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Return probability and scaling exponents in the critical random matrix ensemble

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Abstract

We study an asymptotic behavior of the return probability for the critical random matrix ensemble in the regime of strong multifractality. The return probability is expected to show critical scaling in the limit of large time or large system size. Using the supersymmetric virial expansion, we confirm the scaling law and find analytical expressions for the fractal dimension of the wavefunctions d_2 and the dynamical scaling exponent μ. By comparing them, we verify the validity of Chalker's ansatz for dynamical scaling.

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1. Introduction

It is well known that the wavefunctions at the point of the Anderson metal–insulator transition are *fractal* [1]. Their amplitudes exhibit self-similar fluctuations at different spatial scales. The standard way to quantify such a complicated behavior is to consider the scaling of moments of the wavefunctions $\psi_n(\mathbf{r})$ with the system size L:

$$I_q = \sum_{\mathbf{r}} \langle |\psi_n(\mathbf{r})|^{2q} \rangle \propto L^{-d_q(q-1)}, \tag{1}$$

where $\langle \cdots \rangle$ stands for averaging over disorder realizations and over a small energy window. The fractal dimension d_q, which is different from zero and from the dimensionality of the space d, is a fingerprint of the fractal wavefunctions. For the *multifractal* wavefunctions, d_q depends non-trivially on q; thus, an infinite set of scaling exponents is required for the full description of the wavefunctions in this case.

Additionally to the non-trivial scaling of the moments of the critical wavefunctions taken at a fixed energy, the correlations of wavefunctions at different energies show the critical behavior as well. The simplest correlation function involving two eigenstates corresponding to two different energies E_m and E_n can be defined as

$$C(\omega) = \sum_{\mathbf{r}} \langle |\psi_n(\mathbf{r})|^2 |\psi_m(\mathbf{r})|^2 \delta(E_m - E_n - \omega)\rangle. \tag{2}$$

As any other correlator at criticality $C(\omega)$ is expected to decay in a power-law fashion

$$C(\omega) \propto (E_0/\omega)^{\mu}, \quad \Delta \ll \omega \ll E_0, \tag{3}$$

where Δ is the mean level spacing and E_0 is a high-energy cutoff. What is more surprising is the fact that the dynamical exponent μ is related to the fractal dimension d_2 in a simple way

$$\mu = 1 - d_2/d. \tag{4}$$

This relation was suggested by Chalker and Daniel [2, 3] and confirmed by a great number of computer simulations [2, 4, 5] thereafter. As $E_0/\omega \gg 1$ and $\mu > 0$, equation (3) implies an enhancement of correlations in critical systems [4] which is possible only if there is a strong overlap of different wavefunctions. This rather counterintuitive picture becomes particularly striking in the regime of strong multifractality $d_2 \ll d$, where eigenstates are very sparse [6] and almost localized. We are not aware of any analytical calculations supporting its validity in this case.

The critical enhancement of the correlations plays an important role in the theory of interacting systems (cf 'multifractal' superconductivity [7] and the Kondo effect [8]) and, therefore, the theory of the critical correlations still attracts considerable attention in spite of its long history.

The aim of this work is to demonstrate that the critical scaling holds true and confirm the validity of the dynamical scaling relation (4) with accuracy up to the second order of the perturbation theory. Some results of this work have been announced in a brief form in [12].

The knowledge of the second-order perturbative results is extremely important because it allows one to confirm the critical scaling. Besides, sub-leading terms in the scaling exponents can reveal some model-dependent features in contrast to the leading-order perturbative result which is universal for a wide class of different critical models [9, 10].

In this paper, we consider a particular model relevant for the Anderson metal–insulator transition—the power-law random banded matrix ensemble [11]. The matrix elements H_{mn} of the Hamiltonian are given by the independent, Gaussian distributed random variables with the only constraint that matrix H is Hermitian. Their mean values are equal to zero and the variances are defined as

$$\langle |H_{mn}|^2\rangle = \frac{1}{2}\begin{cases} 1, & n = m \\ \dfrac{b^2}{(n-m)^2}, & |n-m| \gg \max\{b, 1\}. \end{cases} \tag{5}$$

The long-range power-law decay of typical matrix elements $|H_{mn}|^2$ leads to the critical behavior at any value of the parameter b. This allows one to study the model perturbatively either for $b \gg 1$ or for $b \ll 1$. The latter condition corresponds to the regime of strong multifractality investigated in this work. In this regime, both scaling exponents μ and d_2 can be expanded into the power series in b and then compared term by term.

The structure of the paper is as follows. In section 2, we give an equivalent formulation of the Chalker's ansatz in terms of the return probability. In section 3, the integral representation for the first-order result for the return probability is derived. It is used in section 4 to calculate the scaling exponents in the first order in b. The second-order result for the return probability is discussed in section 5. The corresponding second-order expressions for the dynamical exponent μ and fractal dimension d_2 are derived in sections 6 and 7, respectively. All calculations in sections 3–7 are done for the unitary symmetry class. A brief announcement of analogous results for the orthogonal case is presented in appendix B; more detailed study will be published elsewhere.

2. Return probability and scaling exponents

It is convenient to reformulate Chalker's ansatz (4) in terms of the return probability

$$P_N(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} C(\omega), \tag{6}$$

where a matrix size N plays the role of the system size. Using the definition of $C(\omega)$ (2), it is easy to show that in the limit $t \to \infty$ the return probability tends to a finite limit, which is nothing else than the inverse participation ratio I_2 :

$$\lim_{t/N\to\infty} P_N(t) = I_2 \propto N^{-d_2}. \tag{7}$$

Thus, the knowledge of the return probability gives a way to calculate the fractal dimension d_2. On the other hand, considering the limit $N \to \infty$ at a fixed large time t, one finds that

$$\lim_{N/t\to\infty} P_N(t) \propto t^{\mu-1}, \tag{8}$$

as it follows from equation (3). So the dynamical scaling exponent μ can also be extracted from the behavior of the return probability. Thus, we conclude that Chalker's ansatz (4) is equivalent to the following statement:

$$\mu - 1 = \lim_{t\to\infty} \frac{\partial}{\partial \ln t} \lim_{N/t\to\infty} \ln P_N(t) = \lim_{N\to\infty} \frac{\partial}{\partial \ln N} \lim_{t/N\to\infty} \ln P_N(t) = -d_2. \tag{9}$$

In the regime of strong multifractality, the return probability can be calculated perturbatively using the method of the virial expansion in a number of resonant states, each of them being localized at a certain site n. The virial expansion formalism was developed in [13] following the initial idea of [14]. The supersymmetric version of the virial expansion [15, 16] is formulated in terms of integrals over supermatrices. In particular, it allows us to represent $P_N(t)$ as an infinite series of integrals over an increasing number of supermatrices associated with different sites

$$P_N(t) = P^{(1)} + P^{(2)} + P^{(3)} + \cdots \tag{10}$$

with $P^{(1)} = 1$ and $P^{(i)} = b^{i-1} f^{(i)}(bt)$. Functions $f^{(j)}$ are governed by a hybridization of j localized states and can be calculated explicitly by means of integrals over j different supermatrices. The above expansion implies the corresponding expansion for $\ln P_N(t)$:

$$\ln P_N(t) = P^{(2)} + \left(P^{(3)} - \tfrac{1}{2}(P^{(2)})^2\right) + \cdots, \tag{11}$$

where the first term is of the first order in b, two terms in the brackets are of the second order in b and so on. This representation allows one to find the corresponding power-law expansions for the fractal dimensions d_2 and the dynamical exponent μ and hence to check Chalker's ansatz in the form of equation (9) order-by-order.

We emphasize that the scaling exponents are finite constants and hence according to equation (9) the leading at $N \to \infty$ or $t \to \infty$ terms in $\ln P_N(t)$ must diverge *logarithmically* with only linear in $\ln N$ or $\ln t$ terms present. Higher-order terms of the form $\ln^m N$ or $\ln^m t$ $(m > 1)$ would generate divergent contributions to d_2 and μ indicating a violation of the power-law scaling. As a matter of fact, both $P^{(3)}$ and $(P^{(2)})^2$ do contain higher-order terms, such as $\ln^2 N$ or $\ln^2 t$. We prove below that these divergent terms cancel out in the combination $\left(P^{(3)} - \frac{1}{2}(P^{(2)})^2\right)$. This is a necessary condition for the existence of the critical scaling and of Chalker's ansatz.

3. Integral representation for $P^{(2)}$

The return probability $P(t)$ can be expressed in terms of Green's functions as

$$P_N(t) = \frac{\Delta^2}{2\pi^2 N} \sum_{p=1}^{N} \langle\langle \mathcal{G}_{pp}(t) \rangle\rangle, \tag{12}$$

where $\langle\langle ab \rangle\rangle \equiv \langle ab \rangle - \langle a \rangle \langle b \rangle$ and the diagonal matrix elements of time-dependent correlator $\mathcal{G}_{pp}(t)$ are related to its energy-dependent counterparts $\mathcal{G}_{pp}(\omega)$ by the Fourier transform

$$\mathcal{G}_{pp}(t) = \frac{1}{\Delta} \int \mathrm{d}\omega \, \mathrm{e}^{-\imath\omega t} \mathrm{Re}\, \mathcal{G}_{pp}(\omega). \tag{13}$$

For the latter quantity, defined by the product of the matrix elements of the retarded and the advance Green's functions $\mathcal{G}_{pp}(\omega) \equiv G^R_{pp}(E+\omega/2) G^A_{pp}(E-\omega/2)$, the perturbation theory has been developed in [15] . The leading-order term of the perturbation theory, corresponding to the diagonal part of a random matrix, is $\mathrm{Re}\, \mathcal{G}_{pp}(\omega) = (2\pi^2/\Delta)\delta(\omega)$. Substituting this expression into equation (12) yields $P^{(1)} = 1$ reproducing the correct normalization of the return probability. The next-order approximation taking into account an 'interaction' between pairs of resonant states is given by equation (55) of [15] . The corresponding result for the return probability reads[4]

$$P^{(2)} = \frac{2\sqrt{\pi}}{Nt} \sum_{n\neq p}^{N} \sum_{k=1}^{\infty} \frac{(-2b_{pn}t^2)^k}{(k-1)!} \frac{k}{2k-1}. \tag{14}$$

This expression was derived for arbitrary variances of the off-diagonal matrix elements $b_{pn} = \frac{1}{2}\langle |H_{pn}|^2 \rangle$; for the PLBRM model, $b_{pn} = \frac{1}{4}(1+|p-n|/b)^{-2}$. In the large N limit, the double sum in equation (14) may be replaced by the integral:

$$\sum_{p=1}^{N} \sum_{n\neq p}^{N} f(|p-n|) \approx 2 \int_0^N \mathrm{d}y \int_0^y \mathrm{d}x\, f(x) \approx 2N \int_0^N \mathrm{d}x\, f(x), \tag{15}$$

where the last equality is justified in appendix A. In the continuum limit, the counterpart of b_{pn} is given by $b^2/4x^2$ (which is valid for $|p-n| \gg b$); however, this expression leads to the appearance of divergent integrals at $x \to 0$ and hence should be regularized. To this end, we replace b_{pn} by $b^2/4x^{2(1-\epsilon)}$ with $\epsilon > 0$ and take the limit $\epsilon \to 0$ at the end of the calculations. Thus, in the continuum limit, we obtain

$$P^{(2)} = \frac{4\sqrt{\pi}}{t} \int_0^N \mathrm{d}x \sum_{k=1}^{\infty} \left(\frac{-b^2t^2}{2x^{2(1-\epsilon)}} \right)^k \frac{k}{(k-1)!} \frac{1}{(2k-1)}. \tag{16}$$

Now, it is convenient to represent the last fraction as an integral $\frac{1}{2k-1} = \int_0^1 \mathrm{d}\tilde{\beta}\tilde{\beta}^{2k-2}, \quad k \geqslant 1,$ and substitute this formula into equation (16):

$$P^{(2)} = \frac{4\sqrt{\pi}}{t} \int_0^N \mathrm{d}x \int_0^1 \mathrm{d}\tilde{\beta} \frac{1}{\tilde{\beta}^2} \sum_{k=1}^{\infty} \left(\frac{-\tilde{\beta}^2 b^2 t^2}{2x^{2(1-\epsilon)}} \right)^k \frac{k}{(k-1)!}. \tag{17}$$

Changing $\tilde{\beta}$ by $\beta = \tilde{\beta} bt/\sqrt{2}$ and using the fact that $\sum_{k=1}^{\infty} (-y)^k \frac{k}{(k-1)!} = -y(1-y)\,\mathrm{e}^{-y}$, we arrive at the following integral representation for the return probability:

$$P^{(2)} = \int_0^{\tau} \frac{\mathrm{d}\beta}{\beta^2} \int_0^N \mathrm{d}x\, F_2\left(\frac{\beta}{x^{1-\epsilon}} \right), \qquad F_2(y) \equiv -2\sqrt{2\pi} b y^2 (1-y^2)\, \mathrm{e}^{-y^2}, \tag{18}$$

where $\tau = bt/\sqrt{2}$.

[4] The right-hand side of equation (55) should be multiplied by $\sqrt{2}$, as in the present calculations we fix $E = 0$, while the averaging over E was performed in [15] .

4. Scaling exponents: the first-order perturbation theory results

The above representation for $P^{(2)}(t)$ (18) is a convenient starting point for the calculation of d_2 and μ. The exponent μ can be extracted from the limit $N \to \infty$ of equation (18):

$$\Pi_2(\tau) \equiv \lim_{N\to\infty} P^{(2)} = \int_0^\tau \mathrm{d}\beta\beta^{\frac{2\epsilon-1}{1-\epsilon}} \int_0^\infty \mathrm{d}y F_2\left(\frac{1}{y^{1-\epsilon}}\right), \qquad y = \beta^{\frac{1}{\epsilon-1}}x. \tag{19}$$

Differentiating $\Pi_2(\tau)$ with respect to $\ln\tau$, we obtain

$$\frac{\partial \Pi_2}{\partial \ln \tau} = \tau^{\frac{\epsilon}{1-\epsilon}} J, \qquad J \equiv \int_0^\infty \mathrm{d}y F_2\left(\frac{1}{y^{1-\epsilon}}\right). \tag{20}$$

The last step in calculating μ is to take the limit $\epsilon \to 0$ in the expression for $\frac{\partial\Pi_2}{\partial\ln\tau}$. The τ-dependent factor then gives 1 and what we need to know is just the zeroth order, i.e. ϵ-independent term, in the ϵ-expansion of J. The required expansion can be found with the help of the following general formula, which can be proved using the integration by parts:

$$\int_0^\infty \mathrm{d}\beta\beta^{\delta-1} f(\beta) = \frac{1}{\delta} f(0) - \int_0^\infty \mathrm{d}\beta \ln\beta \frac{\mathrm{d}f}{\mathrm{d}\beta} + O(\delta). \tag{21}$$

To this end, we change the integration variable y by $t = 1/y^{2(1-\epsilon)}$ and apply the above formula:

$$J = -\frac{\sqrt{2\pi}b}{(1-\epsilon)} \int_0^\infty \mathrm{d}t \frac{(1-t)\,\mathrm{e}^{-t}}{t^{-\frac{1}{2(1-\epsilon)}}} = -\frac{\pi b}{\sqrt{2}}\left[1 + \epsilon\left(2 + \frac{\gamma}{2} + \ln 2\right)\right] + O(\epsilon^2), \tag{22}$$

where γ is Euler's constant. We keep the first order in ϵ term, as it is important in the second-order perturbation theory. The zeroth-order term gives the exponent μ:

$$1 - \mu = \frac{\pi b}{\sqrt{2}} + O(b^2). \tag{23}$$

Now, we perform similar calculations for the fractal dimension d_2. First, we introduce in equation (18) the new integration variables $\tilde{x}$ and $\tilde{\beta}$ defined by the relations $x = \tilde{\beta}^{\frac{1}{1-\epsilon}} N\tilde{x}$, $\beta = N^{1-\epsilon}\tilde{\beta}$ and then take the limit $\tau \to \infty$:

$$\Pi_2(N) \equiv \lim_{\tau\to\infty} P^{(2)} = N^\epsilon \int_0^\infty \mathrm{d}\tilde{\beta}\tilde{\beta}^{\frac{2\epsilon-1}{1-\epsilon}} \int_0^{\tilde{\beta}^{-\frac{1}{1-\epsilon}}} \mathrm{d}\tilde{x} F_2\left(\frac{1}{\tilde{x}^{1-\epsilon}}\right). \tag{24}$$

Taking the derivative with respect to $\ln N$ and returning to the previous notation for the integration variables $\tilde{x} \to x$, $\tilde{\beta} \to \beta$, we obtain

$$\frac{\partial \Pi_2}{\partial \ln N} = \epsilon N^\epsilon \int_0^\infty \mathrm{d}\beta\beta^{\frac{2\epsilon-1}{1-\epsilon}} \int_0^{\beta^{-\frac{1}{1-\epsilon}}} \mathrm{d}x F_2\left(\frac{1}{x^{1-\epsilon}}\right). \tag{25}$$

Then, we apply equation (21) with $\delta = \frac{\epsilon}{1-\epsilon}$ and find

$$\int_0^\infty \mathrm{d}\beta\beta^{\frac{2\epsilon-1}{1-\epsilon}} \int_0^{\beta^{-\frac{1}{1-\epsilon}}} \mathrm{d}x F_2\left(\frac{1}{x^{1-\epsilon}}\right) = \frac{1-\epsilon}{\epsilon}\left(\int_0^\infty \mathrm{d}x F_2\left(\frac{1}{x^{1-\epsilon}}\right) + \epsilon \int_0^\infty \mathrm{d}\beta \ln\beta \frac{1}{\beta^2} F_2(\beta) + O(\epsilon^2)\right). \tag{26}$$

The ϵ-expansion of the first integral is given by equation (22), while the second integral can be calculated explicitly using the definition of F_2:

$$\int_0^\infty \mathrm{d}\beta \ln\beta \frac{1}{\beta^2} F_2(\beta) = \frac{\pi b}{\sqrt{2}}(1 + \gamma/2 + \ln 2). \tag{27}$$

Thus, we arrive at the following result for $\frac{\partial \Pi_2}{\partial N}$:

$$\frac{\partial \Pi_2}{\partial \ln N} = N^{\epsilon}\left[-\frac{\pi b}{\sqrt{2}} + O(\epsilon^2)\right]. \tag{28}$$

It is interesting to note that the first order in ϵ term is absent in the above expression. The constant term yields the fractal dimension d_2:[5]

$$d_2 = \frac{\pi b}{\sqrt{2}} + O(b^2). \tag{29}$$

Comparing this result with the corresponding expression for $1-\mu$ equation (23), we conclude that Chalker's ansatz is valid in the first-order perturbation theory.

Leading contributions of order of b to the scaling exponents $1-\mu$ and d_2 in the orthogonal case are calculated in appendix B.

5. Integral representations for $P^{(3)}$

The second-order perturbation result for the return probability can be derived from the corresponding expression for the matrix elements of Green's functions given by equation (72) of [15] :

$$P^{(3)} = \frac{\pi}{16t^2N}\sum_{p=1}^{N}\sum_{\{m,n\neq p\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}(-8b_{pm}t^2)^{k_1}(-8b_{pn}t^2)^{k_2}(-8b_{mn}t^2)^{k_3} \\ \times \frac{\Xi(k_1,k_2,k_3)}{\Gamma(2[k_1+k_2+k_3]-1)}(k_1+k_2)(k_1+k_2-1), \tag{30}$$

where

$$\Xi(k_1,k_2,k_3) = \prod_{i=1}^{3}\frac{\Gamma(k_i-1/2)}{\pi^{1/2}k_i!} \\ \times(2k_1k_2k_3 - k_1k_2 - k_1k_3 - k_2k_3)\,\Gamma(k_1+k_2+k_3-1). \tag{31}$$

First, we multiply and divide the last expression by $2(k_1+k_2+k_3-1)$ and then use the identity $\Gamma(2z) = \frac{1}{\sqrt{2\pi}}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2)$ for $z = k_1+k_2+k_3-1/2$, which allows us to cancel $\Gamma(k_1+k_2+k_3)$:

$$P^{(3)} = \sum_{k_{1,2,3}=0}^{\infty}F(k_1,k_2,k_3)(2k_1k_2k_3 - k_1k_2 - k_1k_3 - k_2k_3)(k_1+k_2)(k_1+k_2-1) \tag{32}$$

with

$$F(k_1,k_2,k_3) = \frac{\pi}{16t^2N}\sum_{p=1}^{N}\sum_{\{m,n\neq p\}}^{N}(-8b_{pm}t^2)^{k_1}(-8b_{pn}t^2)^{k_2}(-8b_{mn}t^2)^{k_3} \\ \times\prod_{i=1}^{3}\frac{\Gamma(k_i-1/2)}{\pi^{1/2}k_i!}\frac{\sqrt{2\pi}\;2^{3/2-2(k_1+k_2+k_3)}}{(k_1+k_2+k_3-1)\Gamma(k_1+k_2+k_3-1/2)}. \tag{33}$$

All terms in equation (32) are symmetric functions of k_1, k_2 and k_3, except for the term $(k_1+k_2)(k_1+k_2-1)$. Symmetrizing it we can write $P^{(3)}$ in the following form:

$$P^{(3)} = \sum_{k_{1,2,3}=0}^{\infty}F(k_1,k_2,k_3)\sum_{\alpha_{1,2,3}}A_{\alpha_1\alpha_2\alpha_3}k_1^{\alpha_1}k_2^{\alpha_2}k_3^{\alpha_3}. \tag{34}$$

[5] The first-order result for d_2 was derived in [17].

The coefficients $A_{\alpha_1\alpha_2\alpha_3}$ are invariant under permutations of the indices and hence all non-zero coefficients can be obtained from the following six:

$$\begin{aligned} &A_{012} = 2/3, \qquad A_{013} = -2/3, \qquad A_{111} = 2, \\ &A_{112} = -10/3, \qquad A_{113} = 4/3, \qquad A_{122} = 4/3. \end{aligned} \tag{35}$$

The next step is to replace the summation over m, n, and p by the integration similarly to how it was done in the calculation of $P^{(2)}$. To this end we note that

$$\begin{aligned} \sum_{p\neq m\neq n} f(|p-m|, |p-n|, |m-n|) &\approx 6\int_0^N \mathrm{d}y \int_0^y \mathrm{d}x_1 \int_{x_1}^y \mathrm{d}x_2 f(|x_1|, |x_2|, |x_2-x_1|) \\ &\approx 6N\int_0^N \mathrm{d}q_1 \int_0^{N-q_1} \mathrm{d}q_2 f(|q_1|, |q_2|, |q_2+q_1|), \end{aligned} \tag{36}$$

provided that $f(x_1, x_2, x_3)$ is invariant under permutations of the arguments. The last equality is again justified in appendix A. In order to be able to sum up over k_i we use the following integral representations:

$$\begin{aligned} \frac{1}{\Gamma(k_1+k_2+k_3-1/2)} &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0} \mathrm{d}s\, \mathrm{e}^s s^{-(k_1+k_2+k_3)+1/2}, \\ \frac{1}{2(k_1+k_2+k_3-1)} &= \int_0^1 \mathrm{d}\tilde{\beta}\,\tilde{\beta}^{2(k_1+k_2+k_3)-3}. \end{aligned} \tag{37}$$

The summation over k_1, k_2 and k_3 can easily be done now. All sums over k_i have the form $\sum_{k=0}^{\infty}(-y)^k \frac{\Gamma(k-1/2)}{\sqrt{\pi}\Gamma(k+1)} k^\alpha \equiv f_\alpha(y)$, $\alpha = 0, 1, 2, 3$. The explicit expressions for f_α are given by

$$\begin{aligned} &f_0(y) = -2\sqrt{1+y}, \qquad f_1(y) = -\frac{y}{\sqrt{1+y}}, \\ &f_2(y) = -\frac{y(2+y)}{2(1+y)^{3/2}}, \qquad f_3(y) = -\frac{y(4+2y+y^2)}{4(1+y)^{5/2}}. \end{aligned} \tag{38}$$

Using this notation and changing $\tilde{\beta}$ by $\beta = \tilde{\beta} bt/\sqrt{2}$, we can write $P^{(3)}$ in a compact form

$$P^{(3)} = \int_0^\tau \frac{\mathrm{d}\beta}{\beta^3} \int_0^N \mathrm{d}x \int_0^{N-x} \mathrm{d}y\, F_3\left(\frac{\beta}{x^{1-\epsilon}}, \frac{\beta}{y^{1-\epsilon}}\right), \tag{39}$$

where F_3 is defined as

$$\begin{aligned} F_3(x, y) = \frac{3\pi^{3/2}b^2}{4\pi\mathrm{i}} &\int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0} \mathrm{d}s\, \mathrm{e}^s \sqrt{s} \sum_{\alpha_{1,2,3}} A_{\alpha_1\alpha_2\alpha_3} \\ &\times f_{\alpha_1}(x^2/s) f_{\alpha_2}(y^2/s) f_{\alpha_3}((x^{\epsilon-1}+y^{\epsilon-1})^{2(\epsilon-1)}/s). \end{aligned} \tag{40}$$

Note that equation (39) is similar to the integral representation for $P^{(2)}$, equation (18).

6. Dynamical scaling exponent μ: the second-order perturbation theory result

To calculate the second-order result for the dynamical scaling exponents μ, one first needs to find

$$\Pi_3(\tau) = \lim_{N\to\infty} \left(P^{(3)} - \tfrac{1}{2}(P^{(2)})^2\right). \tag{41}$$

For $P^{(2)}$ one can use representation (19) allowing to integrate over β explicitly. To exploit a similar trick for $P^{(3)}$, we scale the integration variables in equation (39) $x \to \beta^{\frac{1}{1-\epsilon}} x$,

$y \to \beta^{\frac{1}{1-\epsilon}} y$. Then, τ-dependence get factorized for both terms in (41) and the derivative with respect to $\ln\tau$ can now be taken explicitly:

$$\frac{\partial \Pi_3}{\partial \ln \tau} = \tau^{\frac{2\epsilon}{1-\epsilon}} \left[\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y F_3(x^{\epsilon-1}, y^{\epsilon-1}) - \left(\frac{1-\epsilon}{\epsilon}\right) J^2 \right] \tag{42}$$

with J defined in equation (20).

The ϵ-expansion of J is given by equation (22); its leading term is a constant of order of b. Therefore, $\left(\frac{1-\epsilon}{\epsilon}\right) J^2$ diverges in the limit $\epsilon \to 0$ as $1/\epsilon$. We show below that the first term in the brackets on the rhs of equation (42) also contains a divergent contribution of order $1/\epsilon$, so that two divergent contributions *cancel*. This cancellation is another manifestation of the existence of the critical dynamical scaling.

Now, let us find an ϵ-expansion for the double integral $I \equiv \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y F_3\left(x^{\epsilon-1}, y^{\epsilon-1}\right)$. To this end, we change variables x by $s^{-\frac{1}{2(1-\epsilon)}}\tilde{x}$, y by $s^{-\frac{1}{2(1-\epsilon)}}\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are complex. Then, the integration over s in equation (40) leads to the appearance of the inverse Γ-function (37). Next, we deform the contour of the integration over $\tilde{x}$ and $\tilde{y}$ back to the real axis and obtain

$$I = \frac{3\pi^{3/2}}{2} b^2 \frac{1}{\Gamma\left(\frac{1}{1-\epsilon} - \frac{1}{2}\right)} \int_0^\infty \mathrm{d}x \int_0^\infty dy \frac{G(x, y, \epsilon)}{x^{1-\epsilon} y^{1-\epsilon} (x+y)^{1-\epsilon}}$$

$$G(x, y, \epsilon) = \sum_{\alpha_{1,2,3}} A_{\alpha_1\alpha_2\alpha_3} g_{\alpha_1}(x^{2(\epsilon-1)}) g_{\alpha_2}(y^{2(\epsilon-1)}) g_{\alpha_3}((x+y)^{2(\epsilon-1)}), \tag{43}$$

where the functions g_α are related to f_α defined in equation (38) as $g_\alpha(y) = f_\alpha(y)/\sqrt{y}$. In order to find the ϵ-expansion of the integral over x and y in the above equation, it is convenient to change the variables by $q = x + y$ and $z = x/(x + y)$. In terms of the new variables, the integral, which we denote by I_0, takes the form

$$I_0 = 2 \int_0^{1/2} \mathrm{d}z \int_0^\infty \frac{\mathrm{d}q}{q^{2-3\epsilon}} \frac{1}{z^{1-\epsilon}(1-z)^{1-\epsilon}} G(qz, q(1-z), \epsilon). \tag{44}$$

To derive this equation, we used the fact that G is a symmetric function, i.e. $G(x, y) = G(y, x)$. The leading singular term of the ϵ-expansion of I_0 originates from $z \to 0$ and can be extracted by integration by parts:

$$I_0 = \frac{2}{\epsilon} \left[\frac{z^\epsilon}{z^{1-\epsilon}} \int_0^\infty \frac{\mathrm{d}q}{q^{2-3\epsilon}} G(qz, q(1-z), \epsilon)|_{z=0}^{z=1/2} \right.$$
$$\left. - \int_0^{1/2} \mathrm{d}z z^\epsilon \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{1}{(1-z)^{1-\epsilon}} \int_0^\infty \frac{\mathrm{d}q}{q^{2-3\epsilon}} G(qz, q(1-z), \epsilon) \right\} \right]. \tag{45}$$

Thus, the ϵ-expansion of I_0 has the form $I_0 = (2/\epsilon)[A + \epsilon B + O(\epsilon^2)]$. The coefficient A is obtained by setting $\epsilon = 0$ in all the terms in the brackets in equation (45) and it is equal to

$$A = \int_0^\infty \frac{\mathrm{d}q}{q^2} G(0, q, 0) = \frac{\pi}{6}, \tag{46}$$

where the integral is calculated explicitly using the definition of G (43). To calculate B, we collect the first-order terms in ϵ generated by all ϵ-dependent contributions in equation (45) and find

$$B = \frac{\pi}{2} - \frac{\pi}{6} \ln 2 + R. \tag{47}$$

The first two constants in the above formula are equal to integrals over q similar to the one in equation (46). The constant R is given by the two-dimensional integral

$$R = \int_0^{1/2} \frac{1}{z} \left[\frac{1}{(1-z)} \int_0^\infty \frac{\mathrm{d}q}{q^2} G(qz, q(1-z), 0) - \frac{\pi}{6} \right] \approx 0.276, \tag{48}$$

which we were able to compute only numerically. The derived results for A and B along with the ϵ-expansion of the inverse Γ-function in equation (43) yield

$$I = \frac{\pi^2 b^2}{2\epsilon}\left[1+\epsilon\left(3+\gamma+\ln 2+\frac{6R}{\pi}\right)+O(\epsilon^2)\right]. \tag{49}$$

Substituting this formula as well as the expression for J equation (22) into equation (42), we obtain

$$\frac{\partial \Pi_3}{\partial \ln \tau} = \pi^2 b^2\left(\frac{3R}{\pi}-\frac{\ln 2}{2}\right) \approx -0.819 b^2. \tag{50}$$

An alternative integral representation of this answer can be found in equations (22)–(23) of [12]. We emphasize that the singular $1/\epsilon$ terms cancel giving a finite result in the limit $\epsilon \to 0$. Thus, we have demonstrated the existence of the dynamical scaling with the accuracy of the sub-leading terms of the perturbation theory.

7. Fractal dimension d_2: the second-order perturbation theory result

To calculate the fractal dimension d_2, one needs to deal with

$$\Pi_3(N) = \lim_{\tau\to\infty}\left(P^{(3)} - \tfrac{1}{2}(P^{(2)})^2\right). \tag{51}$$

Changing the integration variables $x = \tilde{\beta}^{\frac{1}{1-\epsilon}} N\tilde{x}$, $y = \tilde{\beta}^{\frac{1}{1-\epsilon}} N\tilde{y}$ and $\beta = N^{1-\epsilon}\tilde{\beta}$ in the integral representations (18) and (39) for $P^{(2)}$ and $P^{(3)}$, respectively, and taking the derivative with respect to $\ln N$, we find

$$\frac{\partial \Pi_3}{\partial \ln N} = 2\epsilon N^{2\epsilon}\left[\int_0^\infty \mathrm{d}\beta\beta^{\frac{3\epsilon-1}{1-\epsilon}}\int_0^{\beta^{-\frac{1}{1-\epsilon}}}\mathrm{d}x\int_0^{\beta^{-\frac{1}{1-\epsilon}}-x}\mathrm{d}y\, F_3\left(\frac{1}{x^{1-\epsilon}},\frac{1}{y^{1-\epsilon}}\right)\right.$$
$$\left. -\frac{1}{2}\left(\int_0^\infty \mathrm{d}\beta\beta^{\frac{2\epsilon-1}{1-\epsilon}}\int_0^{\beta^{-\frac{1}{1-\epsilon}}}\mathrm{d}x\, F_2\left(\frac{1}{x^{1-\epsilon}}\right)\right)^2\right], \tag{52}$$

where we returned to the previous notation for x, y, and β. Since we are interested in the limit $\epsilon \to 0$, we can integrate over β using formula (21). In this way, we obtain that $\frac{\partial \Pi_3}{\partial \ln N}$ is very similar to expression (42) for $\frac{\partial \Pi_3}{\partial \ln \tau}$:

$$\lim_{\epsilon\to 0}\frac{\partial \Pi_3}{\partial N} = \lim_{\epsilon\to 0}\frac{\partial \Pi_3}{\partial \tau} + \lim_{\epsilon\to 0}[2\epsilon K_1(\epsilon)] - K_2 \tag{53}$$

with

$$K_1(\epsilon) = \int_0^\infty \mathrm{d}\beta \ln\beta\beta^{\frac{\epsilon-2}{1-\epsilon}}\int_0^{\beta^{-\frac{1}{1-\epsilon}}}\mathrm{d}x\, F_3\left(\frac{1}{x^{1-\epsilon}},\frac{1}{(\beta^{-\frac{1}{1-\epsilon}}-x)^{1-\epsilon}}\right), \tag{54}$$

$$K_2 = 2\int_0^\infty \mathrm{d}x\, F_2\left(\frac{1}{x}\right)\int_0^\infty \mathrm{d}\beta \ln\beta\frac{1}{\beta^2}F_2(\beta). \tag{55}$$

The rhs of equation (53) does not contain divergent contributions of order $1/\epsilon$, as divergences cancel out in $\frac{\partial \Pi_3}{\partial \tau}$ (see the previous section) and $\lim_{\epsilon\to 0}[2\epsilon K_1(\epsilon)]$ is finite (see equation (59)). This fact demonstrates the existence of the spatial scaling $1/N^{d_2}$ with the accuracy of the sub-leading terms of the perturbation theory. The validity of Chalker's ansatz implies that $\lim_{\epsilon\to 0}[2\epsilon K_1(\epsilon)] - K_2 = 0$ and this is what we show below.

Let us first calculate K_2. Using the explicit expression for F_2 (18), one can easily evaluate both integrals

$$\begin{aligned}\int_0^\infty \mathrm{d}x\, F_2\left(\frac{1}{x}\right) &= -\frac{\pi b}{\sqrt{2}},\\ \int_0^\infty \mathrm{d}\beta\, \ln\beta\, \frac{1}{\beta^2} F_2(\beta) &= \frac{\pi b}{\sqrt{2}}(1+\gamma/2+\ln 2);\end{aligned} \tag{56}$$

hence, we obtain

$$K_2 = -\pi^2 b^2 (1+\gamma/2+\ln 2). \tag{57}$$

For K_1 we are interested only in the leading $1/\epsilon$ term of the ϵ-expansion. In order to extract it, we first change the variable β by $y = \beta^{-\frac{1}{1-\epsilon}} - x$ in equation (54) and find that $K_1(\epsilon) = -(1-\epsilon)^2 I_1$ with

$$I_1 = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \ln(x+y) F_3\left(\frac{1}{x^{1-\epsilon}}, \frac{1}{y^{1-\epsilon}}\right). \tag{58}$$

The expression for I_1 has a structure very similar to that of I defined below equation (42) and hence its ϵ-expansion can be found in exactly the same way. Skipping the details of the calculation, we present here only the final result

$$\lim_{\epsilon\to 0}[2\epsilon K_1(\epsilon)] = -\pi^2 b^2 (1+\ln 2+\gamma/2) = K_2. \tag{59}$$

Thus, we conclude that the contributions of the last two terms in equation (53) cancel and $\lim_{\epsilon\to 0}\frac{\partial \Pi_3}{\partial N} = \lim_{\epsilon\to 0}\frac{\partial \Pi_3}{\partial \tau}$. This equality not only proves the validity of Chalker's ansatz in the second-order perturbation theory but also provides the expressions for d_2 and μ:

$$d_2 = 1-\mu = \frac{\pi b}{\sqrt{2}} - \pi^2 b^2 \left(\frac{3R}{\pi} - \frac{\ln 2}{2}\right) + O(b^3), \tag{60}$$

where R is defined in equation (48).

8. Conclusion

In the above calculations, we have demonstrated by expansion in the parameter $b \ll 1$ that the power-law scaling (equations (1) and (3)) holds true as soon as $\ln N \gg 1$ and $\ln(E_0/\omega) \gg 1$, $E_0 \sim b$, even in the perturbative region where $b \ln N \ll 1$ and $b \ln(E_0/\omega) \ll 1$. This statement is verified up to the second order in $b \ll 1$. With the same accuracy, we have shown that the exponents d_2 and μ are connected by the scaling relation equation (4). Moreover, we have found a term $\sim (\pi\, b)^2$ in d_2 (see equation (60)) which appears to enter with an anomalously small coefficient $0.083\,(\pi\, b)^2$.

However, in order to obtain all the above results, we used the analog of the dimensional regularization, replacing $(b/(n-m))^2$ in equation (5) by $(b/|n-m|)^{2(1-\epsilon)}$ and setting $\epsilon \to 0$ at the end of calculation. This trick is well known in quantum field theory and it works well for models whose renormalizability is proven. In other words, it works well if it is known that the critical exponents (and the power laws themselves) do not depend on the short-range details of the system (e.g. on the form of the function $\langle |H_{nm}|^2\rangle$ in equation (5) which interpolates between the well-defined limits at $n = m$ and $|n-m| \gg 1$). We would like to emphasize here that the renormalizability of the long-range model studied in this paper is not proven. That is why it may in principle occur that the results derived in this work depend on the regularization scheme. We only know that this is not the case in the first order in b where all the integrals can be explicitly calculated using any other regularization. Whether or not the universality (independence on the interpolating function) holds in higher orders in b is an interesting open problem.

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Appendix A. Integration over the 'center of mass' coordinate

Passing from discrete sums to integrals in equations (15) and (36), we replaced the integration over the 'center of mass' coordinate y by multiplication by N. The aim of this appendix is to justify that step.

In calculating $P^{(2)}$ we deal with the following integral:

$$I_2(N) = \frac{1}{N}\int_0^N \mathrm{d}y \int_0^y \mathrm{d}x f(x) \equiv \frac{1}{N}\int_0^N \mathrm{d}y F(y). \tag{A.1}$$

Our calculations show that the asymptotic behavior of $F(y)$ is given by

$$F(y) = c\ln y + c_0 + O(1/y); \tag{A.2}$$

hence,

$$I_2(N) = c(\ln N - 1) + c_0 + \cdots = c\ln(N/e) + c_0 + \cdots \approx F(N/e). \tag{A.3}$$

So that replacing $(1/N)\int_0^N \mathrm{d}y$ by 1 in equation (A.1) is equivalent in the asymptotic limit to replacing N by N/e. Now, let us show that the same is true in calculation of $P^{(3)}(t)$. The relevant integral now has the following form:

$$I_3(N) = \frac{1}{N}\int_0^N \mathrm{d}y \int_0^y \mathrm{d}q_1 \int_0^{y-q_1} \mathrm{d}q_2 f(|q_1|, |q_2|, |q_2+q_1|) \equiv \frac{1}{N}\int_0^N G(y). \tag{A.4}$$

According to our results, the asymptotic behavior of G is given by

$$G(y) = d_2\ln^2 y + d_1\ln y + d_0 + O(1/y), \tag{A.5}$$

and then substituting this expansion into the definition of $I_3(N)$, we find

$$\begin{aligned} I_3(N) &= d_2[\ln^2 N - 2\ln N + 2] + d_1[\ln N - 1] + d_0 + \cdots \\ &= d_2\ln^2(N/e) + d_1\ln(N/e) + d_0 + \cdots \approx G(N/e). \end{aligned} \tag{A.6}$$

Thus, the integration over the 'center of mass' can be taken into account in calculations of both $P^{(2)}$ and $P^{(3)}$ by scaling the system size. However, since we are actually interested in the calculation of the scaling exponents, the scaling of the system size by a constant factor is irrelevant as follows from equation (9).

Appendix B. Scaling exponents in the orthogonal symmetry class

The leading contribution to the virial expansion of the return probability in the orthogonal case can be obtained straightforwardly from the results of [16]:

$$P^{(2)}_{\mathrm{orth}}(t) = -\frac{\sqrt{2\pi}}{N}\sum_{n\neq p}^{N} \mathrm{e}^{-2b_{pn}t^2} 2b_{pn}|t| I_0(2b_{pn}t^2). \tag{B.1}$$

Here, I_0 is the modified Bessel function. We have to calculate the double sum in equation (B.1) with logarithmic accuracy at $bt \gg 1$ and $N \gg 1$. Therefore, the formula for $P_{\rm orth}^{(2)}$ can be reduced to a single integral (cf the unitary case):

$$P_{\rm orth}^{(2)} \simeq -2\sqrt{\pi}\, b \int_l^N \frac{\mathrm{d}x}{x}\, \mathrm{e}^{-\frac{\tau^2}{x^2}} \frac{\tau}{x} I_0\left(\frac{\tau^2}{x^2}\right), \qquad \tau \equiv \frac{bt}{\sqrt{2}}, \tag{B.2}$$

where l is a finite constant. The dominant contribution to the integral is governed by the region $x < \tau$ where the asymptote of the Bessel function can be used:

$$I_0(z \gg 1) \approx \frac{\mathrm{e}^z}{\sqrt{2\pi z}}.$$

Thus, we can rewrite equation (B.2) with the logarithmic accuracy as follows:

$$P_{\rm orth}^{(2)} \simeq -\sqrt{2} b \int_l^{\min(N,\tau)} \frac{\mathrm{d}x}{x}. \tag{B.3}$$

Inserting equation (B.3) into equation (9), we find

$$1 - \mu = d_2 = \sqrt{2} b + O(b^2), \tag{B.4}$$

which confirms Chalker's ansatz up to the leading terms of the perturbation theory.

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