \qquad \text{(F.5)}
\end{aligned}
$$

Note that the sign of the contraction is

$$
\overline{\Psi}\Psi\overline{\Psi}\overline{\Psi}\Psi\Psi\overline{\Psi}\Psi = -\overline{\Psi}\Psi\overline{\Psi}\overline{\Psi}\Psi\overline{\Psi}\Psi\overline{\Psi}\Psi = -1. \qquad \text{(F.6)}
$$

Finally, we show that the counterterm in (10.15) indeed cancels the diverging term exactly. Treating it as an interaction, the lowest order contribution to the full vertex factor is given by

$$
\begin{aligned}
&-\left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)S_{int}\right\rangle_0\\
&= -g\int_k \left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)\sigma_y^{\mu\nu}\overline{\Psi}^c_\mu(k)\delta_{cd}\Psi^d_\nu(k)\right\rangle_0 - g\int_k \left\langle \overline{\Psi}^a_\alpha(q)\Psi^b_\beta(q)\sigma_y^{\mu\nu}\overline{\Psi}^c_\mu(k)\delta_{cd}\frac{Z_{g,1}}{\epsilon}\Psi^d_\nu(k)\right\rangle_0\\
&= g\int_k G_{da}(q)\delta_{q,k}\delta_{\alpha\nu}G_{bc}(q)\delta_{q,k}\delta_{\beta\mu}\sigma_y^{\mu\nu}\delta_{cd} + g\frac{Z_{g,1}}{\epsilon}\int_k G_{da}(q)\delta_{q,k}\delta_{\alpha\nu}G_{bc}(q)\delta_{q,k}\delta_{\beta\mu}\sigma_y^{\mu\nu}\delta_{cd}\\
&= G^{ad}(q)\left(g\sigma_y^{\beta\alpha}\delta_{dc} + g\frac{Z_{g,1}}{\epsilon}\sigma_y^{\beta\alpha}\delta_{dc}\right)G^T_{cb}(q) = G^{ad}(q)\left(-g\sigma_y^{\alpha\beta}\mathbb{1} - g\frac{Z_{g,1}}{\epsilon}\sigma_y^{\alpha\beta}\mathbb{1}\right)_{dc}G^T_{cb}(q)\\
&= G^{ad}(q)\left(-g\sigma_y^{\alpha\beta}\mathbb{1} + g\sigma_y^{\alpha\beta}\mathbb{1}\frac{4u_1\lambda^2}{N(1-a)\sqrt{|\tilde{a}|}\epsilon}\right)_{dc}G^T_{cb}(q) \qquad \text{(F.7)}
\end{aligned}
$$

and therefore has the right sign to cancel the ϵ-pole of the one-loop correction.

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Erklärung:

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst zu haben und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt zu haben.

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