One-Dimensional Disordered Systems with Interactions

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The transport properties of a one dimensional disordered system of interacting spin-polarized electrons is studied using a bosonized field theory. In contrast to other approaches in that field [3], [2], a fully bosonized action on the Keldysh contour is used to calculate disorder averaged observables. After a brief introduction to Fermi and Luttinger liquids, we review bosonization, the Keldysh technique and disorder effects in mesoscopic systems. We calculate the first order correction to the clean conductivity in the disorder strength in order to develop a diagrammatic technique for a bosonized system on the Keldysh contour. Furthermore, the dissipative part of the self energy of the retarded plasmonic Green's function is calculated via equations of motion in the high temperature regime. Corrections to the Drude conductivity of order $\epsilon = 1 - K$ (weak interactions) and $\omega\beta$ (high temperature) are calculated representing the terms beyond the saddle point approximation.

0.1 Introduction

Three decades ago Abrahams, Anderson, Licciardello and Ramakrishnan proposed a *Scaling Theory of Localization* [1], indicating that in disordered systems in less than three dimensions all states are localized. In this localized regime electron wave functions are pinned by disorder with an exponentially small overlap between each other. Thus, disorder has a very strong influence on one dimensional systems, rendering metallic transport behaviour impossible.

Explicit calculations in one dimensional disordered systems have shown that the conductivity vanishes like $\sigma(\omega) \sim \omega^2 \left[\ln(\omega)\right]^2$ in the limit $\omega \to 0$ [4]. The so called Mott-conductivity is depicted in Fig. (1).

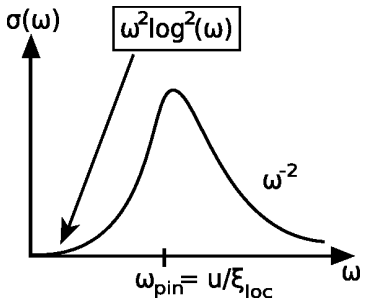


Figure 1: Mott conductivity. See [5]

Hence the theory of transport of non-interacting particles in a disordered one dimensional potential is quite well understood.
On the other hand, interactions among electrons alter a clean one-dimensional electron system drastically compared to higher dimensions. Instead of weakly interacting fermionic quasiparticles, one-dimensional systems exhibit collective bosonic excitations such as spin and charge density waves. The differences between the well-known Fermi liquid, which is applicable to three- and two-dimensional systems, and Luttinger liquids is discussed in section (1.1). Since the Fermi surface is topologically disconnected in 1D, it is convenient to describe low energy physics in terms of right- and left-moving particles with a linear dispersion relation. This is essentially the famous Tomonaga-Luttinger liquid model where in addition, finite range interactions are assumed. Powerful analytical tools like bosonization [11] and functional bosonization [26], [25] have been developed to solve it. They will be briefly introduced in section (2.2) of the first chapter. The Luttinger liquid model describes a whole range of interacting 1D systems. They are all characterized by two parameters: K which ranges from strong repulsive interactions $K < 1$, to non-interacting systems $K = 1$ and to attractive ones $K > 1$. The other parameter is the plasmon velocity u.
Consequently, disorder and interactions alone have a strong influence on the nature of one-dimensional electron systems. In particular, an interesting situation will arise if both are present. They may drive a one-dimensional system into different directions. This case has also been studied in the literature, e.g. [7], [2], and it will be the context of our work. A renormalization group analysis [7] has shown that at zero temperature there is a delocalized regime, far in the attractive (superconducting) region ($K > 3/2$) as well as a localized regime where disorder becomes relevant. At finite temperatures mesoscopic quantum interference effects such as weak localization become less pronounced due to dephasing by inelastic electron-electron interactions. Whether this mechanism is strong enough to release electrons from their localized states was studied in [2], [3] and will be discussed at the end of chapter two.
Regarding transport behaviour, I.V.Gornyi, A.D.Mirlin and D.G.Polyakov (GMP) [2] found that there is a temperature regime where a weakly interacting disordered Luttinger liquid behaves like a Fermi liquid. Although their functional bosonization approach seems to be a useful tool for those problems at weak interactions, it has still left some open questions concerning the disorder influence on the interaction propagators, which we are going to address in chapter three. However, we use full bosonization to analyze the conductivity for

weak disorder and weak interactions. According to [2], full bosonization is often inconvenient since it is hardly possible to relate the bosonic diagrams to those used for mesoscopic electron systems in higher dimension, such as Diffuson and Cooperon. On the other hand, we are able to include disorder from the very beginning in the action of interacting electrons by a random potential. In [2] disorder is included at two points: for electron propagators and interaction propagators separately, assuming that there are no relevant correlations between these two types of impurities. This may be a weak point. In a nutshell, for weak interactions and high temperatures, i.e. $1 \ll T\tau$, but still much lower than the Fermi energy, the system shows Drude like transport behaviour.
Our theory can be relevant for the following experimental realizations of one dimensional systems:

The two most prominent examples are quantum wires fabricated in GaAs/AlGaAs heterostructures, see Fig. (2) and single wall metallic carbon nanotubes. Luttinger liquid behaviour has been proven in both systems [13], [19]. In weakly disordered quantum wires an interaction parameter of $K \approx 0.66$ was found. Here, disorder stems from lithographical imperfections which cause interface roughness and thus potential fluctuations along the axis of the wire [13]. Carbon nanotubes are wrapped single layers of graphite sheets their Luttinger parameter was found to be $K \approx 0.28$ [19].

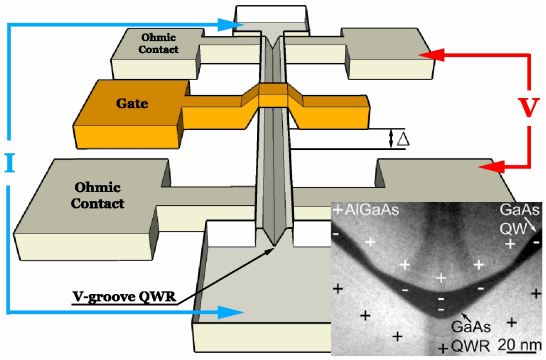


Figure 2: Device for transport measurements of a long V-groove quantum wire. Inset: cross-sectional TEM image. The quantum wire is located at the bottom of the V groove. See [13]

Chapter 1

Concepts in Many-Body Physics

1.1 Fermi liquid vs. Luttinger liquid

Before we discuss properties of interacting one dimensional systems we briefly summarize the concepts of many-body physics in the context of interaction effects in higher dimensions. We start with free fermions since they provide the most simple and familiar system. From the basics of quantum statistics we know that non-interacting electrons obey the Fermi-Dirac distribution. At zero temperature the occupation with respect to momentum is just a step function as depicted in Fig.(1.1). All states up to the Fermi energy ϵ_F are occupied. Extra electrons cannot be added into that Fermi sea due to the Pauli principle. Furthermore the probability to find a state at momentum $\vec{k}$ and frequency ω, the spectral function, is a delta function

$$A(\vec{k},\omega) = \delta(\omega - (\epsilon(\vec{k}) - \mu)) \tag{1.1}$$

That is a consequence of the trivial time evolution $e^{i\xi(\vec{k})t}$ which does not decay in time since $\xi(\vec{k})$ is an eigenenergy of the system.

Now what happens if we turn on interactions? Surprisingly in three dimensions interactions change not much. There is still the concept of single-particle excitations as well as a Fermi sphere. Solving the Schroedinger equation of that complicated and strongly-coupled electron system is certainly not feasible and one should employ the many-body formalism instead. Fortunately there is the quasi-particle picture of Landau's Fermi liquid theory which provides also an intuitive understanding of the physical mechanisms.
In principle, the quasi particles are electrons/holes dressed by a cloud of particle-hole excitations. These quasi particles only survive near to the Fermi sphere. Moreover they are not exact eigenstates of the Hamiltonian. Hence they decay with time τ_k. This is described by the spectral function in figure(1.1) which is now a broadened Lorentzian peak with width $1/\tau_k$ and area $Z < 1$. Z denotes the fraction of particles which are in the quasi-particle state. Moreover Z appears in the occupation distribution of the interacting bare electrons at $T = 0$. It gives the height of the discontinuous step, Fig.(1.1). Since the quasi particles are living near the Fermi edge their energy can be expressed as $\epsilon_k = k_F \frac{k-k_F}{m^*} + \epsilon_F$ with renormalized mass m^* that is different from the bare electron mass m. k_F is the Fermi momentum.
The lifetime τ_k always exceeds the period of oscillation of the excitation $1/(\epsilon_k - \epsilon_F)$ meaning that the quasi-particles become better defined near the Fermi-level. This can be seen by a simple phase space argument due to Migdal [27]. We consider the setting in figure (1.1) with a quasi-particle at $|\vec{k}_1| > k_F$ (filled circle). It scatters with an other quasi-particle with mo-

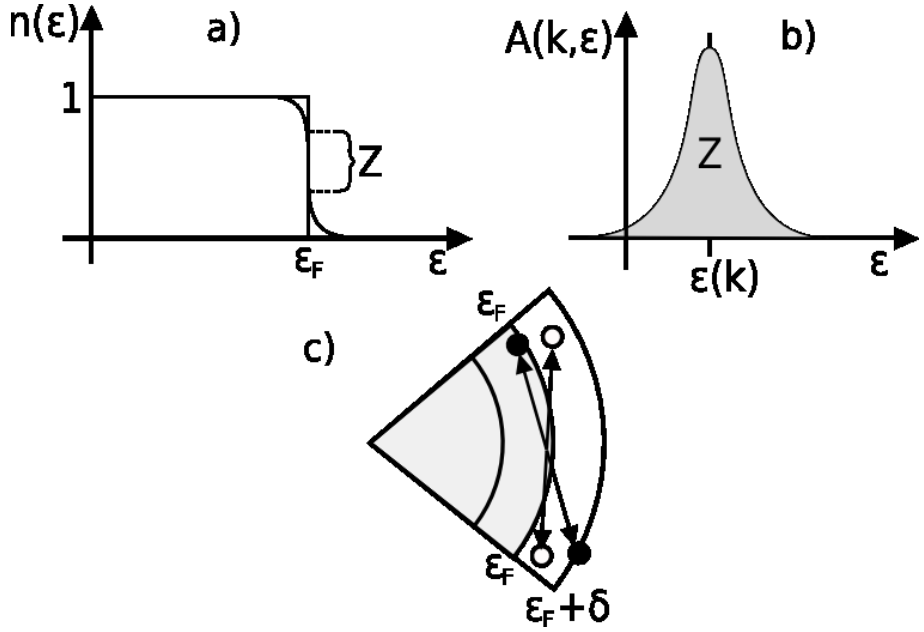


Figure 1.1: a) The electron distribution function $n(\epsilon)$ at zero temperature. In the non-interacting case it is a step function. Even in the interacting case, there is a jump of size $Z < 1$. b) Spectral function of an interacting electron gas. The area under the curve equals Z. c) Part of the Fermi sphere. Two quasi-particles scatter out of their initial states depicted by filled circles. This can happen only within a shell of width $\delta = \epsilon(\vec{k}_1) - \epsilon_F$.

mentum $\vec{k}_2$ (the other filled circle). Due to energy and momentum conservation and the Pauli principle the resulting states at $\vec{k}_3$ and $\vec{k}_4$ (open circles) lie in a shell of thickness $\delta = |\vec{k}_1| - k_F$ around the Fermi-sphere.
The decay time τ_k is the inverse of the decay rate of state $\vec{k}$. Since a quasi particle will be scattered out of state $\vec{k}$ by a collision with other quasi-particles, the scattering amplitude is proportional to the volume of the shell $\propto 4\pi k_F^2(|\vec{k}_1| - k_F)$. The probability is the square of it, hence

$$\frac{1}{\tau_k} \propto (|\vec{k}_1| - k_F)^2 \sim (\epsilon_k - \epsilon_F)^2 \tag{1.2}$$

Thus, near the Fermi surface the decay time Eq.(1.2) is bigger than the period of oscillation $1/(\epsilon_k - \epsilon_F)$ of the excitation:

$$\tau_k \gg |\epsilon_k - \epsilon_F|^{-1} \tag{1.3}$$

This means that these quasi-particles are well defined near the Fermi surface.
In total the picture of single particle excitations is still present in the interacting case. As we will see later this is not the case in one dimension. Landau's Fermi-liquid is good at not to high temperatures[1] and can even be used for strong coupling. So either very very strong interactions or some instabilities are needed in order to leave that framework.
So far we discussed single particle properties of free and interacting systems in higher dimensions which are not present in one dimension. In addition to these single particle excitations there are also collective excitations such as density waves.

[1] $\sim 1 - 100K$ since the Fermi energy is around $10^5 K$. [5]

Since in one dimension electron interactions are strong, collectivism is a ubiquitous feature of one dimensional systems. The naive picture that a single electron has to push all the other electrons further to be able to move works pretty well in that sense. According to that picture, any individual excitation should become a collective one in 1D. In higher dimensions ($d \geq 2$) the Fermi surface is a connected object. An electron which is excited from slightly below the Fermi sphere to slightly above it represents a particle-hole excitation with zero energy $\epsilon_k(q) = \xi(k+q) - \xi(k) \approx 0$. In picture Fig.(1.2) there is a particle-hole excitation drawn. This particle-hole excitation has a momentum q. Although the excitation's energy is zero there are various possibilities for the momentum q. See Fig.(1.2). This leads to a continuum of possible particle-hole excitations, which makes them not very well defined 'particles'. In one dimension our Fermi sphere is disconnected. Thus the momentum states of the same energy ϵ_F just represent two distinct points at $\pm k_F$. For small momenta $q \sim 0$ our particle-hole excitations do become well defined quasi-particles. Since there are only two possibilities for q when $\epsilon_k(q) = 0$, $q = 0$ and $q = 2k_F$, we get a different diagram in Fig.(1.2) for 1D. Consequently, at $q = 0$ particle-hole excitations are bosonic quasi particles with linear dispersion relation $\epsilon(q) = v_F q$. This is already a good hint for a theory in 1D which is formulated in terms of bosonic degrees of freedom. These degrees of freedom are charge and spin density waves.

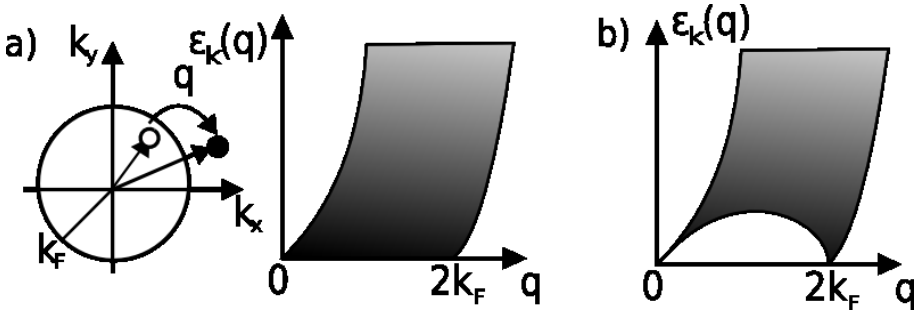


Figure 1.2: a) particle-hole excitations in dimension $d \geq 2$. b) particle-hole excitations in 1D. For small q these excitations have a well defined energy-momentum relation.

A more quantitative analysis using many - body theory and linear response [31] will show that one dimensional systems have a critical behaviour. From statistical mechanics it is known that there are no phase transitions into an ordered state in one dimension. Quantum fluctuations will emphasize this statement even more.
A diverging susceptibility χ is usually an indication of a phase transition. For example, there is the Cooper instability in the theory of superconductors where the pair susceptibility χ_{pp} diverges. It means that it is favorable to create Cooper pairs and the ordered state is the superconducting state. Besides the particle-particle susceptibility χ_{pp} there is also a particle-hole susceptibility χ_{ph} and in one dimensional electron systems χ_{ph} is always diverging. The ordered state would correspond to some order in the density, a charge density wave or spin density wave. The instabilities in χ_{pp} and χ_{ph} want to push the 1D system into different ordered states, without being able to order completely. In this sense one expects the system to behave critically.

1.2 Operator bosonization

Bosonization stands for rewriting correlation functions of a fermionic model in terms of bosonic degrees of freedom. We will see that this is possible in an effective low-energy theory in one dimension. For the moment we restrict ourselves to spinless Fermions[2]. The goal is to be able to diagonalize the full Hamiltonian with interactions.
The general Hamiltonian of an interacting electron gas in second quantized formulation is:

$$\mathcal{H} = \frac{1}{2m}\int \mathrm{d}x \partial_x\psi^\dagger(x)\partial_x\psi(x) + \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}x' \psi^\dagger(x)\psi^\dagger(x')V(x-x')\psi(x')\psi(x) \tag{1.4}$$

$\psi^\dagger, \psi$ are the electron creation and annihilation operators, $V(x-x')$ is the bare electron-electron interaction potential. In higher dimensions it is in general impossible to diagonalize this Hamiltonian. However in the following we will show that we can solve practically any Hamiltonian that describes a one dimensional system of interacting electrons.

Phenomenological bosonization

In the last section it was mentioned that in one dimension the fundamental excitations are collective ones. A collective excitation is for example a charge-density wave and it is characterized by the particle density:

$$\rho(x) = \sum_{i=1}^{N} \delta(x - x_i) \tag{1.5}$$

The next step is to define a labelling field or a cumulative particle-number field $\phi_l(x)$. At the position of the j-th particle the labelling field is $\phi_l(x_j) = j \cdot 2\pi$. Moreover it is an increasing monotonic function of x. Furthermore by using the following property of the Dirac delta

$$\delta(f(x)) = \sum_{zeros(f)} \frac{1}{|f'(x)|}\delta(x - x_i) \tag{1.6}$$

we can rewrite the density in terms of the labelling field.

$$\rho(x) = \sum_n |\nabla\phi_l(x)|\delta(\phi_l(x) - 2\pi n) = \frac{\nabla\phi_l(x)}{2\pi}\sum_p e^{ip\phi_l(x)} \tag{1.7}$$

If ρ_0 is the average particle density in the system then $\rho_0^{-1} = d$ is the average distance between two neighbouring particles. Hence $\phi_l(x) = 2\pi\rho_0 x$ would represent a perfect lattice.
In order to look at deviations from this perfect lattice we introduce another field $\phi(x)$ by

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x) \tag{1.8}$$

and hence

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{i2p(\pi\rho_0 x - \phi(x))} \tag{1.9}$$

[2] just think about a quantum wire with spin-polarized electrons (due to a magnetic field)

In a low energy approximation we neglect the higher harmonic terms and only take $p = 0$. Thus the low energy particle density is:

$$\rho(x) \approx \rho_0 - \frac{1}{\pi}\nabla\phi(x) \tag{1.10}$$

What is an intuitive physical interpretation of the field ϕ? To answer this question we add a particle at point x_0 and integrate the density of excitations up to x_0.

$$\int_{-\infty}^{x>x_0} (\rho(x') - \rho_0)dx' = -\frac{1}{\pi}[\phi(x) - \phi(-\infty)] \tag{1.11}$$

Quantized steps in ϕ are a measure of the total charge added to the system. Moreover a kink in $\phi(x)$ at x_0 is a delta-like peak in $\nabla\phi(x)$ at x_0 which means that at x_0 there is an extra particle since $\rho(x_0) - \rho_0 \neq 0$. Solitonic kink excitations in $\phi(x)$ define in that way the number of quasi-particles in the system. This classical discussion will help us at least to have an intuitive understanding of bosonization and the boson field $\phi(x)$ which we will encounter throughout the whole text. In the rest of the section we will follow a constructive procedure[3].

Constructive approach to Boson operators

Particle-hole excitations are bosonic since they consist of two fermions: electrons and holes. Moreover, as discussed above, they are well defined quasi particles for small momenta q, Fig. (1.2). A superposition of particle-hole excitations yields the density operator.

$$\rho^\dagger(q) = \sum_k c^\dagger_{k+q} c_k \tag{1.12}$$

Consequently, a theory expressed in terms of ρ operators turns the cumbersome four fermion interaction vertex into a simple quadratic one.

$$\mathcal{H}_{int} = \frac{1}{2\Omega}\sum_q V(q)\rho(q)\rho^\dagger(q) \tag{1.13}$$

This relation is important since we are able to take into account the full interactions just by changing the basis. However, an exact diagonalization of $\mathcal{H}$ in terms of particle-hole excitations makes sense as long as they are well defined quasi particles, i.e. they have a well defined energy momentum relation $\epsilon(q) \sim q$.
On the other hand in general we have

$$\epsilon_k(q) = \xi(k+q) - \xi(k) \tag{1.14}$$

Hence $\epsilon_k(q)$ will depend on k for a parabolic dispersion relation $\xi(k) = (k^2/2m - \mu)$ of the electrons. However, for a linear dispersion relation particle-hole excitations have a well defined momentum:

$$\epsilon_k(q) = v_F(k+q) - v_F k = v_F q \tag{1.15}$$

[3] see chapter2 in [5] or [11]

Since the Fermi sphere is represented just by two points at $\pm p_F$, it makes sense to linearize the spectrum and obtain a right and a left mover branch. We rewrite the electronic field operator in the following way:

$$\begin{aligned}\psi_{phys}(x) &= \sqrt{\frac{2\pi}{L}}\sum_p e^{ipx}c_p \\ &= \sqrt{\frac{2\pi}{L}}\sum_{k>-p_F}\left(e^{-i(p_F+k)x}\underbrace{c_{-p_F-k}}_{=:c_{L,k}}+e^{i(k+p_F)x}\underbrace{c_{k+p_F}}_{=:c_{R,k}}\right)\end{aligned} \tag{1.16}$$

In figure (1.3) this construction is shown schematically. Neglecting large momenta k in Eq. (1.16), we describe electrons only around the Fermi points. Furthermore, we seperate the fast

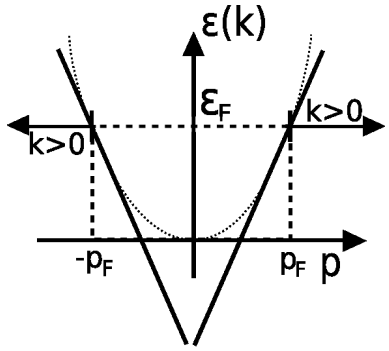


Figure 1.3: Schematic representation of the bosonization construction. Electrons with a quadratic dispersion relation and momentum p are decomposed into right and left movers with momentum $k := |p| - k_F$.

oscillations at the Fermi momentum p_F from the slow ones by introducing slowly oscillating electron fields for left and right moving particles:

$$\psi_{phys}(x) = e^{-ip_Fx}\psi_L(x) + e^{ip_Fx}\psi_R(x) \tag{1.17}$$

After linearizing the spectrum around the Fermi points, the noninteracting part of the Hamiltonian, Eq. (1.4), becomes

$$\mathcal{H}_0 = \sum_k \Big[v_F(-k-k_F)c^\dagger_{L,k}c_{L,k} + v_F(k-k_F)c^\dagger_{R,k}c_{R,k}\Big] \tag{1.18}$$

The particle-hole excitations are now everywhere well defined quasiparticles with an energy given in (1.15). Equation (1.18) is a Dirac Hamiltonian for massless fermions. We decomposed the electrons with the originally quadratic dispersion relation into two species of electrons with a linear dispersion relation. States are extended to energies $\epsilon = -\infty$. Furthermore, we assume that these states are filled up to the Fermi energy ϵ_F.
This is a dangerous construction and one may expect that singularities and ill defined expressions will appear at some points in calculations. To avoid these problems we consider normal ordered versions of operators. Normal ordering is denoted by points on the left and right of the operators, for example:

$$:\rho(q) := \sum_k : c^\dagger_{k+q}c_k : \tag{1.19}$$

A normal ordered product of operators $:AB:$ puts all destruction operators to the right of all creation operators. This is equivalent to subtracting the average value in the vacuum [5].

$$:AB:=AB-\langle 0|AB|0\rangle \tag{1.20}$$

Here the vacuum is the Fermi sea. The action of the fermionic creation and annihilation operators upon that vacuum is

$$c^\dagger_{k\eta}|0\rangle = 0 \text{ in the filled sea } k<0 \tag{1.21}$$
$$c_{k\eta}|0\rangle = 0 \text{ for empty states } k>0 \tag{1.22}$$

Thus it depends on the momentum index k whether $c^\dagger_k$ and c_k actually act as creation or annihilation operators. As an example for normal ordering let us consider the following product of operators: ($k'<0$, $k<0$)

$$:c^\dagger_k c_{k'}:=-c_{k'}c^\dagger_k = c^\dagger_k c_{k'}-\delta_{kk'} \tag{1.23}$$

This is consistent with equation (1.20) since $\delta_{kk'}=\langle 0|c^\dagger_k c_{k'}|0\rangle$.
In order to rewrite the problem in terms of bosons we still have to check whether the density operator $\rho(x)=\psi^\dagger(x)\psi(x)$ is indeed a bosonic one. Consequently, we have to check if bosonic commutation relations are fulfilled. Since $\rho(x)$ is real the Fourier transform of the density fulfills $\rho^\dagger(q)=\rho(-q)$. In the following we only consider density operators of the low energy electron fields: $\rho_{R/L}=\psi^\dagger_{R/L}(x)\psi_{R/L}(x)$. Moreover the density operator can destroy the vacuum state:

$$\begin{gathered}\rho^\dagger_L(p>0)|0\rangle=\textstyle\sum_k(:c^\dagger_{L,k+p}c_{L,k}:+\langle 0|c^\dagger_{L,k+p}c_{L,k}|0\rangle)=0\\ \rho^\dagger_R(p<0)|0\rangle=0\end{gathered} \tag{1.24}$$

$$\begin{aligned}\left[\rho^\dagger_r(p),\rho^\dagger_r(-p')\right] &= \textstyle\sum_{k,q}\left[c^\dagger_{r,k+p}c_{r,k},c^\dagger_{r,q-p'}c_{r,q}\right]\\ &= \textstyle\sum_{k,q}\left(c^\dagger_{r,k+p}c_{r,q}\delta_{k,q-p'}-c^\dagger_{r,q-p'}c_{r,k}\delta_{k+p,q}\right)\\ &= \textstyle\sum_q\left(c^\dagger_{r,q+p-p'}c_{r,q}-c^\dagger_{r,q-p'}c_{r,q-p}\right)\end{aligned} \tag{1.25}$$

Where $r=R/L$ denotes right and left movers.
To get a well defined result out of the last line in equation (1.25) we have to use the normal ordering relation.

$$\begin{aligned}\left[\rho^\dagger_r(p),\rho^\dagger_r(-p')\right] = \;& \textstyle\sum_q\left(:c^\dagger_{r,q+p-p'}c_{r,q}:-:c^\dagger_{r,q-p'}c_{r,q-p}:\right)\\ &+\textstyle\sum_q\left(\langle 0|c^\dagger_{r,q+p-p'}c_{r,q}|0\rangle-\langle 0|c^\dagger_{r,q-p'}c_{r,q-p}|0\rangle\right)\end{aligned} \tag{1.26}$$

Due to normal ordering we can safely perform a change of variables $q\to q+p$ in the first term on the right hand side. Thus the right hand side of the first line vanishes. The expressions between the vacuum states in the second line of (1.26) can only be nonzero if $p=p'$. They give $\langle 0|c^\dagger_{r,k}c_{r,k}|0\rangle=1$ if the state is occupied and otherwise zero. During the calculation we always assumed the momentum k to be quantized. In a system of size L with periodic boundary conditions $k=2\pi n/L$ where $n\in\mathbb{Z}$. Hence the commutator is basically a difference between occupation numbers: $n_{R,k}-n_{R,k-p}=\frac{pL}{2\pi}$ for right movers and $n_{L,k}-n_{L,k-p}=-\frac{pL}{2\pi}$ for left movers. Thus:

$$\left[\rho^\dagger_R(p),\rho^\dagger_R(-p')\right]=-\delta_{p,p'}\frac{pL}{2\pi} \tag{1.27}$$

$$\left[\rho_L^\dagger(p), \rho_L^\dagger(-p')\right] = \delta_{p,p'}\frac{pL}{2\pi} \tag{1.28}$$

The commutator is zero for density operators of different chiralities R, L [5][4].
Equation (1.27) says that the density operators $\rho(p)$ obey Bose commutation relations up to a normalization factor. Based on the ρ-operators, we can define operators $b_q, b_q^\dagger$ which fulfill bosonic canonical commutation relations

$$\begin{aligned} b_q^\dagger &= \left(\tfrac{2\pi}{L|q|}\right)^{1/2}\left(\theta(q)\rho_R^\dagger(q) + \theta(-q)\rho_L^\dagger(q)\right) = b_{R,q}^\dagger + b_{L,q}^\dagger \\ b_q &= \left(\tfrac{2\pi}{L|q|}\right)^{1/2}\left(\theta(q)\rho_R^\dagger(-q) + \theta(-q)\rho_L^\dagger(-q)\right) = b_{R,q} + b_{L,q} \end{aligned} \tag{1.29}$$

Where $\theta(q)$ is a theta step function which is unity for $q > 0$ and zero otherwise.
These operators are creation and annihilation operators of particle hole excitations. Moreover b_q and $b_q^\dagger$ preserve the total particle number.

The Hamiltonian in terms of the $b, b^\dagger$ operators can be constructed from the commutator of $b, b^\dagger$ with the Hamiltonian $\mathcal{H}_0$.

$$\begin{aligned} [b_q, \mathcal{H}_0] &= v_F q b_{q_0} \\ \left[b_q^\dagger, \mathcal{H}_0\right] &= v_F q b_{q_0}^\dagger \end{aligned} \tag{1.30}$$

These two equations completely define the Hamiltonian in terms of the $b, b^\dagger$ operators [5]p.34.

$$\mathcal{H}_0 = \sum_{q\neq 0} v_F|q| b_q^\dagger b_q + \frac{\pi v_F}{L}\sum_r \hat{N}_r^2 \tag{1.31}$$

So far we found a notation in terms of bosonic operators for $\mathcal{H}_0$ which is diagonal. Or in other words, we found that the kinetic energy term of $\mathcal{H}$ which is quadratic in Fermion operators can be transformed to a term quadratic in Boson operators that means quartic in Fermion operators. This will be helpful when we include the interaction term which is also quartic in Fermion operators. However, at first we show that every fermionic operator can be expressed in the bosonic operator basis. To obtain a direct mapping between Fermion operators ψ and Boson operators b_q we evaluate their commutator.

$$\begin{aligned} [b_{r,q}, \psi_{r'}(x)] &= \tfrac{1}{\sqrt{\Omega}}\left(\tfrac{2\pi}{L|q|}\right)^{1/2}\textstyle\sum_{k,k'}\left[c_{r,k-q}^\dagger c_{r,k}, e^{-ik'x}c_{r',k'}\right] \\ &= -\delta_{r,r'}\tfrac{1}{\sqrt{\Omega}}\left(\tfrac{2\pi}{L|q|}\right)^{1/2} e^{iqx}\textstyle\sum_k e^{-ikx}c_{r,k} \\ &= -\left(\tfrac{2\pi}{L|q|}\right)^{1/2}\psi_r(x) \end{aligned} \tag{1.32}$$

Since $b_{r,q}$ is an annihilation operator of the vacuum $|0\rangle$ we can derive from this expression that $\psi_r(x)|0\rangle$ is an eigenstate of $b_{r,q}$ with eigenvalue $-\sqrt{2\pi/L|q|}exp(iqx)$. An eigenstate of a boson annihilation operator directly brings us to the notion of a coherent state $|\lambda\rangle$.
A coherent state $|\lambda\rangle$ is defined by $b_{r,q}|\lambda\rangle = \lambda|\lambda\rangle$. The state $|\lambda\rangle$ on the other hand can be expressed in terms of a superposition of number states $|n\rangle$ which represent states with n bosons respectively. In our system we would call $|n\rangle$ the state with n particle-hole excitations. The

[4] [5] p.32

vacuum state with zero particle-hole excitations coincides with the Fermi ground state $|0\rangle$, $b_{r,q}|0\rangle = 0$. The coherent state can be expressed in terms of $|0\rangle$ and $b^\dagger_{r,q}$.

$$|\lambda\rangle = e^{\lambda b^\dagger_{r,q}}|0\rangle \tag{1.33}$$

Now we are able to identify the Fermion operator $\psi_r(x)$ with the Boson operator $b^\dagger_{r,q}$.
Since we showed in Eq. (1.32) that $b_{r,q}\psi_r(x)|0\rangle \propto \psi_r(x)|0\rangle$ we can write:

$$\psi_r(x) \sim e^{\sum_q \lambda_q(x) b^\dagger_{r,q}} \tag{1.34}$$

The problem that on the left hand side of equation (1.34) we have a Fermion operator whereas on the right hand side there is a Boson can be solved by introducing so called Klein factors $U_r, U_r^\dagger$ which contain the fermionic nature but no space-time dependences. See [11] for a more rigorous derivation of constructive bosonization. This mapping makes bosonization an exact operator identity. Moreover the two Hilbertspaces $\mathcal{F}_{fermion} = span\{\{n_{k,r}\}, n_{k,r} \in 0, 1\}$ and $\mathcal{F}_{boson} = span\{\{N_r, m_{q,r}\}, N_r \in \mathbb{Z}, m_{q,r} \in \mathbb{N}_+\}$ are both complete and equivalent[5].
The exact mapping [5] is:

$$\psi_{R/L}(x) = U_{R/L} \lim_{\alpha\to 0} \frac{1}{\sqrt{2\pi\alpha}} e^{i\pm(k_F - \pi/L)x} e^{-i(\pm\phi(x) - \theta(x))} \tag{1.35}$$

Note that, although the limit $\alpha \to 0$ is explicitly mentioned, α has to be finite in the relation above in order to avoid singularities. It can be shown ([5] p.45) that if an interaction of finite range ξ_{int} is present, we can safely take the limit $\alpha \to 0$ without any singularities. Thus ξ_{int} plays the role of α. However it is technically simpler to assume pointlike interactions and let α be of the order of the lattice spacing. Moreover α proposes a UV cutoff in our theory reflecting a finite bandwidth $1/\alpha$.
The bosonic operators ϕ and θ are defined as[6]

$$\begin{aligned}
\phi(x) &= -(\hat{N}_R + \hat{N}_L)\tfrac{\pi x}{L} - \tfrac{i\pi}{L}\textstyle\sum_{p\neq 0} \frac{1}{p} e^{-\alpha|p|/2 - ipx} \left(\rho^\dagger_R(p) + \rho^\dagger_L(p)\right) \\
\theta(x) &= +(\hat{N}_R - \hat{N}_L)\tfrac{\pi x}{L} + \tfrac{i\pi}{L}\textstyle\sum_{p\neq 0} \frac{1}{p} e^{-\alpha|p|/2 - ipx} \left(\rho^\dagger_R(p) - \rho^\dagger_L(p)\right)
\end{aligned} \tag{1.36}$$

Let us examine the commutation relations between the new bosonic fields ϕ and θ. The operators ϕ, θ in terms of $b, b^\dagger$ are

$$\begin{aligned}
\phi(x) &= -(\hat{N}_R + \hat{N}_L)\tfrac{\pi x}{L} - \tfrac{i\pi}{L}\textstyle\sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{1}{p} e^{-\alpha|p|/2 - ipx} \left(b^\dagger_p + b_p\right) \\
\theta(x) &= (\hat{N}_R - \hat{N}_L)\tfrac{\pi x}{L} + \tfrac{i\pi}{L}\textstyle\sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2 - ipx} \left(b^\dagger_p - b_p\right)
\end{aligned} \tag{1.37}$$

We want to write the Hamiltonian which is quadratic in $b, b^\dagger$ in terms of ϕ, θ. Hence it is important that ϕ and θ are canonically conjugate operators. The commutator between ϕ and θ is

$$\begin{aligned}
[\phi(x), \theta(y)] &= \textstyle\sum_{p\neq 0} \frac{\pi}{Lp} e^{ip(y-x) - \alpha|p|} \\
&\to i \textstyle\int_0^\infty \frac{dp}{p} \sin(p(y-x)) e^{\alpha|p|}
\end{aligned} \tag{1.38}$$

[5] [11] Appendix B
[6] See [5] p.35

In the limit $\alpha \to 0$ the integral yields $sign(y-x) \times \pi/2$ and we are left with

$$[\phi(x), \theta(y)] = i\frac{\pi}{2} sign(y-x) \tag{1.39}$$

This is certainly not the outcome of a commutator of two canonical conjugate fields. If we take the derivative with respect to y we will get a delta function on the right hand side.

$$\left[\phi(x), \frac{1}{\pi}\nabla\theta(y)\right] = i\delta(y-x) \tag{1.40}$$

Consequently, $\Pi(x) = \frac{1}{\pi}\nabla\theta(x)$ is the canonically conjugate to the ϕ field. Now we are able to rewrite the Hamiltonian in (1.31) in terms of ϕ and Π fields.

$$\mathcal{H}_0 = \frac{1}{2\pi}\int \mathrm{d}x v_F \left[(\pi\Pi(x))^2 + (\nabla\phi(x))^2\right] \tag{1.41}$$

Furthermore the derivatives of these fields can be related to the density operators

$$\begin{aligned} \nabla\phi(x) &= -\pi\left[\rho_R(x) + \rho_L(x)\right] \\ \nabla\theta(x) &= \pi\left[\rho_R(x) - \rho_L(x)\right] \end{aligned} \tag{1.42}$$

From these equations we can see that $\nabla\theta(x)$ is the difference between left and right movers. Whereas from Eq. (1.9) and (1.42) we see that, since $\rho_{R/L}$ describes only the density of right and left movers and not of $2k_F$ terms like: $\exp(i2k_F)\psi_R^\dagger\psi_L$, $\nabla\phi(x)$ is related to the $q \sim 0$ part of the density fluctuations at point x [5].

Equation (1.41) is a very important result since it allows us to rewrite a Hamiltonian quadratic in Fermion operators in terms of an operator being proportional to four Fermion operators. We only used the fact that the Fermi sphere is disconnected in one dimensions which makes particle-hole excitations well defined quasiparticles in a certain regime. We focused on that regime and linearized the spectrum at the two Fermi points. This allows one to consider four Fermion interaction terms without difficulties.

Diagonalizing Hamiltonians with interactions

We consider an interaction Hamiltonian of the general form

$$\mathcal{H}_{int} = \int dx dy V(y-x)\rho(x)\rho(y) \tag{1.43}$$

For the following discussion we assume the interaction to be q independent which means we consider a short-range interaction potential.
In one dimension there are three different types of interaction processes, combined, they are known under the term g-ology. Figure (1.4) shows the three different types of interactions. g_4 is a forward scattering event since it couples two Fermions on the same side of the Fermi sphere. g_2 couples Fermions with different chiralities of the Fermi sphere. However, g_2 processes preserve the direction of movement of each Fermion. That means after the interaction a right or left mover stays a right or left mover. Finally, g_1 processes are backscattering processes because they change the direction of motion of the Fermions. For a spinless system g_1 and

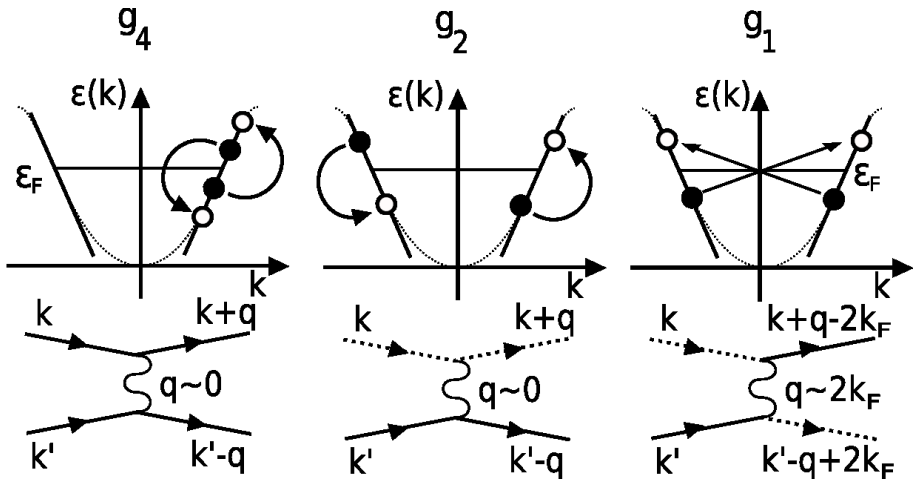


Figure 1.4: Three different low-energy scattering processes that occur in one dimensional systems [5]. Full and dotted lines represent right and left movers. Here the spin is omitted so g_2 and g_1 processes are indistinguishable.

g_2 processes are the same due to the indistinguishability in quantum mechanics. That will simplify the whole problem a lot. Therefore, from now on g_1 processes are omitted.
With the help of equation (1.42) we can rewrite the four Fermion interaction.
Let us write down the g_4 processes in terms of bosonic fields for right movers and for left movers:

$$\frac{g_4}{2}\int dx \underbrace{\psi_R^\dagger(x)\psi_R(x)}_{\rho_R(x)}\underbrace{\psi_R^\dagger(x)\psi_R(x)}_{\rho_R(x)} = \frac{g_4}{2}\frac{1}{(2\pi)^2}\int dx(\nabla\phi-\nabla\theta)^2 \tag{1.44}$$

$$\frac{g_4}{2}\int dx \underbrace{\psi_L^\dagger(x)\psi_L(x)}_{\rho_L(x)}\underbrace{\psi_L^\dagger(x)\psi_L(x)}_{\rho_L(x)} = \frac{g_4}{2}\frac{1}{(2\pi)^2}\int dx(\nabla\phi+\nabla\theta)^2 \tag{1.45}$$

Summing up Eq. (1.44) and (1.45) we obtain:

$$\frac{g_4}{(2\pi)^2}\int dx\left[(\nabla\phi)^2+(\nabla\theta)^2\right] \tag{1.46}$$

This contribution can be easily included in our noninteracting Hamiltonian (1.41) by simply adding the constant g_4/π to v_F. Thus g_4 processes only change the velocity of the excitations. The g_2 process is

$$\begin{aligned} g_2\int dx\psi_R^\dagger(x)\psi_R(x)\psi_L^\dagger(x)\psi_L(x) &= g_2\int dx\rho_R(x)\rho_L(x) \\ &= \tfrac{g_2}{(2\pi)^2}\int dx(\nabla\phi-\nabla\theta)(\nabla\phi+\nabla\theta) \\ &= \tfrac{g_2}{(2\pi)^2}\int dx\left[(\nabla\phi)^2-(\nabla\theta)^2\right] \end{aligned} \tag{1.47}$$

If we include these two interaction terms in the Hamiltonian $\mathcal{H}_0$ we get

$$\mathcal{H} = \frac{1}{2\pi}\int dx\left[uK(\pi\Pi(x))^2+\frac{u}{K}(\nabla\phi(x))^2\right] \tag{1.48}$$

Where we have introduced two independent parameters, the velocity u

$$u = v_F\sqrt{(1+y_4/2)^2-(y_2/2)^2} \tag{1.49}$$

and the dimensionless Luttinger parameter K

$$K = \left(\frac{1 + y_4/2 - y_2/2}{1 + y_4/2 + y_2/2} \right)^{1/2} \tag{1.50}$$

$y_i = g_i/(\pi v_F)$ is a dimensionless coupling constant. For repulsive interactions $g_2, g_4 > 0$ the Luttinger interaction parameter is $K < 1$. For a non interacting system $K = 1$, and quite generally for an attractive system $g_2, g_4 < 0$: $K > 1$.

Physical properties of the Luttinger liquid model

In order to discuss the physical consequences, we have to consider the following correlation functions for zero temperatures, see [5] appendix C:

$$\begin{aligned} \langle [\phi(x,\tau) - \phi(0)]^2 \rangle &= K \cdot F_1(x,\tau) \\ \langle [\theta(x,\tau) - \theta(0)]^2 \rangle &= \frac{1}{K} F_1(x,\tau) \\ F_1(x,\tau) = \frac{1}{2} \ln &\left[\frac{x^2 + (u|\tau| + \alpha)^2}{\alpha^2} \right] \end{aligned} \tag{1.51}$$

$\tau = it$ is the imaginary time.
In the first section the critical behaviour of the particle-particle and particle-hole susceptibility of an interacting one dimensional system was mentioned. In the following, we examine this situation more quantitatively. The full density operator is:

$$\begin{aligned} \rho(x,\tau) &= \psi^\dagger_{phys} \psi_{phys} = \rho_R + \rho_L + \left(e^{i2k_F} \psi^\dagger_L \psi_R + h.c. \right) \\ &= -\frac{1}{\pi} \nabla \phi(x,\tau) + \frac{1}{2\pi\alpha} \left[e^{i2k_F x} e^{-i2\phi(x,\tau)} + h.c. \right] \end{aligned} \tag{1.52}$$

The density-density correlation function is, see [5] p.44:

$$\langle \rho(x,\tau)\rho(0) \rangle = \frac{K}{2\pi^2} \frac{(u\tau + \alpha \cdot sign(\tau))^2 - x^2}{x^2 + (u\tau + \alpha \cdot sign(\tau))^2} + \frac{2}{(2\pi\alpha)^2} \cos(2k_F x) \left(\frac{\alpha}{\sqrt{x^2 + (u|\tau| + \alpha)^2}} \right)^{2K} \tag{1.53}$$

The second term is a non-universal power law with an interaction dependent exponent. This term is responsible for Luttinger liquid behaviour. From the exponent of the power law term it can be seen that the correlation will decay faster when interactions are attractive, $K > 1$. The Fourier transformed expression of (1.53), the so called susceptibility, diverges indicating that there is a tendency to order into a phase where the density is periodically modulated with a wave vector $2k_F$. This charge modulation is called charge density wave.
For a classical field ϕ the density would be:

$$\rho(x) = \frac{1}{\pi\alpha} \cos(2k_F x - 2\phi) \tag{1.54}$$

ϕ plays the role of the phase of the charge-density wave (CDW). For a perfect wave ϕ orders and thus would be constant. This certainly minimizes the ϕ part of the Hamiltonian (1.41) but since Π is the canonical conjugate field it would have large fluctuations and in total the

Hamiltonian, Eq. (1.41), is not minimized by this configuration.
Furthermore we are going to analyze the correlation function of the pairing operator:

$$O_{SC}(x,\tau) = \psi^\dagger(x,\tau)\psi^\dagger(x,\tau) \tag{1.55}$$

The corresponding correlation function at temperature $T = 0$ is:

$$\langle O_{SC}(x,\tau)O^\dagger_{SC}(0)\rangle \approx \frac{1}{(\pi\alpha)^2}\left(\frac{\alpha}{\sqrt{x^2+(u|\tau|+\alpha)^2}}\right)^{\frac{2}{K}} \tag{1.56}$$

Equation (1.56) describes superconducting fluctuations. From the exponent of this correlation function it can be seen that superconducting fluctuations are stronger in an attractive interacting system $K > 1$ where the system tends do order into a superconducting state.
Thus superconducting fluctuations and density fluctuations are dual in this sense. However, due to the fact that a one dimensional system has no symmetry broken phase, a perfect order is always destroyed by quantum fluctuations.
The "phase diagram" of spinless Luttinger liquids is shown in Fig. (1.5).

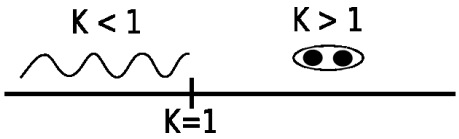


Figure 1.5: Phase diagram of the dominant behaviour of a Luttinger liquid. For $K < 1$ the system tends to order into a CDW. For attractive interactions $K > 1$ superconducting behaviour is dominant.

1.3 Functional bosonization

The approach we used so far to diagonalize the Hamiltonian of a one dimensional interacting electron system was an operator based approach. In this part we will show that there is another way to solve one dimensional systems with interaction by using a functional integral approach. It is a very short way to obtain a formulation of an interacting fermionic system in terms of bosons and it is commonly known as functional bosonization. In the following we are going to use the imaginary time Matsubara formalism and functional integrals to compute expectation values. τ denotes the imaginary time.
Our starting point is the general Hamiltonian (1.4) of an interacting electron gas. We derive the general action $\mathcal{S}$ of an interacting electron system via a Legendre transformation from the Hamiltonian:

$$\mathcal{S} = \hat{\mathcal{H}} - \int \mathrm{d}x \int \mathrm{d}\tau \psi(x,\tau)\dot{\psi}(x,\tau) \tag{1.57}$$

Thus the action is:

$$\begin{aligned}\mathcal{S}[\psi,\bar{\psi}] = S_0 + S_{int} &= \int \mathrm{d}x\mathrm{d}\tau\bar{\psi}(x,\tau)\left[\partial_\tau + \xi\right]\psi(x,\tau) \\ &+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\mathrm{d}\tau\bar{\psi}(x,\tau)\bar{\psi}(x',\tau)V_0(x-x')\psi(x',\tau)\psi(x,\tau)\end{aligned} \tag{1.58}$$

Where $\xi = -\frac{1}{2m}\partial_x^2 - \mu$ and μ is the chemical potential and ψ, $\bar{\psi}$ are fermion field and conjugate fermion field respectively[7]. Due to the Matsubara periodicity: $\psi(\beta+\tau) = -\psi(\tau)$ all τ-integrations are running from 0 to β.
The thermal imaginary time ordered Green's function reads:

$$\mathcal{G}(x-x',\tau-\tau') = \langle \mathcal{T}_\tau \psi(x,\tau)\psi^\dagger(x',\tau')\rangle \tag{1.59}$$

It can be calculated from the action by computing the following expectation value:

$$\mathcal{G}(x-x',\tau-\tau') = \frac{\int \mathcal{D}[\psi,\bar{\psi}]\psi(x,\tau)\bar{\psi}(x',\tau')e^{-\mathcal{S}[\psi,\bar{\psi}]}}{\int \mathcal{D}[\psi,\bar{\psi}]\exp(-\mathcal{S}[\psi,\bar{\psi}])} \tag{1.60}$$

Quadratic terms in $\mathcal{S}[\psi,\bar{\psi}]$ lead to analytically exact answers. Therefore we decouple S_{int} into a quadratic in ψ term. This decoupling is performed via a Hubbard-Stratonovich transformation by introducing an auxiliary field φ that is conjugated to the electron density $\psi^\dagger\psi$. The following relation is easily proven by integrating out the φ field:

$$e^{-\frac{1}{2}\bar{\psi}\bar{\psi}'V_0\psi'\psi} = \frac{1}{\mathcal{Z}_\varphi}\int \mathcal{D}[\varphi]e^{-\frac{1}{2}\varphi V_0^{-1}\varphi + i\varphi\bar{\psi}\psi} \tag{1.61}$$

where

$$\bar{\psi}\bar{\psi}'V_0\psi'\psi = \int \mathrm{d}x\mathrm{d}x'\mathrm{d}\tau\bar{\psi}(x,\tau)\bar{\psi}(x',\tau)V(x-x')\psi(x',\tau)\psi(x,\tau) \tag{1.62}$$

$$\varphi V_0^{-1}\varphi := \int \mathrm{d}x\mathrm{d}x'\mathrm{d}\tau\varphi(x,\tau)V_0^{-1}(x-x')\varphi(x',\tau) \tag{1.63}$$

$$\varphi\bar{\psi}\psi := \int \mathrm{d}x\mathrm{d}\tau\varphi(x,\tau)\bar{\psi}(x,\tau)\psi(x,\tau) \tag{1.64}$$

$$\mathcal{Z}_\varphi = \int \mathcal{D}[\phi]\exp\left(-\frac{1}{2}\varphi V_0^{-1}\varphi\right) \tag{1.65}$$

Note that the Hubbard-Stratonovich field φ is different from the Boson field ϕ in the full bosonization technique. Later we will identify φ with the propagator of the screened Coulomb field. Now the full action of the system is quadratic in Fermion fields:

$$\mathcal{S}[\psi,\bar{\psi},\varphi] = \frac{1}{2}\varphi V_0^{-1}\varphi' + \int \mathrm{d}x\mathrm{d}\tau\bar{\psi}(x,\tau)\left[\partial_\tau + \xi - i\varphi(x,\tau)\right]\psi(x,\tau) \tag{1.66}$$

The second term can be integrated out which yields a functional determinant. The Determinant can be rewritten as the exponential of the Trace of a logarithm, see [8] p.252.

$$\begin{aligned}\int \mathcal{D}[\psi,\bar{\psi}]e^{-\int \mathrm{d}x\mathrm{d}\tau\bar{\psi}[\partial_\tau+\hat{\xi}-i\varphi]\psi} &= Det\left[\partial_\tau + \hat{\xi} - i\varphi\right] \\ &= \exp\left(Tr\ln\left[\partial_\tau + \xi - i\varphi\right]\right)\end{aligned} \tag{1.67}$$

In this way the Green's function (1.60) can be written in the following form:

$$\mathcal{G}(x-x',\tau-\tau') = \frac{1}{\mathcal{D}[\varphi]e^{-\mathcal{S}[\varphi]}}\int \mathcal{D}[\varphi]\mathcal{G}_0(x-x',\tau-\tau',[\varphi])e^{\mathcal{S}[\varphi]} \tag{1.68}$$

[7] ψ and $\bar{\psi}$ are anticommutating fields

Where $\mathcal{G}_0(x-x',\tau-\tau',[\varphi])$ is the Green's function of a non-interacting system in an external potential $i\varphi$:

$$\mathcal{G}_0(x-x',\tau-\tau',[\varphi]) = \int \mathcal{D}[\psi,\bar{\psi}]\psi(x,\tau)\bar{\psi}(x',\tau')e^{-\int \bar{\psi}[\partial_\tau+\xi-i\varphi]\psi} \tag{1.69}$$

The full action (1.66) is now written entirely in terms of the Boson field φ.

$$\mathcal{S}[\varphi] = \frac{1}{2}\varphi V_0^{-1}\varphi - Tr\ln\left[\partial_\tau+\xi-i\varphi\right] \tag{1.70}$$

The Green's function $\mathcal{G}_0$ in equation (1.68) is explicitly dependent on the field φ. It is the Green's function of free Fermions in an external potential $\varphi(x,\tau)$.
So far the derivation was completely general and not fixed to one dimensions.
As we have seen above, Eq. (1.17), in one dimensional systems we are able to decompose the Fermion field into fast and slow modes and express it in terms of left and right movers. We linearize the spectrum:

$$\xi_{R/L} = \pm iv_F\partial_x \tag{1.71}$$

We insert this spectrum in equation (1.70) and obtain:

$$\mathcal{S}[\varphi] = \frac{1}{2}\varphi V_0^{-1}\varphi - Tr\ln\begin{pmatrix} \partial_\tau - iv_F\partial_x - i\varphi & 0 \\ 0 & \partial_\tau + iv_F\partial_x - i\varphi \end{pmatrix} \tag{1.72}$$

Now we calculate $\mathcal{G}_0$, Eq. (1.69). $\mathcal{G}_{0,R/L}$ satisfies the following equation:

$$\left(\partial_\tau \mp iv_F\partial_x - i\varphi(x,\tau)\right)\mathcal{G}_{0,R/L}(x-x',\tau-\tau';[\varphi]) = \delta(x-x')\delta(\tau-\tau') \tag{1.73}$$

The solution can formally be written in the following way

$$\mathcal{G}_{0,R}(x-x',\tau-\tau';[\varphi]) = g_R(x-x',\tau-\tau')e^{i\theta_R(x,\tau)-i\theta_R(x',\tau')} \tag{1.74}$$
$$\mathcal{G}_{0,L}(x-x',\tau-\tau';[\varphi]) = g_L(x-x',\tau-\tau')e^{i\theta_L(x,\tau)-i\theta_L(x',\tau')} \tag{1.75}$$

The field $\theta(x,\tau)$ has to fulfill the following equation:

$$(\partial_\tau - iv_F\partial_x)\theta_R(x,\tau) = \varphi(x,\tau) \tag{1.76}$$
$$(\partial_\tau + iv_F\partial_x)\theta_L(x,\tau) = \varphi(x,\tau) \tag{1.77}$$

The free Green's function $g_{R/L}$ can be obtained from the following Fourier transformation:

$$g_{R/L}(x,\tau) = \sum_{k_n} e^{-ik_nx}\sum_{\omega_n} e^{-i\omega_n\tau}\frac{1}{i\omega_n \mp v_Fk}$$

$$g_{R/L}(x,\tau) = \pm\frac{i}{2v_F\beta}\frac{1}{\sinh\left(\frac{\pi}{\beta}\left(\frac{x}{v_F}\pm i\tau\right)\right)} \tag{1.78}$$

Here $k_n = 2\pi(n-1/2)/L$ and $\omega_n = 2\pi n/\beta$, n is an integer. It can be shown [25] that:

$$Tr\ln(\partial_\tau \pm iv_F\partial_x - i\varphi(x,\tau)) = Tr\ln(\partial_\tau \pm iv_F\partial_x) + \frac{1}{2}\varphi\pi_{R/L}\varphi \tag{1.79}$$

Where $\pi_{R/L}$ is the polarization operator for right and left movers:

$$\pi_R = \frac{1}{2\pi}\frac{q}{qv_F - i\Omega_n} \tag{1.80}$$

$$\pi_L = \frac{1}{2\pi}\frac{q}{qv_F + i\Omega_n} \tag{1.81}$$

That means that the random phase approximation (RPA) becomes exact in one dimensional systems with linearized spectrum. Finally, we are able to calculate the Green's function of a one dimensional system with interactions.

$$\langle \mathcal{G}_{0,\mu}(x,\tau;[\varphi])\rangle_{S_\varphi} = g_\mu(x,\tau)e^{-\mathcal{B}_{\mu,\mu}(x,\tau)} \tag{1.82}$$

where

$$\begin{aligned}\mathcal{B}_{\mu\nu}(x,\tau) &= \frac{1}{2}\langle[\theta_\mu(x,\tau)-\theta_\nu(0,0)]^2\rangle \\ &= \frac{1}{\beta}\sum_\Omega\int\frac{\mathrm{d}q}{2\pi}\frac{\left[e^{iqx-i\Omega\tau}-1\right]}{(i\Omega_n \mp v_F q)(i\Omega_n \mp v_F q)}V_{\mu\nu}(q,\Omega)\end{aligned} \tag{1.83}$$

$\mu,\nu = R,L$ and $\mathcal{B}_{LL}(x,\tau) = \mathcal{B}_{RR}(-x,\tau)$ and $\mathcal{B}_{LR}(x,\tau) = \mathcal{B}_{RL}(x,\tau)$.

$$V_{RR} = \left[g_4 + (g_4^2 - g_2^2)\frac{1}{2\pi v_F}\frac{qv_F}{i\Omega_n + qv_F}\right]\frac{\Omega_n^2 + q^2v_F^2}{\Omega_n^2 + q^2u^2} \tag{1.84}$$

$$V_{RL} = g_2\frac{\Omega_n^2 + q^2v_F^2}{\Omega_n^2 + q^2u^2} \tag{1.85}$$

Furthermore we obtained the same Plasmon velocity u as in (1.49):

$$u = v_F\sqrt{\left(1+\frac{g_4}{2\pi v_F}\right)^2 - \left(\frac{g_2}{2\pi v_F}\right)^2}$$

The result of equation (1.83) is:

$$\mathcal{B}_{RR}(x,\tau) = -\ln\eta_R(x,\tau) - \frac{(1-K)^2}{4K}\mathcal{L}(x,\tau) \tag{1.86}$$

$$\mathcal{B}_{RL}(x,\tau) = -\frac{(1-K^2)}{4K}\mathcal{L}(x,\tau) \tag{1.87}$$

where

$$\mathcal{L}(x,\tau) = \ln\frac{(\pi\alpha/\beta)^2}{\sinh\left[\frac{\pi}{\beta}\left(\frac{x}{u}+i\tau\right)\right]\sinh\left[\frac{\pi}{u}-i\tau\right]} \tag{1.88}$$

$$\eta_{R/L}(x,\tau) = \frac{v_F}{u}\frac{\sinh\left[\frac{\pi}{\beta}\left(\frac{x}{v_F}\pm i\tau\right)\right]}{\sinh\left[\frac{\pi}{\beta}\left(\frac{x}{u}\pm i\tau\right)\right]} \tag{1.89}$$

The role of $\eta_{R/L}$ is to replace the Fermi velocity v_F in the free electron propagator (1.78) by the Plasmon velocity u. Thus it directly shows the velocity renormalization due to interactions. As we will see later, the functional bosonization technique is sometimes more convenient since the fundamental object is still the electron Green's function. Moreover, the electron Green's function clearly decomposes into the free Green's function times a phase that accounts for interactions. Therefore, it is a more transparent approach when we want to compare one dimensional effects with mesoscopic effects in higher dimension, such as weak localization.

1.4 Keldysh formalism

The Keldysh formalism is like its imaginary time counterpart, the Matsubara technique, a method to evaluate thermal and quantum expectation values of observables with quantum field theory tools. Both techniques are well-known but Matsubara calculations seemed to be more convenient for systems in equilibrium since the Keldysh approach requires twice the number of degrees of freedom. On the other hand it is only possible to use the Matsubara method close to the linear response regime whereas the Keldysh technique can be extended beyond it. Since non-equilibrium physics has become more important the Keldysh technique became more famous. Unlike the Matsubara approach, the Keldysh technique is formulated in real time and hence, an analytic continuation in the end of the calculation is not necessary. But there are even more advantages in equilibrium which outweigh the extra effort due to the doubled amount of fields. In disordered systems for example one can evaluate observables with respect to a specific disorder configuration as well as impurity averaged expectation values of observables. The latter case means, we have to evaluate an observable, like the conductivity, for a specific disorder pattern and average over all possible configurations of the impurities. This is needed for observables in nano structures which behave self-averaging, i.e. they are composed of many similar but statistically independent parts. Technically it is quite hard to perform the disorder average after the thermal/quantum average due to normalization factors and one would like to do it the other way round. Three methods: supersymmetry, Keldysh and replicated field theory have been invented to be able to average over the impurity configurations first. In the Keldysh technique this problem is solved since by definition this normalization factor is unity. A detailed discussion of the advantages of the Keldysh technique in disordered systems is given in the next chapter.
In the following we are going to construct the Keldysh formalism.

A major problem of the many body theory is that in general we do not know the eigenstates of the full interacting system. However we need them in order to calculate expectation values of observables. The idea of adiabatic switching on of interactions was invented to be able to average over known eigenstates of the noninteracting system. It means that the system starts at time $-\infty$ with no interactions and then the interaction strength is slowly increased to the required value at time t where the observable is measured. Let us consider the expectation value for some observable $\hat{\mathcal{O}}(t)$.

$$\langle \hat{\mathcal{O}}(t) \rangle = \frac{Tr[\hat{\mathcal{O}}\rho(t)]}{Tr[\rho(t)]} \tag{1.90}$$

The noninteracting ground state at $t = -\infty$ evolves to the ground state of the interacting system $|GS\rangle = \hat{\mathcal{U}}_{t,-\infty}|0\rangle$ at time t, see Fig. (1.6). Here $\hat{\mathcal{U}}_{t,-\infty}$ is the time evolution operator that takes a state at $t = -\infty$ and moves it to time t. The observable $\hat{\mathcal{O}}$ is measured at time t.

$$\langle GS|\hat{\mathcal{O}}|GS\rangle = \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \tag{1.91}$$

This construction involves a time contour similar to Fig. (1.6,a), which goes from $-\infty$ to t and back to $-\infty$. In the so called $T = 0$ method of quantum field theory a simpler time contour is used, see Fig. (1.6,b). Here one averages over initial and final states at $t = \pm\infty$. This is done by a small manipulation of Eq. (1.91). Due to the Gell-Mann-Low theorem, [18]

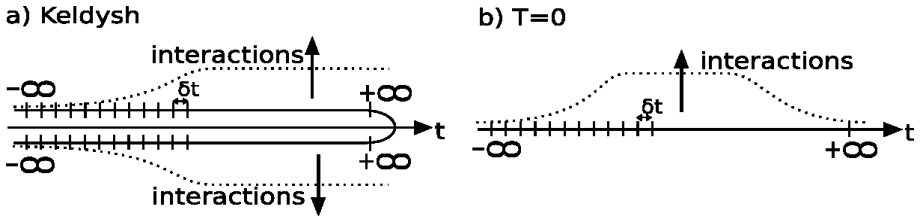


Figure 1.6: The solid line is the time contour on which the system evolves. The dotted line represents the strength of the interactions in the system. At $t = -\infty$ the system is always non-interacting. The observable $\mathcal{O}(t)$ is evaluated at time t where full interactions are present. To construct the free action we discretize time steps of length δ_t

p.91, we can do the following replacement:

$$\langle 0|\hat{\mathcal{U}}_{-\infty,0} = e^{-i\varphi}\langle 0|\hat{\mathcal{U}}_{\infty,-\infty}\hat{\mathcal{U}}_{-\infty,0} = \frac{\langle 0|\hat{\mathcal{U}}_{\infty,0}}{\langle 0|\hat{\mathcal{U}}_{\infty,-\infty}|0\rangle} \tag{1.92}$$

Thus the system collects a phase factor $e^{i\varphi} = \langle 0|\hat{\mathcal{U}}_{\infty,-\infty}|0\rangle$ during the evolution from $t = -\infty$ to ∞. Moreover interactions are adiabatically decreased by going to $t = \infty$. Thus, in the $T = 0$ method Eq. (1.91) becomes:

$$\langle GS|\hat{\mathcal{O}}|GS\rangle = \langle 0|\hat{\mathcal{U}}_{\infty,-\infty}\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle/e^{i\varphi} = \langle 0|\hat{\mathcal{U}}_{\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle/e^{i\varphi} \tag{1.93}$$

In the language of perturbation theory the normalization factor $e^{i\varphi}$ represents a sum of all disconnected diagrams. In the following we consider an alternative construction of expectation values known as the Keldysh technique. Let us rewrite equation (1.90):

$$\langle\hat{\mathcal{O}}(t)\rangle = \frac{Tr[\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(t)]} \tag{1.94}$$

The density matrix at time t can be obtained from the initial density matrix
$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)[\hat{\mathcal{U}}_{t,-\infty}]^\dagger$. In other words, in order to get the expectation value we insert the initial distribution of states $\hat{\rho}(-\infty)$, let the system evolve from $-\infty$ to time t, measure the observable and go back to $-\infty$. However it is convenient to go from t further up to ∞ and then back to $-\infty$ like it is depicted in figure (1.6).

$$\langle\hat{\mathcal{O}}(t)\rangle = \frac{Tr[\hat{\mathcal{U}}_{-\infty,\infty}\hat{\mathcal{U}}_{\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(-\infty)]} \tag{1.95}$$

Here the observable $\hat{\mathcal{O}}$ is inserted at time t on the forward contour. However, $\hat{\mathcal{O}}$ can be inserted on the backward contour as well since $\hat{\mathcal{U}}_{t,\infty}\hat{\mathcal{U}}_{\infty,t} = 1$.
The closed Keldysh contour is shown in figure (1.6). We denote the time evolution operator which moves the state along the whole Keldysh contour $\mathcal{C}$ by
$\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,\infty}\hat{\mathcal{U}}_{\infty,-\infty} = 1$.
Furthermore the partition sum is

$$\mathcal{Z} = \frac{Tr[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(-\infty)]} = 1 \tag{1.96}$$

Equation (1.96) is fulfilled as long as all external fields are exactly the same on the forward and backward branches, as will be shown in the next subsection. Moreover Eq. (1.96) is the reason why the Keldysh approach is useful for disordered systems. However, we postpone the applications to disordered systems to the next chapter, section (2.1).
In the next section the functional field approach to many body physics is constructed.

1.4.1 Bosonic action in the Keldysh technique

In the following subsection we will explicitly construct an action of free Bosons in discretized time. After that, the continuum limit is taken.
In order to evaluate the partition sum $\mathcal{Z} = Tr[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}]/Tr[\hat{\rho}]$ we divide the Keldysh contour $\mathcal{C}$ in $2(N-1)$ small time steps of length δ_t as it is shown in figure (1.6). Since the time contour is closed the first and last point coincide $t_1 = t_{2N} = -\infty$. Moreover there is no physical difference between time t_N and t_{N+1}, since $t_N = t_{N+1} = \infty$. Hence there is no evolution from t_N till t_{N+1}.
At each time step t_i we insert the resolution of unity in the coherent state basis [12], [8][8]:

$$\hat{1} = \iint \frac{d(\mathcal{R}e\phi_i)d(\mathcal{I}m\phi_i)}{\pi} e^{-|\phi_i|^2} |\phi_i\rangle\langle\phi_i| \tag{1.97}$$

The numerator in the partition sum (1.96) becomes:

$$Tr[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}] = \int \langle\phi_{2N}|\hat{\mathcal{U}}_{-\delta_t}|\phi_{2N-1}\rangle \cdot \langle\phi_{2N-1}|\hat{\mathcal{U}}_{-\delta_t}|\phi_{2N-2}\rangle \cdot \ldots \cdot \langle\phi_{N+1}|\hat{1}|\phi_N\rangle \cdot \ldots$$
$$\ldots \cdot \langle\phi_2|\hat{\mathcal{U}}_{\delta_t}|\phi_1\rangle\langle\phi_1|\hat{\rho}|\phi_{2N}\rangle \tag{1.98}$$

Where the time evolution operator on the Keldysh contour $\hat{\mathcal{U}}_{\mathcal{C}}$ is split into a product of evolution operators $\hat{\mathcal{U}}_{\pm\delta_t}$. Between each time step we inserted equation (1.98) and thus we have to integrate over all ϕ-fields at intermediate time steps.
Since coherent states are eigenstates of the bosonic annihilation operator b,
$b|\phi_j\rangle = \phi_j|\phi_j\rangle$ we obtain the following relation in the limit $\delta_t \to 0$:

$$\langle\phi_{j+1}|\hat{\mathcal{U}}_{\pm\delta_t}|\phi_j\rangle = \langle\phi_{j+1}|e^{\mp i\hat{\mathcal{H}}(b^\dagger,b)\delta_t}|\phi_{2N-1}\rangle \approx \langle\phi_{j+1}|\phi_j\rangle e^{\mp i\hat{\mathcal{H}}(\bar{\phi}_{j+1},\phi_j)\delta_t} \tag{1.99}$$

where $\hat{\mathcal{H}}(b^\dagger, b)$ is a normally-ordered Hamiltonian expressed in the Boson fields $b^\dagger, b$. Furthermore, we use the following property of coherent states: $\langle\phi_{j+1}|\phi_j\rangle = e^{\bar{\phi}_{j+1}\phi_j}$

Let us consider a simple example of a Hamiltonian: $\mathcal{H}_0 = \omega b^\dagger b$. Therefore $\langle\phi_1|\hat{\rho}|\phi_{2N}\rangle$ in equation (1.98) becomes:

$$\langle\phi_1|e^{-\beta(\omega-\mu)b^\dagger b}|\phi_{2N}\rangle = e^{\bar{\phi}_1\phi_{2N}\rho(\omega)} \tag{1.100}$$

where $\rho(\omega) = \exp(-\beta(\omega-\mu))$.
Furthermore, equations (1.99) and (1.100) will be inserted in Eq. (1.98). Thus, we obtain the partition sum $\mathcal{Z} = Tr[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}]/Tr[\hat{\rho}]$:

$$\mathcal{Z} = \frac{1}{Tr[\hat{\rho}]} \iint \prod_{j=1}^{2N} \left[\frac{d(\mathcal{R}e\phi_j)d(\mathcal{I}m\phi_j)}{\pi}\right] \exp\left(i \sum_{j,j'=1}^{2N} \bar{\phi}_j G^{-1}_{jj'} \phi_{j'}\right) \tag{1.101}$$

[8] chapter 4

Since we work in a discretized framework the inverse propagator $G^{-1}_{jj'}$ is a $2N \times 2N$ matrix. To underline the structure of the inverse propagator we consider a three time step example, i.e. $N = 3$.

$$iG^{-1}_{jj'} = \begin{pmatrix} -1 & & & & & \rho(\omega) \\ 1 - i\omega\delta_t & -1 & & & & \\ & 1 - i\omega\delta_t & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 + i\omega\delta_t & -1 & \\ & & & & 1 + i\omega\delta_t & -1 \end{pmatrix} \tag{1.102}$$

To show that the partition sum is indeed unity we integrate out the ϕ fields in equation (1.101) by using the result of a Gaussian integration in higher dimensions $\int_{-\infty}^{\infty} d^N x e^{-x^\dagger Ax} = \sqrt{\pi^N / det(A)}$. Hence equation (1.101) reduces to:

$$\mathcal{Z} = \frac{Det^{-1}[i\hat{G}^{-1}]}{Tr[\hat{\rho}]} = 1 \tag{1.103}$$

Based on the matrix in equation (1.102) the determinant can be calculated explicitly

$$Det[i\hat{G}^{-1}] = (-1)^{2N} - \rho(\omega)(1 - (i\omega\delta_t)^2)^{N-1} \approx 1 - \rho(\omega)e^{(\omega\delta_t)^2(N-1)} \to 1 - \rho(\omega) \tag{1.104}$$

This approximation is valid for $\delta_t \to 0$, $N \to \infty$ and $N\delta_t \to$ const. Moreover $Tr[\hat{\rho}]$ in equation (1.101) is:

$$Tr[\hat{\rho}] = \sum_{n=0}^{\infty} e^{-\beta(\omega-\mu)n} = \frac{1}{1 - \rho(\omega)} \tag{1.105}$$

According to (1.102) the action in equation (1.101) can also be rewritten in the following form:

$$S[\bar{\phi}, \phi] = \sum_{j=2}^{2N} \left[i\bar{\phi}_j \frac{\phi_j - \phi_{j-1}}{\delta t_j} - \omega\bar{\phi}_j\phi_{j-1} \right] \delta t_j + i\bar{\phi}_1 \left[\phi_1 - \rho(\omega)\phi_{2N}\right] \tag{1.106}$$

where $\delta t_j = \pm\delta t$ depending on the fields being on the upper or lower branch of the Keldysh contour.
For further calculations it is necessary to derive Green's functions. For the 2-point function we have to calculate the following expectation value:

$$\langle \phi_j \bar{\phi}_{j'} \rangle = \int \mathcal{D}[\bar{\phi}\phi] \phi_j \bar{\phi}_{j'} exp \left(i \sum_{j,j'=1}^{2N} \bar{\phi}_j G^{-1}_{jj'} \phi_{j'} \right) = iG_{jj'} \tag{1.107}$$

We remind that $\mathcal{Z} = 1$ and thus in contrast to the Matsubara formalism there is no normalization factor. The next step will be the calculation of the inverse of $iG^{-1}_{jj'}$ of equation (1.102). For the $N = 3$ example we get:

$$iG_{jj'} = \frac{1}{1 - \rho} \begin{pmatrix} 1 & \rho e^h & \rho e^{2h} & \rho e^{2h} & \rho e^h & \rho \\ e^{-h} & 1 & \rho e^h & \rho e^h & \rho & \rho e^{-h} \\ e^{-2h} & e^{-h} & 1 & \rho & \rho e^{-h} & \rho e^{-2h} \\ e^{-2h} & e^{-h} & 1 & 1 & \rho e^{-h} & \rho e^{-2h} \\ e^{-h} & 1 & e^h & e^h & 1 & \rho e^{-h} \\ 1 & e^h & e^{2h} & e^{2h} & e^h & 1 \end{pmatrix} \tag{1.108}$$

We can divide the matrix in (1.108) according to the position of the time arguments j, j' on the upper (+) and lower branch (−) in four different sectors. The corresponding correlation functions are

$$\langle \phi_{j+} \bar{\phi}_{j'-} \rangle = iG^{<}_{jj'} = n_B \cdot exp[-(j-j')h] \tag{1.109}$$

$$\langle \phi_{j-} \bar{\phi}_{j'+} \rangle = iG^{>}_{jj'} = (n_B+1) \cdot exp[-(j-j')h] \tag{1.110}$$

$$\langle \phi_{j+} \bar{\phi}_{j'+} \rangle = iG^{T}_{jj'} = \frac{1}{2}\delta_{jj'} + \theta(j-j')iG^{>}_{jj'} + \theta(j'-j)iG^{<}_{jj'} \tag{1.111}$$

$$\langle \phi_{j-} \bar{\phi}_{j'-} \rangle = i\tilde{G}^{\tilde{T}}_{jj'} = \frac{1}{2}\delta_{jj'} + \theta(j'-j)iG^{>}_{jj'} + \theta(j-j')iG^{<}_{jj'} \tag{1.112}$$

Here n_B is the Bose-Einstein distribution function: $\rho/(1-\rho) = 1/(\exp[-\beta(\omega-\mu)]-1)$. The fields of the lesser Green's function $G^{<}_{jj'}$ are ordered such that the field with the later time argument j' is on the right of the field with the earlier time argument[9] j and vice versa for $iG^{>}_{jj'}$. $G^{T}_{jj'}$ is the time ordered Green's function which means that it will place the field of the later time argument to the left. $\tilde{G}^{\tilde{T}}_{jj'}$ is the anti time ordered Green's function and it places the field with the earlier time argument to the left. The θ step function in equations (1.109) to (1.112) is defined such that $\theta(0) = 1/2$ and $\theta(j) + \theta(-j) = 1$.
The matrix in equation (1.108) of the example ($N = 3$) can be characterized in the following way:

$$iG_{jj'} = \begin{pmatrix} iG^{T}_{jj'} & iG^{<}_{jj'} \\ iG^{>}_{jj'} & i\tilde{G}^{\tilde{T}}_{jj'} \end{pmatrix} \tag{1.113}$$

In Eq. (1.108) $G_{jj'}$ is a matrix in discretized time indices j, which are taken from the interval $[0, \ldots, 2N]$. Equations (1.109) to (1.112) suggest a more compact notation where $G_{jj'}$ is written as a 2×2 matrix as it is done in Eq. (1.113). The entries are now the Green's functions defined above and the indices j, j' are running from $0 \to N$. The four components of Eq. (1.113) define the so called Keldysh space. From equation (1.109) to (1.112) it is obvious that not all four components of the matrix (1.113) are independent. There is a relation between them:

$$G^{T}_{jj'} + \tilde{G}^{\tilde{T}}_{jj'} - G^{<}_{jj'} - G^{>}_{jj'} = -i\delta_{jj'} \tag{1.114}$$

$$\Rightarrow G^{T}_{jj'} - \tilde{G}^{\tilde{T}}_{jj'} = sign(j-j')(G^{>}_{jj'} - G^{<}_{jj'}) \tag{1.115}$$

It is convenient to perform a rotation in the 2×2 Keldysh space such that one entry is always zero by definition. Let's define the following linear transformation in Keldysh space:

$$\begin{aligned} &\phi^{cl}_j = \phi_{j+} + \phi_{j-} \qquad\qquad \phi^{q}_j = \phi_{j+} - \phi_{j-}; \\ &\phi_{j+} = \tfrac{1}{2}\left(\phi^{cl}_j + \phi^{q}_j\right); \quad \phi_{j-} = \tfrac{1}{2}\left(\phi^{cl}_j - \phi^{q}_j\right) \end{aligned} \tag{1.116}$$

This transformation may seem asymmetric, and it is indeed not the standard convention used in [12]. However, for later calculations of disordered systems it is more convenient.

[9] Note that j' is on the (-) backward contour

For the time being we just take that transformation as a definition and apply it to the matrix in equation (1.113):

$$\begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

Thus:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} iG^T_{jj'} & iG^<_{jj'} \\ iG^>_{jj'} & i\tilde{G}^T_{jj'} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} iG^K_{jj'} & iG^R_{jj'} \\ iG^A_{jj'} & \frac{1}{4}\delta_{jj'} \end{pmatrix} \tag{1.117}$$

Here we introduced three new Green's functions and for the lower right entry we used relation (1.114). Throughout the whole text we will use these three Greens functions, they are the basic objects in the Keldysh technique.

$$G^R_{jj'} = -i\langle \phi^{cl}_j \bar{\phi}^q_{j'} \rangle = \theta(j-j')2\left(G^>_{jj'} - G^<_{jj'}\right) = -i\theta(j-j')e^{-(j-j')h} \tag{1.118}$$

$$G^A_{jj'} = -i\langle \phi^q_j \bar{\phi}^{cl}_{jj'} \rangle = \theta(j'-j)2\left(G^<_{jj'} - G^>_{jj'}\right) = i\theta(j'-j)e^{-(j-j')h} \tag{1.119}$$

$$G^K_{jj'} = -i\langle \phi^{cl}_j \bar{\phi}^{cl}_{j'} \rangle = -i\delta_{jj'} + 2G^>_{jj'} + G^<_{jj'} = -i\delta_{jj'} - i2(2n_B+1)e^{-(j-j')h} \tag{1.120}$$

Due to the θ-function the retarded and advanced Green's functions are lower and upper triangular matrices in time indices j,j'. Hence if two retarded/advanced matrices are multiplied one obtains again a lower/upper triangular matrix. This conservation of retardation reflects the important property that causality is preserved when multiplying two retarded propagators. Moreover, since $(G^<)^\dagger = -G^>$ the Green's functions, Eq.(1.118) and (1.119), are related by transposition of the time arguments j,j':

$$G^A = \left(G^R\right)^\dagger \tag{1.121}$$

The Keldysh Green's function is anti-Hermitian:

$$G^K = -\left(G^K\right)^\dagger \tag{1.122}$$

As argued above the continuum limit is given by the conditions $\delta_t \to 0$, $N \to \infty$ while $N\delta_t \to const.$ Furthermore $t_j = j \cdot \delta_t$ and $-(j-j')h = -i\omega(t-t')$. As concerns the $\delta_{jj'}$ in equation (1.117), we choose it to be zero in the continuum limit. This is a consistent choice for two reasons. Firstly all important physical observables are given by off-diagonal elements, even observables like the particle occupation number: $\langle n_B(t)\rangle = i\lim_{\epsilon\to 0} G^T(t, t' = t+\epsilon)$ which corresponds to $\langle n_B(t_j)\rangle = iG^T_{jj+1} = iG^<_{jj+1}$. Secondly, due to matrix multiplication of several Green's functions in perturbation series, the intermediate expressions contain multiple sums[10] of the form: $\delta_t^2 \sum_{j,j'} \delta_{jj'} G^{R/A/K}_{j'j} \to \delta_t^2 N \to 0$. This is the case for example in more complex diagrams which consist of many Green's functions [12].

All calculations in discretized time are really necessary in order to keep all factors which are important for convergence of the expression in the continuum limit. If we encounter uncertainties in an expression given in continuous time we can resolve it by going to discretized time.

To conclude, the Green's functions in the continuum limit are:

$$-i\langle \phi_\alpha(t)\bar{\phi}_\beta(t')\rangle = D^{\alpha\beta}(t,t') = \begin{pmatrix} D^K(t,t') & D^R(t,t') \\ D^A(t,t') & 0 \end{pmatrix} \tag{1.123}$$

[10] or multiple integrals since $\sum_j \delta t \to \int \mathrm{d}t$

The Fourier transformed elements are given by:

$$D^R(\epsilon) = (\epsilon - \omega + i\delta)^{-1} \tag{1.124}$$

$$D^A(\epsilon) = (\epsilon - \omega - i\delta)^{-1} \tag{1.125}$$

$$D^K(\epsilon) = -2\pi i[2n_B(\epsilon) + 1]\delta(\epsilon - \omega) \tag{1.126}$$

$\langle \phi_q(t)\bar{\phi}_q(t')\rangle = 0$ is a fundamental property in the Keldysh technique.

Note that the retarded and advanced Green's functions contain only information about the spectrum, they are independent of the occupation number. The Keldysh component introduces the occupation of the states into the theory. This seperation is good if we are not too far from equilibrium. In thermal equilibrium we can relate the Keldysh component to the retarded and advanced parts which results from the fluctuation dissipation theorem.

$$D^K(\epsilon) = \left[D^R(\epsilon) - D^A(\epsilon)\right] \coth\left(\frac{\epsilon}{2T}\right) \tag{1.127}$$

In general the Keldysh Green's function is given by the retarded, advanced and the Hermitian matrix $F(t,t') = F^\dagger(t,t')$ which is the distribution function $f(\tau,\epsilon)$ after a Wigner transformation[11].

$$D^K_{t,t'} = D^R_{t,t''} \circ F_{t'',t'} - F_{t,t''} \circ D^A_{t'',t'} \tag{1.128}$$

The circles $\circ$ denote an integral over times t'' for discretized time it would be the usual matrix product. As a result the action in the continuum is given by:

$$S[\phi^{cl}, \phi^q] = \iint_{-\infty}^{\infty} dt dt' (\bar{\phi}_{cl}, \bar{\phi}_q)_t \begin{pmatrix} 0 & (D^{-1})^A \\ (D^{-1})^R & (D^{-1})^K \end{pmatrix}_{t,t'} \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}_{t'} \tag{1.129}$$

where

$$[D^{-1}]^{R(A)} = [D^{R(A)}]^{-1} = \epsilon - \omega \pm i\delta \to \delta_{t,t'}(i\partial_t - \omega \pm i\delta) \tag{1.130}$$

$$[D^{-1}]^K = [D^R]^{-1} \circ F - F \circ [D^A]^{-1} \tag{1.131}$$

So far we discussed the non-interacting case. Now we are going to include vertices which are not quadratic in the fields like a ϕ^4-term. Then the problem is in general no longer analytically solvable and one has to employ approximations like perturbation theory instead. In the following we want to show that $\mathcal{Z} = 1$ in a perturbation series. Suppose we add an additional ϕ^4-term $\mathcal{H}_{int}$ to the quadratic Hamiltonian $\mathcal{H}_0 = \omega b^\dagger b$, this yields the following action:

$$\mathcal{H}_{int} = \lambda b^\dagger b^\dagger b b \Rightarrow S_{int} = \int_{\mathcal{C}_K} dt \lambda \bar{\phi}\bar{\phi}\phi\phi \tag{1.132}$$

Here, λ is the coupling constant. It could play the role of a charging energy in quantum dots. Usually it is more convenient to use the action than the Hamiltonian of a quantum system in quantum field theory. Instead of explicitly constructing the Green's function structure in discretized time we use a common recipe[12] to go from an action of fields ϕ given on the Keldysh contour to an action in terms of ϕ_{cl}, ϕ_q. Thus the action in Eq. (1.132) becomes:

$$S_{int} = \lambda \int_{-\infty}^{\infty} dt \left[\bar{\phi}_{cl}\bar{\phi}_q \left(\phi_{cl}\phi_{cl} + \phi_q\phi_q\right) + c.c.\right]. \tag{1.133}$$

[11] The Wigner transformation of a function $F(t,t')$ is defined as [12],p.17 $f(\rho,\tau) := \int F\left(\rho + \frac{\tilde{t}}{2}, \rho - \frac{\tilde{t}}{2}\right) e^{i\tau\tilde{t}}$

[12] [12] section 3, p.12

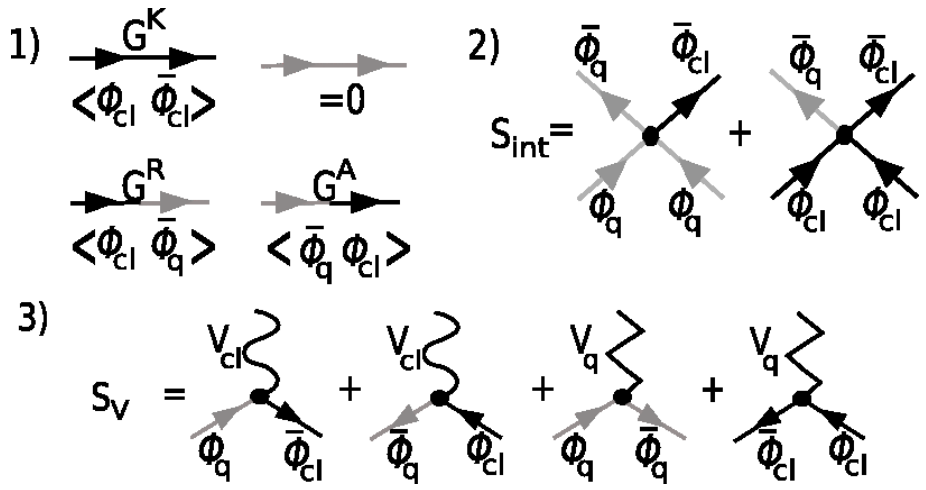


Figure 1.7: 1) The four possible Green's functions in Keldysh space. Classical ϕ_{cl} and quantum fields ϕ_q are represented by black and grey lines respectively. As mentioned in the text, the definition of these fields are such that the quantum-quantum correlator $\langle\phi_q\phi_q\rangle$ is zero. 2) Interaction vertices $S_{int} \sim \bar{\phi}\bar{\phi}\phi\phi$. 3) S_V couples source fields V to the bosonic fields.

The interaction terms are depicted in figure (1.7,2). In figure (1.8,1) it is shown, up to the first order, that the property $\mathcal{Z} = 1$ still holds in a perturbation expansion since disconnected diagrams vanish.
For calculating expectation values in the functional fields formalism, it is convenient to introduce source fields V which are coupled to the ϕ fields. Furthermore we rewrite the correlation functions as functional derivatives with respect to the source fields. Then we are able to calculate correlation functions by taking functional derivatives of the partition sum with respect to the source field V. As an example of that procedure imagine we add a source action to our quadratic action $S_0 = \int dt\omega\bar{\phi}\phi$:

$$S_V = \int (\phi V + \bar{\phi}\bar{V})\mathrm{d}t \tag{1.134}$$

The partition function is

$$\mathcal{Z}[V, \bar{V}] = \int \mathcal{D}[\phi, \bar{\phi}] e^{iS_0 + iS_V} \tag{1.135}$$

It can be easily seen that we can use the partition function as a generating function for correlation functions, for example:

$$\langle\phi\bar{\phi}\rangle = \frac{\delta^2}{\delta\bar{V}\delta V}\mathcal{Z}[V, \bar{V}]|_{\bar{V}=0} \tag{1.136}$$

If the source field V preserves forward-backward symmetry on the Keldysh contour then $\mathcal{Z} = 1$, as shown in Fig. (1.8,2) up to first order in S_V.

In conclusion, all important properties of the Keldysh approach to quantum field theory have been discussed. In the next chapters we are going to apply this tool to one dimensional systems with disorder.

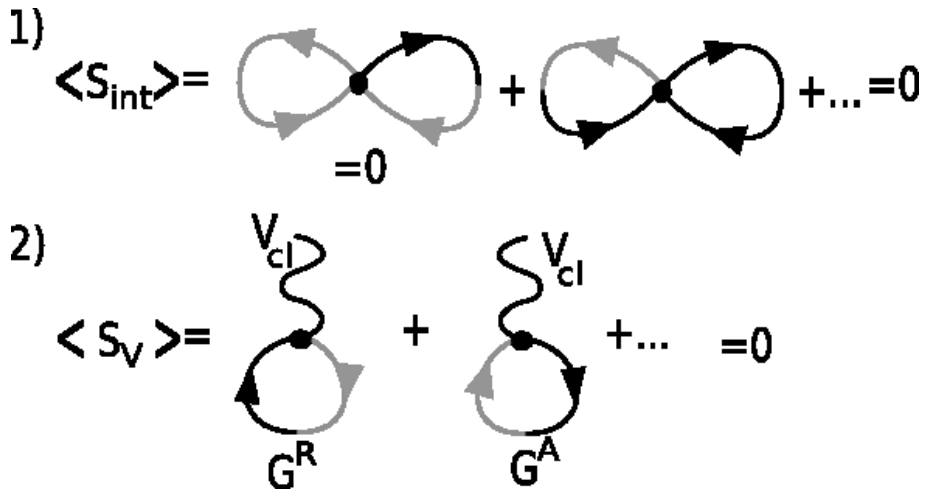


Figure 1.8: 1) Vanishing first order correction in S_{int} to the partition function $\mathcal{Z}_0$. This is an example showing that in perturbation theory the fundamental property, that the partition function is $\mathcal{Z} = 1$, is preserved. 2) As long as we do not explicitly break the forward-backward symmetry, i.e. $V_q = 0$, the first order correction in V_{cl} is zero, since $\langle S_V \rangle \sim G^A(t,t) + G^R(t,t) = 0$.

1.5 Transport in 1D systems

The main object of our considerations will be the frequency dependent Drude conductivity σ_D. The conductivity is the proportionality coefficient between the electric current density $ej(x,t)$ and the applied electromagnetic field $E(t) = E_0 e^{-i(\omega+i\delta)t}$: $ej(x,t) = \sigma E(t)$. In the noninteracting case the expansion of σ_D in powers of the inverse scattering time τ reads

$$\sigma(\omega) = \frac{\sigma_0}{1 + i/(\omega\tau)} \approx \sigma_0 \left(1 - \frac{i}{\omega\tau} + \dots\right) \tag{1.137}$$

where $\sigma_0 = \frac{e^2}{\hbar\pi} \frac{iv_F}{\omega+i\delta}$ is the conductivity of a clean system and δ is an infinitesimal. For one dimensional interacting systems σ can be calculated within the linear response theory [5]. We start with the expectation value of the electric current density $ej(x,t)$.

From linear response theory [18](p.185) we know that

$$\langle ej(x_j,t_f)\rangle = -D \cdot A(x_f,t_f) + i \int_{-\infty}^{t_f} dt_i \int dx_i \langle [ej(x_f,t_f), ej(x_i,t_i)]\rangle A(x_i,t_i) \tag{1.138}$$

Where $[\dots,\dots]$ is the commutator.
$D = e^2 uK/\pi\hbar$ denotes the diamagnetic term which can be derived from the Hamiltonian when electric fields are included through minimal coupling of a vector potential A to the momentum. In the Hamiltonian of the bosonized system (1.48) the field $\Pi(x,t)$ plays the role of the momentum. Hence, the common procedure: $\Pi(x,t) \to \Pi(x,t) - \frac{e}{\pi\hbar}A$ (with an extra factor due to the definition of Π) introduces electromagnetic fields. The diamagnetic term is obtained by derivating the Hamiltonian two times with respect to A.

$$D = -\frac{\partial^2 \mathcal{H}}{\partial A \partial A} = \frac{e^2 uK}{\pi\hbar} \tag{1.139}$$

Let us rewrite the second term of equation (1.138) in a more convenient form for the Keldysh technique.

$$\langle ej(x_f,t_f)\rangle = -D\cdot A(x_f,t_f) + i\int dx_i \int_{\mathcal{C}_K} dt_i \langle ej(x_f,t_f)ej(x_i,t_i)\rangle A(x_i,t_i) \tag{1.140}$$

$\mathcal{C}_K$ denotes the Keldysh contour which starts from $-\infty$, then goes to t_f and finally back to $-\infty$.
We need an expression for the current in terms of the Boson fields.
The continuity equation relates the electric current density $ej(x,t)$ to the charge density $e\rho(x,t)$.

$$\partial_t e\rho(x,t) = -\nabla ej(x,t) \tag{1.141}$$

Furthermore we use the relation from the bosonization section between the density $\rho(x,t)$ and the Boson field $\phi(x,t)$:

$$\rho(x) = -\frac{1}{\pi}\nabla\phi(x). \tag{1.142}$$

If we insert (1.142) into (1.141) we get.

$$\nabla\left(-\frac{e}{\pi}\partial_t\phi(x) + ej(x,t)\right) = 0 \tag{1.143}$$

The expression in brackets has to be constant and we choose this constant to be zero:

$$ej(x,t) = \frac{e}{\pi}\partial_t\phi(x,t). \tag{1.144}$$

In order to get the time evolution of the electric current operator we solve the Heisenberg equation of motion[13]

$$\partial_t ej(x,t) = i/\hbar[\mathcal{H}, ej(x,t)] = \frac{i}{\hbar}\left[\mathcal{H}, \frac{e}{\pi}\partial_t\phi(x,t)\right] \tag{1.145}$$

Since Π and ϕ are canonically conjugate fields

$$[\phi(x),\Pi(y)] = i\delta(x-y) \tag{1.146}$$

we get

$$\partial_t ej(x,t) = \frac{uKe}{\hbar}\partial_t\Pi(x,t) \tag{1.147}$$

$$\Rightarrow ej(x,t) = \frac{e}{\hbar}(uK)\Pi(x,t) \tag{1.148}$$

To summarize, the electric current can be identified with the $\Pi(x,t)$ field from the bosonized quadratic Hamiltonian. However, it is more convenient to work with the action in the language of Feynman path integrals. The quadratic action of the bosonized model is:

$$S_{\Pi,\phi} = \int_{-\infty}^{\infty} dt \int dx \left[\Pi(x,t)\partial_t\phi(x,t) - \frac{1}{2\pi}\left(uK(\pi\Pi(x,t))^2 + \frac{u}{K}(\partial_x\phi(x,t))^2\right)\right] \tag{1.149}$$

If we define this action on the Keldysh contour $\mathcal{C}_K$ we get[14] for the first term

$$\int_{\mathcal{C}_K}\Pi\partial_t\phi \to \int_{-\infty}^{\infty} dt\frac{1}{2}\left(\Pi_{cl}\partial_t\phi_q + \Pi_q\partial_t\phi_{cl}\right) \tag{1.150}$$

[13] $\mathcal{H} = \frac{1}{2\pi}\int dx[uK(\pi\Pi(x))^2 + \frac{u}{K}(\partial\phi(x))^2]$, units $[\Pi] = \sqrt{J\cdot s}/m$, $[\phi] = \sqrt{J\cdot s}$
[14] some of the steps in the calculation are made in Appendix A.1

Using (1.148), we can calculate the retarded current-current correlation function in (1.140). In the following we focus on the part of equation (1.140) with the current-current correlation function. As we will see, it contains a term that exactly cancels the diamagnetic term $D \cdot A$.

$$i \int dx_i \int_{\mathcal{C}_K} dt_i \langle ej(x_f, t_f) ej(x_i, t_i) \rangle A(x_i, t_i) \tag{1.151}$$

$$= i \left(\frac{euK}{\hbar} \right)^2 \int dx_i \int_{\mathcal{C}_K} dt_i \langle \Pi(x_f, t_f) \Pi(x_i, t_i) \rangle A(x_i, t_i) \tag{1.152}$$

Where we used Eq. (1.148) for the current.
Now we rewrite the contour integral in a normal one using $\pm$-fields on the forward and backward contour, see section (1.4).

$$\int_{\mathcal{C}_K} dt_i \langle \Pi(x_f, t_f) \Pi(x_i, t_i) \rangle = \int_{-\infty}^{t_f} dt_i \langle \Pi_+(x_f, t_f) \Pi_+(x_i, t_i) \rangle + \int_{t_f}^{-\infty} dt_i \langle \Pi_+(x_f, t_f) \Pi_-(x_i, t_i) \rangle \tag{1.153}$$

Note that the observation time t_f belongs to the upper contour. However, it does not change the final result (1.158) if we had put t_f on the lower and write $\Pi_-(x_f, t_f)$ instead[15]. The correlation functions in equation (1.153) are essentially the bigger $D^>$ and lesser $D^<$ Green's functions of Π-fields. Therefore, equation (1.153) can be simplified:

$$\int_{-\infty}^{\infty} dt_i \theta(t_f - t_i) \left(i D_\Pi^>(x_f - x_i, t_f - t_i) - i D_\Pi^<(x_f - x_i, t_f - t_i) \right) \tag{1.154}$$

As long as $\langle \Pi_q(x_f, t_f) \Pi_q(x_i, t_i) \rangle = 0$, equation (1.154) is equal to the retarded Green's function:

$$\frac{1}{2} \int_{-\infty}^{\infty} dt_i \langle \Pi_{cl}(x_f, t_f) \Pi_q(x_i, t_i) \rangle A(x_i, t_i) \tag{1.155}$$

The $\theta(t_f - t_i)$ function is already incorporated in the retarded correlation function. So far we did not care about the time structure of $A(x_i, t_i)$ since it is simply a classical field in the Keldysh formalism.

Note that $S_{\Pi,\phi}$ (1.150) contains terms $\Pi \cdot \partial_t \phi$ we can obtain the correlator of Π fields by taking functional derivatives with respect to $\partial_t \phi$.

$$\langle \Pi_{cl}(x,t) \Pi_q(x',t') \rangle = 4 \left(\frac{\hbar}{i} \right)^2 \frac{\delta^2}{\delta(\partial_t \phi_q(x,t)) \delta(\partial_{t'} \phi_{cl}(x',t'))} \mathcal{Z}[\Pi_{cl}, \Pi_q, \phi_{cl}, \phi_q] \tag{1.156}$$

Furthermore we integrate out the Π fields in (1.149) and obtain the non-interacting action S_0 which will be extensively used below.

$$S_0 = \frac{1}{4} \int d(x,t) \int d(x',t') (\phi_{cl}, \phi_q)_{x,t} \begin{pmatrix} 0 & D_A^{-1} \\ D_R^{-1} & (D^{-1})_K \end{pmatrix} \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}_{x',t'} \tag{1.157}$$

Where $D_{R/A}^{-1} = \frac{1}{uK\pi}(-\partial_t^2 + \partial_x^2)\delta(x - x')\delta(t - t')$. The partition function $\mathcal{Z}$ with the action introduced in equation (1.156) is not changed by the partial trace over Π fields. After tracing out Π, we obtained a quadratic term in $\partial_t \phi$ in the action. Hence a functional derivative

[15] This statement is true as long as $\langle \Pi_q(x,t) \Pi_q(x',t') \rangle \rangle = 0$. It is shown in the following subsection that this is indeed the case.

with respect to $\partial_t \phi_{cl/q}$ in (1.156) now yields $\partial_t \phi$ fields instead of $\Pi_{cl/q}$-fields. Evaluating (1.156) using the action (1.157) gives the following expression for the retarded current-current correlation function $\frac{i}{2}\int dt_i \int dx_i \langle \Pi_{cl}(x_f,t_f)\Pi_q(x_i,t_i)\rangle A(x_i,t_i)$:

$$\frac{e^2 uK}{\pi\hbar} A(x_f,t_f) + i\frac{e^2}{2\hbar^2\pi^2}\int_{-\infty}^{\infty} dt_i \int dx_i \langle \partial_{t_f}\phi_{cl}(x_f,t_f)\partial_{t_i}\phi_q(x_i,t_i)\rangle A(x_i,t_i) \tag{1.158}$$

It is easy to see that the first term in (1.158) cancels the diamagnetic part $D = e^2 uK/\pi\hbar$ in equation (1.138). Moreover we can deconvolute the second term in (1.158) into a product by using Fourier representations.

$$\langle ej(x_f,t_f)\rangle = \int \frac{dk}{2\pi}\int \frac{d\omega}{2\pi} e^{i(kx_f-\omega t_f)}\langle e\tilde{j}(k,\omega)\rangle \tag{1.159}$$

The second term in (1.158) becomes

$$\frac{e^2}{2\hbar^2\pi^2}\int \frac{dk}{2\pi}\int \frac{d\omega}{2\pi}\int \frac{dq}{2\pi}\int \frac{d\nu}{2\pi} e^{i(kx_f-\omega t_f)}\omega\langle \phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle E_{-\nu} \tag{1.160}$$

Where we used the Fourier transformed electric field E_ν instead of the vector potential $A(q,\nu) = E_\nu/i\nu$. The final result for the Fourier transformed electric current density is:

$$\langle e\tilde{j}(k,\omega)\rangle = \frac{e^2}{2\hbar^2\pi^2}\int \frac{dq}{2\pi}\int \frac{d\nu}{2\pi}\omega\langle \phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle E_{-\nu} \tag{1.161}$$

1.6 Clean conductivity

As a first example let us calculate the conductivity of a clean system. The conductivity $\sigma(k,\omega)$ can be calculated by dividing the Fourier transformed current (1.161) by the electric field E_ω.

We will use the retarded correlation function in energy-momentum representation which are derived in appendix A:

$$\langle \phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle = 2\hbar i\frac{\pi uK}{(\omega+i\delta)^2-u^2k^2}\delta_{k,-q}\delta_{\omega,-\nu} \tag{1.162}$$

Inserting Eq. (1.162) into Eq. (1.161), we obtain the result for the zeroth order of the conductivity[16]:

$$\sigma(\omega) = \frac{e^2}{\pi\hbar}\frac{iuK}{\omega+i\delta} = \sigma_0(\omega) \tag{1.163}$$

[16] leading order of the expansion of the Drude conductivity in $1/\tau$, see Eq. (1.137)

Chapter 2

Disordered Luttinger Liquids

2.1 Review of known results in disordered systems

The clean Luttinger liquid, which we considered so far, is often not a sufficient description of real one dimensional nano devices. Dislocations, vacancies and magnetic or charged impurities for example, may be sources of disorder.
In the following we do not examine the origin of the external random potential $V(\mathbf{r})$. Instead we impose some general properties of the disorder field $V(\mathbf{r})$. First of all we consider static impurities which do not have any internal degrees of freedom which could be excited in a scattering event. Consequently, there are only elastic scattering processes possible and hence there is no loss of phase coherence of the electrons. In the following we use two important parameters. The coherence length L_ϕ describes the length scale on which the phase of the electrons is not randomized during propagation. Secondly, the mean free path l is the average length between two successive scattering events. It is very important to stress that static disorder does not explain a finite phase coherence length L_ϕ and that L_ϕ and the mean free path l are fundamentally different parameters. Reasons for a destruction of phase coherence are mechanisms like inelastic electron - electron (e-e) scattering or coupling to phonons.
In figure (2.1) there are three examples of possible disorder models[1].

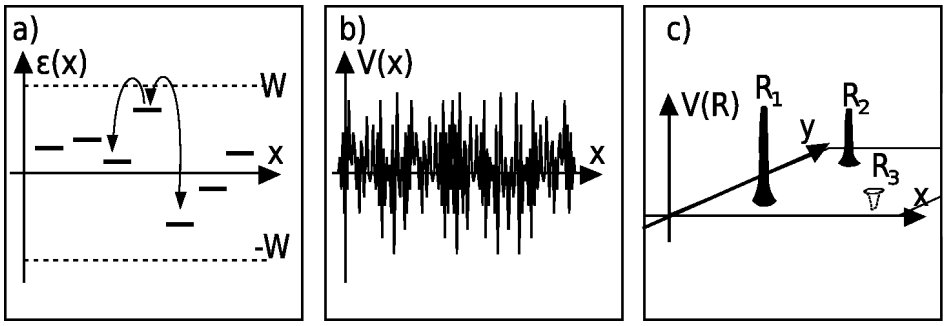


Figure 2.1: Illustration of the various types of disorder models: a) The Anderson model is a tight binding model where the on-site energies ϵ_i are randomly distributed in an interval $-W < \epsilon_i < W, \forall i$. b) The Gaussian white noise model and c) the localized impurity model.

The localized impurity model

This is the most intuitive and straight forward realization of disorder. It is depicted in figure (2.1,c). Here, impurities are randomly distributed on positions $\mathbf{R}_i$. The disorder potential reads:

$$V(\mathbf{r}) = \sum_{i=1}^{N} v(\mathbf{r} - \mathbf{R}_j)$$

Here N is the number of impurities in the considered volume Ω. In the thermodynamic limit $N \to \infty$ and $\Omega \to \infty$ the impurity density stays constant $n_i = \frac{N}{\Omega}$.

[1] see [6]

The Gaussian model

The Gaussian model is described in terms of a continuous, but random disorder field $V(\mathbf{r})$. This is the type of disorder model which is employed in the following sections. We assume that our physical observables vary on a much larger length scale than the distance between impurities. In that case we always have to take into account a whole ensemble of impurities that act on an observable in an area where the observable does not vary. The central limit theorem then guarantees, for a large range of single impurity distributions, the Gaussian character of the continuous disorder field $V(\mathbf{r})$.
The mean value and variance are given in the following equations.

$$\overline{V(\mathbf{r})} = 0 \tag{2.1}$$

$$\overline{V(\mathbf{r})V(\mathbf{r}')} = \Delta(\mathbf{r}-\mathbf{r}') \tag{2.2}$$

Here the line over the random potential means an average with respect to a probability density $\mathcal{P}[V(\mathbf{r})]$. The disorder average can be described in terms of a functional integral over all possible disorder realizations. The probability of a specific random potential is:

$$\mathcal{P}[V(\mathbf{r})]\mathcal{D}[V(\mathbf{r})] = \frac{1}{\mathcal{N}} e^{-\frac{1}{2}\iint d\mathbf{r}d\mathbf{r}' V(\mathbf{r})\Delta^{-1}(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')}\mathcal{D}[V(\mathbf{r})] \tag{2.3}$$

Here, $\mathcal{N}$ is a normalization factor: $\mathcal{N} = \int \mathcal{P}[V(\mathbf{r})]\mathcal{D}[V(\mathbf{r})]$.
A simple and often convenient choice for the variance is $\Delta(\mathbf{r}-\mathbf{r}') = D_{dis}\cdot\delta(\mathbf{r}-\mathbf{r}')$, where the disorder potential at a position $\mathbf{r}$ is not correlated with any other point $\mathbf{r}'$. Here D_{dis} denotes the disorder strength. Therefore we assume that the characteristic decay length of the correlation function of disorder potentials happens on a much smaller length scale compared to the wavelength of the electrons.
In figure (2.1,b) the Gaussian white-noise model is depicted.
In the Hamiltonian we account for the disorder field in the usual manner:

$$\mathcal{H}_{dis} = \int dx V(x)\psi^\dagger(x)\psi(x) \tag{2.4}$$

$\psi^\dagger$ and ψ are the electron creation and annihilation operators respectively.
Note that in the limit of many weakly interacting impurities, that means $v(\mathbf{r}) \to 0$ and $n_i \to \infty$, the localized impurity model and the Gaussian model become equivalent. Moreover, assuming the scattering potential to be smooth on the momentum shell we set the Fourier component of $v(q)$ constant, $v_{k,k'} \approx V_0$ and get the following relation[2]: $D_{dis} = n_i v_0^2$.

Disorder average

Every observable that is calculated with respect to the disorder Hamiltonian will depend on the specific realizations of the scattering potential $V(\mathbf{r})$. However, at not too low temperatures and large enough samples, the phase coherence L_ϕ is much smaller than the sample size L. Hence it is possible that self averaging of observables occurs [10] p.211. It is also possible to think of an ensemble of similar samples which are measured or a sample which is measured several times. Between each measurement it is heated up such that the position of the impurities change from one measurement to another. In the end the mean of all measurement

[2] See [6] p.41.

outcomes is taken.
Consequently, a disorder average over all possible disorder realizations makes sense for certain experimental situations.
In the following we are going to take two subsequent averages, a disorder average and an average over thermal and quantum fluctuations.
Let us briefly discuss the technical implications of taking two subsequent averages. In the last chapter two finite temperature techniques have been mentioned. Namely the real time Keldysh and the imaginary time Matsubara technique.
The Matsubara technique requires the partition function: $\mathcal{Z} = \int \mathcal{D}[\phi] e^{-S_V(\phi)}$ as a normalization factor, when averaging an observable $\hat{\mathcal{O}}$ over thermal and quantum fluctuations of the fields $\phi(x, \tau)$.

$$\langle \hat{\mathcal{O}} \rangle_V = \frac{\int \mathcal{D}[\phi] \hat{\mathcal{O}}(\phi) e^{-S_V(\phi)}}{\int \mathcal{D}[\phi] e^{-S_V(\phi)}} \tag{2.5}$$

Here $S_V(\phi) = S_0(\phi) + \int dx d\tau V(x) F[\phi(x)]$ contains a coupling to a specific realization of the external disorder potential $V(\mathbf{r})$. $F(\phi)$ is an arbitrary analytic function of the fields. The next step is to average over all possible random potentials V.

$$\overline{\langle \hat{\mathcal{O}} \rangle} = \frac{\int \mathcal{D}[V] \langle \hat{\mathcal{O}}(\phi) \rangle_V \mathcal{P}[V]}{\int \mathcal{D}[V] \mathcal{P}[V]} \tag{2.6}$$

Without the normalization factor in the denominator of Eq. (2.5) it would be easy to interchange the averages in Eq. (2.6) and perform the disorder average at first.
In order to get rid of the denominator one can basically choose between three techniques. The first one to mention is a supersymmetric ansatz. But since it applies mainly to non-interacting theories it is not helpful for a description of strongly correlated one dimensional electron systems. The second one is the Keldysh formalism where the denominator $\mathcal{Z} = \int \mathcal{D}[\phi] e^{-S_V(\phi)} = 1$ is one by construction, see section (1.4). This type of theory is used later, but for the moment we will have a look at the replica technique which is often used in the context of the imaginary time formalism.
Although being mathematically a little bit sloppy the replica trick is conceptually simple. At first, note that the role of the denominator of equation (2.5) is to cancel disconnected diagrams [8]. Expectation values of fields ϕ can be generated by taking functional derivatives with respect to source fields J from the logarithm of the action:

$$\frac{\delta}{\delta J} \ln \mathcal{Z} \Big|_{J=0} = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\phi] \phi e^{-S(\phi)} = \langle \phi \rangle \tag{2.7}$$

Here the source field J is coupled to ϕ by the following term in the action: $\iint \mathrm{d}x \mathrm{d}\tau J\phi$.
If we find a relation where the logarithm of the partition function $\ln \mathcal{Z}$ can be expressed by a monomial of the partition function: $\mathcal{Z}^R$, there will be no normalization factor since a functional derivative of $\mathcal{Z}^R$ does not produce a factor $1/\mathcal{Z}$ anymore. When R is very small the following relation is a good approximation

$$z^R = e^{R \ln z} \approx 1 + R \ln z \tag{2.8}$$

On the other hand we can use this to rewrite the logarithm:

$$\ln z = \lim_{R \to 0} \frac{1}{R} \left(z^R - 1 \right) \tag{2.9}$$

Furthermore we neglect -1 in the brackets. When R is an integer it is simple to interpret the following partition sum as a theory with R different fields ϕ_α.

$$\mathcal{Z}^R = \int \prod_{\alpha=1}^{R} e^{\sum_{\alpha=1}^{R} S(\phi_\alpha)} \tag{2.10}$$

Now we are able to interchange the disorder average with the thermal and quantum average. The disorder average is performed with respect to the probability measure $\mathcal{P}[V(\mathbf{r})]\mathcal{D}[V(\mathbf{r})]$ from above:

$$\frac{1}{\int \mathcal{P}[V(\mathbf{r})]\mathcal{D}[V(\mathbf{r})]} \int \mathcal{D}[V(\mathbf{r})]\mathcal{P}[V(\mathbf{r})] e^{-\sum_{\alpha=1}^{R} \int dx d\tau V(x)\cdot F[\phi]} = e^{\sum_{\alpha,\beta}^{R} S_{dis}(\phi_\alpha,\phi_\beta)} \tag{2.11}$$

where $S_{dis}(\phi_\alpha, \phi_\beta) = \frac{D_{dis}}{2} \int dx \int d\tau d\tau' F[\phi_\alpha(x,\tau)] F[\phi_\beta(x,\tau')]$.
A problem of this construction is that $f(R) = \frac{\mathcal{Z}^R}{R}$ is well defined only on integer values of R. An analytic continuation of the result to $R = 0$ need not exist. Moreover we have a model with an arbitrary number of fields ϕ_α, which can be a source of difficulties. Note that the Keldysh approach elegantly avoids all these uncertainties which is one of the reasons why we are going to use it in later sections.
The general recipe [3] for disorder averaged quantities in the replica theory is to calculate the following expression:

$$\overline{\hat{\mathcal{O}}} = \lim_{R\to 0} \frac{1}{R} \sum_{\alpha=1}^{R} \langle \mathcal{O}(\phi_\alpha) \rangle_{S_{repl}} \tag{2.12}$$

S_{repl} is the replicated action after the disorder average.

2.1.1 Drude conductivity and Diffuson in a non-interacting system

In this subsection the disorder averaged single-particle Green's function of a non-interacting system is calculated. We have already introduced the mean free path l which characterizes transport of particles through a disordered medium. l is the average distance travelled by an electron between two scattering events. The mean free path gives rise to a characteristic time $\tau = l/v$, the collision time. v is the group velocity of the electron wave. Although the energy is conserved during a scattering event the momentum of the scattered particle changes and hence a plane wave $|\mathbf{k}\rangle$ has only a finite lifetime τ. The collision time τ can be estimated from the lifetime τ_k of a plane wave $|\mathbf{k}\rangle$ by using Fermi's golden rule:

$$\frac{1}{\tau} = 2\pi \sum_{\mathbf{k}'} |\langle \mathbf{k}|V|\mathbf{k}'\rangle|^2 \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \tag{2.13}$$

Here V denotes the Gaussian disorder field. In the beginning of this section we have introduced the Gaussian white-noise model with variance given by $\overline{V(\mathbf{r})V(\mathbf{r}')} = D_{dis}\delta(\mathbf{r}-\mathbf{r}')$ and zero mean value. Performing the disorder average in Fermi's golden rule, equation (2.13) we have to insert the relation $\Omega\overline{|\langle \mathbf{k}|V|\mathbf{k}'\rangle|^2} = D_{dis}$. Where Ω is the volume of the system and D_{dis} is the disorder strength. Hence the average lifetime of a state with energy $\epsilon_{\mathbf{k}}$ is given by:

$$\frac{1}{\tau} = 2\pi \frac{\nu(\epsilon_{\mathbf{k}})}{\Omega} D_{dis} \tag{2.14}$$

[3] see [8]

Here we introduced the density of states at energy $\epsilon_{\mathbf{k}}$: $\nu(\epsilon_{\mathbf{k}}) := \sum_{\mathbf{k}'} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})$. In order to describe the evolution of a plane wave through disordered media we need to disorder average the single-particle Green's function $G(\omega, \mathbf{k})$. The diagrammatic expansion of the Green's function is given in figure (2.2): The corresponding formula is:

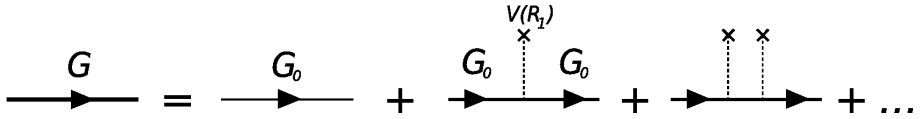


Figure 2.2: Diagrammatic expansion of the single particle Green's function $G(\omega, \mathbf{k})$ in a disordered medium with a specific random field $V(\mathbf{r})$. The dotted line represents a scattering event, the solid line on the right hand side is the electron Green's function G_0 of a clean system.

$$G(\mathbf{r}_i, \mathbf{r}_f, t) = G_0(\mathbf{r}_i, \mathbf{r}_f, t) + \int G_0(\mathbf{r}_i, \mathbf{r}, t) V(\mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_f, t) d\mathbf{r} + \dots$$

$$= G_0(\mathbf{r}_i, \mathbf{r}_f, t) + \int G_0(\mathbf{r}_i, \mathbf{r}, t) V(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_f, t) d\mathbf{r} \tag{2.15}$$

This is known as the Dyson equation.
After performing the disorder average: $\overline{V(\mathbf{r})} = 0$, $\overline{V(\mathbf{r})V(\mathbf{r}')} = D_{dis}\delta(\mathbf{r} - \mathbf{r}')$ we get the following diagrammatic expansion, see Fig (2.3).

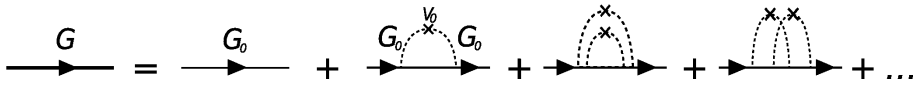


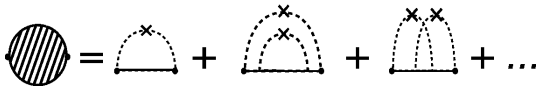
Figure 2.3: Diagrammatic expansion of the full disorder averaged single particle Green's function $G(\omega, \mathbf{k})$ in a disordered medium.

Disorder averaging is represented by a connection of two dotted lines.
We reformulate the diagrammatic expansion (2.15) in energy-momentum space:

$$G(\omega, \mathbf{k}) = G_0(\omega, \mathbf{k}) + G_0(\omega, \mathbf{k})\Sigma(\omega, \mathbf{k})G(\omega, \mathbf{k}) \tag{2.16}$$

Where $\Sigma(\omega, \mathbf{k})$ denotes the self energy, that means the sum of all irreducible diagrams without external links:

$\Sigma(\omega, \mathbf{k}) =$



(2.17)

Now we solve the Dyson equation (2.16) and write the full disordered Green's function in terms of the self energy.

$$G(i\omega_n, \mathbf{k}) = \frac{G_0(i\omega_n, \mathbf{k})}{1 - G_0(i\omega_n, \mathbf{k})\Sigma(i\omega_n, \mathbf{k})} = \frac{1}{i\omega_n - \xi(\mathbf{k}) - \Sigma(i\omega_n, \mathbf{k})} \tag{2.18}$$

Where we used $G_0^{-1}(i\omega_n, \mathbf{k}) = i\omega_n - \xi(\mathbf{k})$. $i\omega_n$ is a Matsubara frequency. Note that we arrive at the corresponding retarded expression by putting $i\omega_n \to \omega + i\delta$, where δ is an infinitesimal.

It can be shown[4] that in dimensions higher than one all crossing diagrams are smaller by a factor of $1/k_F l$ than the non-crossing diagrams. Thus in two and three dimensions the third diagram on the right hand side of equation (2.17) is much smaller compared to the second one. For the remaining part of this section we exclude the discussion about disorder effects in 1D and postpone it to the section about disorder in Luttinger liquid.
We are mainly interested in the imaginary part of the self energy since the real part $\mathcal{R}e\Sigma$ yields only a renormalization of the energy $\epsilon(\mathbf{k})$. Furthermore we only account for the non-crossing diagrams and neglect contributions from the crossed ones. This is known as the Born approximation [10](p.220). As a further approximation we calculate the self energy by taking into account only the first diagram of (2.17).

$$\Sigma(i\omega_n, \mathbf{k}) \approx n_i \sum_{\mathbf{k}'} |V_{\mathbf{k}-\mathbf{k}'}|^2 \frac{1}{i\omega_n - \xi_{\mathbf{k}'}} \approx v_0^2 n_i \nu(\epsilon_{\mathbf{k}}) \int \mathrm{d}z \frac{1}{i\omega_n - z} \tag{2.19}$$

Here we used that the scattering potential $V_{\mathbf{k}-\mathbf{k}'}$ is smooth on the Fermi momentum shell and thus: $v_0 \approx V_{\mathbf{k}-\mathbf{k}'}$.
Performing the integral in Eq. (2.19) and using Eq. (2.14) we obtain the self energy:

$$\Sigma(i\omega_n, \mathbf{k}) = -i \cdot sign(\omega_n) \frac{1}{2\tau} \tag{2.20}$$

Finally, we get the following expression for the disorder averaged single particle Green's function:

$$G(i\omega_n, \mathbf{k}) = \frac{1}{i\omega_n - \xi(\mathbf{k}) + sign(\omega_n)\frac{i}{2\tau}} \tag{2.21}$$

We get the retarded Green's function from the Matsubara Green's function by performing the analytic continuation: $i\omega_n \to \omega + i\delta$.
To conclude, the discussion of the single particle Green's function, we give the retarded Green's function in energy-momentum and space-energy representation [6].

$$G^R(\epsilon, \mathbf{k}) = \frac{1}{\epsilon - \xi(\mathbf{k}) + \frac{i}{2\tau}} \tag{2.22}$$

$$G^R(\mathbf{r}, \mathbf{r}', \epsilon) = G_0^R(\mathbf{r}, \mathbf{r}', \epsilon) e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{2l}} \tag{2.23}$$

Equation (2.23) shows that correlations are decaying on a length scale of the order of the mean free path. In the last chapter we already introduced the Kubo formula for calculating the current density, Eq. (1.138). In contrast to equation (1.144), the current operator, written in terms of electron fields $\hat{\psi}$, $\hat{\psi}^\dagger$ is given by:

$$\hat{\mathbf{j}}(\mathbf{r}, t) = \frac{e}{2mi} \left(\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}'}\right) \hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) \Big|_{\mathbf{r}'=\mathbf{r}} \tag{2.24}$$

Thus, the current-current correlation function in Eq. (1.138) yields a four Fermion correlation function. Using Wick's theorem the four Fermion correlation function can be expressed as a product of two Green's functions. Diagrammatically, the product of two Green's functions corresponds to a bubble, see Fig. (2.4). The expansion of the conductivity in terms of diagrams is given in figure (2.4). Furthermore this product of Green's functions has to

[4] [10] p. 223

be disorder averaged. In general the disorder average over a product of Green's functions $\overline{G_0(\mathbf{p})G_0(\mathbf{p'})}$ is different from the product of two averaged Green's functions $\overline{G_0(\mathbf{p})} \cdot \overline{G_0(\mathbf{p'})}$. We remind that the product $G^R(\mathbf{r},\mathbf{r'},\epsilon)G^A(\mathbf{r'},\mathbf{r},\epsilon-\omega)$ represents the probability of quantum diffusion $P(\mathbf{r},\mathbf{r'},\omega)$ which is the probability for a wave packet to travel from $\mathbf{r}$ to $\mathbf{r'}$ [6]. The so called Drude-Boltzmann approximation is to approximate the average of the product of two Green's functions by the product of two averaged Green's functions:

$$\overline{G^R(\mathbf{r},\mathbf{r'},\epsilon)G^A(\mathbf{r'},\mathbf{r},\epsilon-\omega)} \approx \overline{G^R(\mathbf{r},\mathbf{r'},\epsilon)} \cdot \overline{G^A(\mathbf{r'},\mathbf{r},\epsilon-\omega)} \tag{2.25}$$

Since the right hand side of (2.25) corresponds to an empty bubble of two disorder averaged Green's functions, this approximation neglects diagrams where upper and lower Green's functions of the bubble are connected by an impurity line, such as the second bubble in Fig (2.4). There are also higher order diagrams in impurity lines possible. These diagrams are called particle-hole ladder diagrams, see Fig. (2.5,c).
Although Eq. (2.25) seems to be a crude approximation, it can be shown, [6] p.277, that for isotropic impurity scattering the contribution of the particle-hole ladder to the current vanishes. The physical picture behind this cancellation is that a scattered electron has completely lost its memory about the direction of the current as long as the impurity has an isotropic scattering potential. Hence higher correlations of scattering events do not contribute.

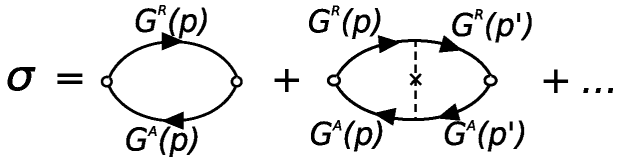


Figure 2.4: Classical contributions to the conductivity. The solid line is the disorder averaged Green's function $G(\mathbf{p})$, the dashed line is a scattering event. The empty bubble on the right hand side represents the Drude-Boltzmann approximation of incoherent collisions. The second bubble is the first order contribution in the disorder strength D_{dis}. It contributes to the so called particle-hole ladder.

The particle-hole ladder diagrams, Fig (2.5), are called Diffuson contribution and they do matter in a calculation of the density-density correlation function. In figure (2.5 b) the meaning of the Diffuson contribution is shown in terms of propagating particles and holes. The ladder approximation corresponds to the weak disorder limit $kl \gg 1$ [6](p.102). Moreover it is an important contribution to the probability $P(\mathbf{r},\mathbf{r'},\omega)$ which we mentioned above. Together with the empty bubble the Diffuson yields the classical contributions. The first two Green's functions $|G^R(\mathbf{r},\mathbf{r}_1)|^2 = G^R(\mathbf{r},\mathbf{r}_1)G^A(\mathbf{r}_1,\mathbf{r})$ are the probability for a particle to propagate from $\mathbf{r}$ to $\mathbf{r}_1$ without any scattering event, [6] p.96. The evolution of the wave packet from $\mathbf{r}_1$ to $\mathbf{r}_2$ is described by $\Gamma(\mathbf{r}_1,\mathbf{r}_2)$ in the diagrammatic language. The vertex function Γ takes into account all possible ways to get from $\mathbf{r}_1$ to $\mathbf{r}_2$ with an arbitrary number of scattering events. Furthermore, the probability to go from point $\mathbf{r}_2$ to $\mathbf{r'}$ is computed by Green's functions in the same way as from $\mathbf{r}$ to $\mathbf{r}_1$. Finally we integrate over all possible points $\mathbf{r}_1$ and $\mathbf{r}_2$. The computation of the vertex function is depicted in (2.5 c). Γ can be evaluated self consistently by the so called Bethe-Salpeter equation which is a Dyson equation for vertex functions. It can be shown that after a large number of collisions, i.e. $t \gg \tau$, the vertex function fulfills a classical diffusion equation [6](4.5).

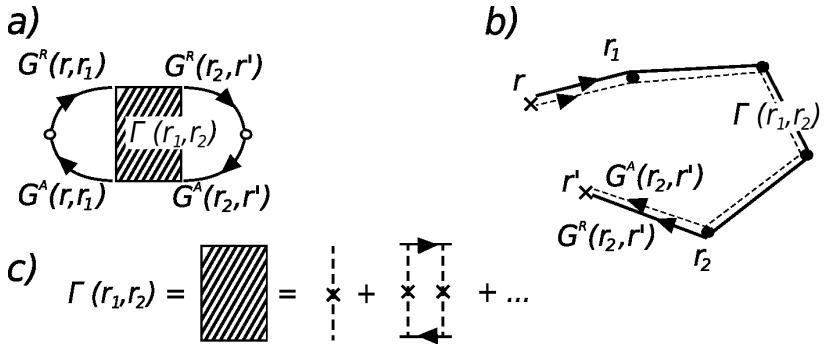


Figure 2.5: Diffuson contribution. a) the diagrammatic representation of the Diffuson contribution to the pair bubble is shown in terms of disorder averaged Green's functions G and the vertex function Γ. The corresponding intuitive picture of propagating particles and holes is depicted in b). Solid lines represent G^R and dashed lines describe G^A. The structure of the vertex function Γ is shown in c). The diagrams that contribute to Γ are called particle-hole ladder diagrams.

2.1.2 Weak localization

The last section was about classical contributions to the conductivity. However, at low temperatures the coherence length L_ϕ becomes longer and thus interference effects become more important. If the coherence length is bigger than the mean free path, two electron wave functions that scatter on several impurities will interfere. Note that at low temperatures dephasing occurs mostly via inelastic electron-electron collisions since all other degrees of freedom that may constitute sources of dephasing, like phonons are frozen out. Since the electron-electron scattering time behaves like $\sim 1/T^\alpha$ one expects that the dephasing time[5] is $\tau_\phi \propto 1/T^\alpha$, where $\alpha > 0$. However, some of the interference effects vanish in samples where the sample size L is much larger than the coherence length. In that case, we can divide the sample into many small subsystems with a subsystem size comparable to the coherence length. When observables like the conductivity are calculated we have to average over these subsystems. Since an electron cannot propagate from one subsystem to the other coherently, interference effects vanish due to averaging over the subsystems. In order to see coherence effects in larger samples, we have to consider an interference mechanism which does not depend on the impurity positions. Such an interference effect will be introduced in the following by a short example, see [10](p.299).
Consider two plane waves, for example light rays with amplitudes t_1 and t_2 and phases ϕ_1 and ϕ_2. These two light rays meet at a certain position on a screen. The intensity at this point on the screen is given by the square of the absolute value of the superposition $t_1 + t_2$:

$$|t_1 + t_2|^2 = |t_1|^2 + |t_2|^2 + \underbrace{2|t_1 t_2| \cos(\phi_1 - \phi_2)}_{interference-term} \tag{2.26}$$

The cosine term which depends on the phase difference is responsible for interference effects. Suppose that two waves interfere and each wave has travelled exactly the same path as the other but in the reversed direction. They will interfere constructively since their phase difference is zero. As long as the path length is smaller than the coherence length it does not matter how many scattering events occurred on that path. However, the picture in figure

[5]For experimental evidence see [9], Fig(8.6), p.229

(2.6,b) shows that, if there is a big difference in the initial and end point, r and r', the dashed path may collect a much larger phase when going from r to the first impurity and from the last impurity to r'. Thus the phase difference need not to vanish. Of course $\phi_1 - \phi_2$ is always zero when $r = r'$. Hence we expect an enhanced probability to return to the same point. Since an enhanced probability means that electrons tend to localize at their starting point the term weak localization is used for that phenomenon.

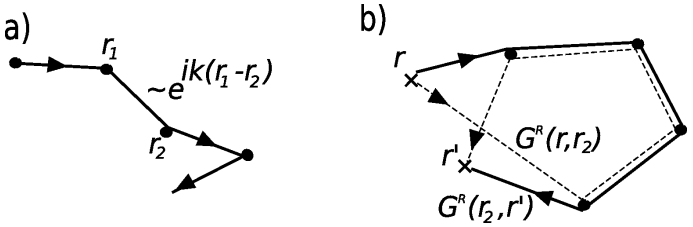


Figure 2.6: The mechanism behind weak localization. a) An electron collects a certain phase on its way from one impurity to the other. b) A path and its time reversed path constitute an interference effect that is independent of the impurity positions and thus survives self averaging over many subsystems.

Since a magnetic field breaks time reversal invariance, the time reversed path collects a different phase and the weak localization effect becomes less pronounced as magnetic fields increase. In contrast to two dimensions and quasi one dimensional systems, weak localization is not a strong effect in three dimensions, which is related to the fact that random walks are transient in three and higher dimensions compared to one and two where they are recurrent. In a strict one dimensional system already weak disorder pins the electron wave functions as mentioned in the introduction. Consequently, if there is no dephasing, quantum interference effects won't be a small correction any more and a diffusive regime is absent. However, in the next section we are going to see that an interacting one dimensional system behaves quite different and the term weak localization becomes applicable in one dimension [2].

The weak localization correction

The weak localization correction can be calculated using the many-body formalism. In principle this contribution is hidden in a full expansion of the pair bubble in orders of the disorder strength D_{dis}. Keeping the physical mechanisms from above in mind, we will explain the diagrammatic representation of the weak localization effect.
Let us consider a two dimensional sample and an electron path which is scattered twice. On the right picture of figure (2.6) one can see that we need at least two scattering events in order to have a time reversed path with equal final and initial points. The single particle Green's function that describes this process is:

$$G^{R(2)}(\mathbf{r}, \mathbf{r}', t) := G_0^R(\mathbf{r}, \mathbf{R}_1, t) v_0 G_0^R(\mathbf{R}_1, \mathbf{R}_2, t) v_0 G_0^R(\mathbf{R}_2, \mathbf{r}', t) \tag{2.27}$$

$\mathbf{R}_1, \mathbf{R}_2$ are the positions of the impurities, v_0 is the disorder potential and G_0^R is the retarded Green's function of a clean system.
In order to calculate the probability we need the absolute square of this expression.

$$\tilde{P}(\mathbf{r}, \mathbf{r}', t) \sim G_0^R(\mathbf{r}, \mathbf{R}_1) v_0 G_0^R(\mathbf{R}_1, \mathbf{R}_2) v_0 G_0^R(\mathbf{R}_2, \mathbf{r}') \times$$

$$\times G_0^A(\mathbf{R}_1, \mathbf{r}')v_0 G_0^A(\mathbf{R}_2, \mathbf{R}_1)v_0 G_0^A(\mathbf{r}, \mathbf{R}_2) \tag{2.28}$$

Here we can interpret the first and the second line as time reversed pairs. Note that in the product of the retarded Green's function we start at the initial point $\mathbf{r}$ and scatter at $\mathbf{R}_1$ whereas in the product of the advanced Green's functions the last scattering event is at $\mathbf{R}_1$. When this is drawn as a diagram we arrive exactly at the maximally crossed impurity-pair bubbles, see Fig. (2.7). In the last subsection we neglected these contributions by arguing that they are small compared to the Drude conductivity.
The weak localization correction can be calculated in a similar way as the Diffuson from the last subsection by dividing it into three parts. At first, we account for the probability to go from the initial point $\mathbf{r}$ to the first scattering event. Then, this probability is multiplied by the vertex correction which is responsible for an arbitrary number of scattering events. Finally, we need the probability to go from the last scattering event to the final point $\mathbf{r}'$. The significant difference to the classical contributions is technically hidden in the vertex correction.
As shown in figure (2.7) we can unfold the maximally crossed diagram and get a ladder like series of diagrams shown in (c). Contrary to the particle-hole ladder of the Diffuson contribution, the Green's functions now describe propagation in the same direction. For this reason the particle-particle ladder is called Cooperon similar to particle-particle correlations of Cooper pairs. The Cooperon is denoted by $\mathcal{C}$.
The Cooperon can be formulated in a so called Bethe-Salpeter equation (2.7,c). The corresponding algebraic expression is:

$$\mathcal{C}(\mathbf{Q}, i\omega_n + i\epsilon_n, i\omega_n) = \int \frac{d\mathbf{p}}{(2\pi)^d} v_0 G(\mathbf{Q}-\mathbf{p}, i\omega_n + i\epsilon_n) G(\mathbf{p}, i\omega_n) v_0$$

$$+ \int \frac{d\mathbf{p}}{(2\pi)^d} \mathcal{C}(\mathbf{Q}, i\omega_n + i\epsilon_n, i\omega_n) v_0 G(\mathbf{Q}-\mathbf{p}, i\omega_n + i\epsilon_n) G(\mathbf{p}, i\epsilon_n) \tag{2.29}$$

G is the impurity averaged Matsubara Green's function, Eq. (2.21). Since the Cooperon depends only on $\mathbf{Q}$ and elastic impurity scattering conserves energy, the Bethe-Salpeter equation can easily be solved:

$$\mathcal{C}(\mathbf{Q}, i\omega_n + i\epsilon_n, i\omega_n) = \frac{v_0 \zeta(\mathbf{Q})}{1 - \zeta(\mathbf{Q})} \tag{2.30}$$

Where we used the following short cut notation:

$$\zeta(\mathbf{Q}) = v_0 \int \frac{d\mathbf{p}}{(2\pi)^d} \mathcal{G}(\mathbf{Q}-\mathbf{p}, i\omega_n + i\epsilon_n) G(\mathbf{p}, i\epsilon_n) \tag{2.31}$$

The Matsubara Green's functions can be expressed as a retarded and advanced Green's function by using the following analytic continuation ($T \to 0$).

$$i\omega_n + i\epsilon_n \to \epsilon + \omega + i\delta \quad i\epsilon_n \to \epsilon - i\delta$$

δ is an infinitesimal.
Using the analytic continuation in (2.31) and considering the DC-limit: $\omega, \epsilon \to 0$ we get:

$$\zeta(\mathbf{Q}) = V_0 \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{1}{-\xi(\mathbf{Q}-\mathbf{p}) + i/2\tau} \cdot \frac{1}{-\xi(\mathbf{p}) - i/2\tau} \tag{2.32}$$

For $\mathbf{Q} = 0$ we obtain $\zeta(\mathbf{Q} = 0) = 1$ and thus the Cooperon diverges. Consequently, the main contribution of the weak localization correction comes from small momenta $\mathbf{Q}$. An expansion of $\zeta(\mathbf{Q})$ for small momenta yields the following expression[6].

$$\zeta(\mathbf{Q}) \approx 1 - D_{diff}\tau\mathbf{Q}^2 \tag{2.33}$$

Where $D_{diff} = v_F^2\tau/d$ is the diffusion constant, d is the dimension.
Furthermore we insert (2.33) into equation (2.30) and obtain the Cooperon vertex correction:

$$\mathcal{C}(\mathbf{Q}, 0, 0) = \frac{V_0(1 - D_{diff}\tau\mathbf{Q}^2)}{D_{diff}\tau\mathbf{Q}^2} \tag{2.34}$$

Thus, the relevant contribution of the Cooperon is a $1/\mathbf{Q}^2$ term.
To conclude, the weak localization correction is [10] p.306:

$$\delta\sigma^{WL} = -\frac{e^2}{\pi}\left(\frac{k_F}{m}\right)^2 \frac{2\tau}{d}\int_{Q<l^{-1}} \frac{d\mathbf{Q}}{(2\pi)^d}\frac{1}{DQ^2} \tag{2.35}$$

The magnitude of the weak localization effect is given by the Q-integral and thus $\delta\sigma^{WL}$ depends on the dimension of the system. Furthermore the Q-integral is cut by the inverse mean free path l^{-1} since the physical mechanism of the weak localization effect happens on length scales larger than the average distance between two impurities. For three dimension the integral is perfectly regular. For one and two dimensions this integral has an infra-red divergence. The infra-red divergence indicates that we consider this effect on arbitrary large length scales which is certainly not true. In the beginning of this section we have noted that the electrons have a certain coherence length L_ϕ which is due to dephasing mechanisms. Thus, the lower cut-off of the momentum integral is the coherence length.

$$\mathcal{C}(\mathbf{Q}, 0, 0) = \frac{V_0}{\tau}\frac{1}{D_{diff}/L_\phi^2 + D_{diff}\mathbf{Q}^2} \tag{2.36}$$

Finally, the weak localization correction of the conductivity is, [6] p.282:

$$\delta\sigma^{WL} \propto \begin{array}{c} -(L_\phi - l), 1D \\ -\ln\left(\frac{L_\phi}{l}\right), 2D \\ -\left(\frac{1}{l} - \frac{1}{L_\phi}\right), 3D \end{array} \tag{2.37}$$

Note that 1D in Eq. (2.37) should be understood as quasi one dimensional. A quasi one dimensional system describes a three dimensional wire which is so narrow that effectively only one dimensional diffusion is possible. The diameter of the wire is large compared to the Fermi wave length λ_F of an electron. That means quantum effects due to the boundaries are not important. Strictly one dimensional systems without interactions do not have a diffusive regime. In the introduction it was mentioned that arbitrary weak disorder immediately localizes the charge carriers. In the next sections we analyze the behaviour of one dimensional disordered systems with interactions.

[6] See [10] p.305 for a detailed calculation

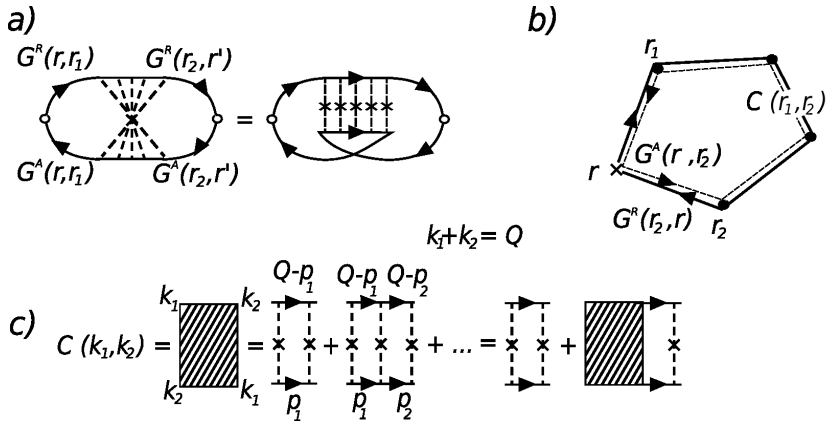


Figure 2.7: a) Maximally crossed diagrams represent the weak localization correction. b) The constructive interference between electrons of time-reversed pairs of paths are responsible for the enhanced probability to remain at point **r**. c) Bethe Salpeter equation for the Cooperon. In contrast to the Diffuson, the Green's functions in the ladder point in the same direction.

2.1.3 Disorder in Luttinger liquids

In one dimensional systems the general disorder term in the Hamiltonian, Eq. (2.4) can be simplified. We assume the disorder strength D_{dis} to be much smaller than the Fermi energy. In that case one can basically distinguish between two different scattering mechanisms. Forward scattering $\eta(x)$ with a momentum transfer of $q \sim 0$, and backward scattering $\xi(x)$ which changes the momentum by $-2k_F$.
The two components η and ξ can be related to the general disorder field $V(x)$ by its Fourier coefficients:

$$\begin{aligned} \eta(x) &= \frac{1}{L}\sum_{q\sim 0} V_q e^{iqx} \\ \xi(x) &= \frac{1}{L}\sum_{q\sim -2k_F} V_q e^{iqx} \end{aligned} \tag{2.38}$$

Note that for $\eta(x)$ the Fourier phase factor is practically 1 and thus η is a real field in contrast to $\xi(x)$, which is in general a complex field.
In the section about bosonization we decomposed the Fermion field ψ into right and left movers as well as fast and slow oscillations.

$$\psi(x) = e^{-i2k_F x}\psi_L(x) + e^{i2k_F x}\psi_R(x)$$

We use the slowly varying chiral fields $\psi_{L/R}(x)$ to couple them to the scattering potentials η, ξ. The fast oscillating phase factors of the backscattering term are absorbed in ξ. Thus the disorder part of the Hamiltonian, Eq. (2.4) becomes:

$$\begin{aligned} \mathcal{H}_{dis} = &\int \mathrm{d}x \eta(x) \left[\bar{\psi}_R(x)\psi_R(x) + \bar{\psi}_L(x)\psi_L(x)\right] \\ &+ \int \mathrm{d}x \left[\xi(x)\bar{\psi}_L(x)\psi_R(x) + \xi^*(x)\bar{\psi}_R(x)\psi_L(x)\right] \end{aligned} \tag{2.39}$$

The disorder fields are uncorrelated since η and ξ are independent random variables. This can easily be seen from the Fourier coefficients of the original disorder field $V(x)$. They are delta correlated: $\overline{V_q V_{q'}^*} = D_{dis}\delta_{q,q'}$, and thus η and ξ are uncorrelated:

$$\overline{\eta(x)\xi(x')} = 0$$

Moreover, from $\overline{V(x)V(x')} = D_{dis}\delta(x-x')$ we can derive the following relations of the disorder fields:

$$\begin{aligned} \overline{\eta(x)\eta(x')} &= D_f\delta(x-x') \\ \overline{\xi(x)\xi(x')} &= 0 \\ \overline{\xi(x)\xi^*(x')} &= D_b\delta(x-x') \end{aligned} \tag{2.40}$$

D_f, D_b denote the disorder strength of forward and backward scattering potentials respectively. From equation (1.42) of the section about bosonization we have seen how to relate the Boson field $\phi(x)$ to the density of left and right movers $\rho_{L/R} = \bar{\psi}_{L/R}\psi_{L/R}$[7]. Moreover the relations for $\psi(x)$ are:

$$\psi_R(x) = U_R \lim_{\alpha\to 0} \frac{1}{\sqrt{2\pi\alpha}} e^{i(k_F - \frac{\pi}{L})x} e^{-i(\phi(x)-\theta(x))} \tag{2.41}$$

$$\psi_L(x) = U_L \lim_{\alpha\to 0} \frac{1}{\sqrt{2\pi\alpha}} e^{-i(k_F + \frac{\pi}{L})x} e^{i(\phi(x)+\theta(x))} \tag{2.42}$$

Inserting these relations into equation (2.39), the full Hamiltonian of the system reads:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{forward} + \mathcal{H}_{backward} \\ &= \frac{1}{2\pi}\int \mathrm{d}x \left[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2\right] - \int \mathrm{d}x\eta(x)\left[\frac{1}{\pi}\nabla\phi\right] \\ &\qquad + \int \mathrm{d}x \frac{\xi^*(x)}{2\pi\alpha} e^{i2\phi(x)} + h.c. \end{aligned} \tag{2.43}$$

Here $\mathcal{H}_0$ is the Hamiltonian of the clean interacting system, Eq. (1.48) and $\mathcal{H}_{dis} = \mathcal{H}_{forward} + \mathcal{H}_{backward}$.
From the expression of the transport time τ_{trans} in the Boltzmann equation we know that forward scattering does not affect transport, see [10] p.275. It is instructive to analyze the role of $\eta(x)$ while keeping the Boltzmann result in mind.
The fact that $\eta(x)$ does not affect the transport behaviour can be easily seen after applying the following gauge transformation in Eq. (2.43):

$$\tilde{\phi}(x) = \phi - \frac{K}{u}\int^x \mathrm{d}y\eta(y) \tag{2.44}$$

Transforming ϕ in that way, $\mathcal{H}_{forward}$ vanishes but the transformation has generated an additional factor in the back scattering term $\mathcal{H}_{backward}$.

$$\mathcal{H}_{backward} = \int \mathrm{d}x \frac{\xi^*(x) e^{i\frac{2K}{u}\int^x \mathrm{d}y\eta(y)}}{2\pi\alpha} e^{i2\tilde{\phi}(x)} + h.c.$$

[7] $-\frac{1}{\pi}\nabla\phi(x) = \rho_L(x) + \rho_R(x)$

However this is not a problem since we can introduce a random field $\tilde{\xi}(x) = \xi(x)e^{-i\frac{2K}{u}\int^x \mathrm{d}y\eta(y)}$. For $\tilde{\xi}(x)$, relation (2.40) remains valid and the distribution of $\tilde{\xi}(x)$ is still gaussian.
We have seen[8] that the current is related to the time derivative of the Boson field $j \sim \partial_t\phi$. The transformation in (2.44) introduces only a shift in space and therefore the additional term drops out after derivating with respect to the time. Hence the transformation, Eq. (2.44) does not change the current. As expected, the conductivity, which is calculated from the current-current correlation function, is not affected by forward scattering.
Nevertheless, forward scattering has influence on correlation functions like $\langle\bar{\psi}\psi\rangle$ which decays exponentially with the forward disorder strength D_f[9].
Henceforth, we write $\tilde{\xi}$ and $\tilde{\phi}$ without tilde and assume always that forward scattering is gauged out.
The action of the disordered Luttinger liquid reads:

$$\begin{aligned} S &= S_0 + S_{backward} \\ &= \frac{1}{2\pi L}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \left[\frac{1}{u}\left(\partial_\tau\phi(x,\tau)\right)^2 + u\left(\partial_x\phi(x,\tau)\right)^2\right] \\ &\quad + \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \left[\frac{\xi^*(x)}{2\pi\alpha}e^{i2\phi(x)} + h.c.\right] \end{aligned} \tag{2.45}$$

Using the replica technique we are able to perform the disorder average at first. We use the Debye-Waller relation[10]:

$$\langle e^{S_{backward}}\rangle_{dis} = e^{-\frac{1}{2}\langle S^2_{backward}\rangle_{dis}}$$

Thus we obtain the following effective disorder action in the imaginary time Matsubara formalism[11]:

$$S_D^{Matsubara} = -\frac{D_b}{(2\pi\alpha)^2}\sum_{a,b=1}^{n}\int \mathrm{d}x\mathrm{d}\tau\mathrm{d}\tau' \cos(2\phi_a(x,\tau) - 2\phi_b(x,\tau')) \tag{2.46}$$

Here τ is the imaginary time coordinate and not the scattering time.

2.1.4 Giamarchi-Schulz RG & phase diagram of disordered Luttinger liquids

In the following we are going to analyze the interplay between interactions and disorder, [5](chapter 9), and [7]. The renormalization group (RG) is a procedure to study the low energy physics of a model by successively integrating out the high energy degrees of freedom. This partial summation is done in a way that preserves the partition function and thus the thermodynamic behaviour remains unchanged. The low energy model can be quite different from the microscopic model from which one has started. The transition from the initial microscopic model to the effective low energy model is described in terms of a flow of coupling constants in a space of all possible couplings. The goal is to obtain physical properties directly from the flow. In order to perform the average over high energy modes we have to decompose the fields into fast and slow modes. Moreover we introduce a momentum cut off Λ which

[8] See section (1.5)
[9] See [5] chapter 9.2
[10] $\langle\ldots\rangle_{dis}$ indicates the disorder average
[11] For a detailed derivation see [5], chapter 9.

corresponds to the smallest length scale in the system. For example the lattice spacing might play such a role. Here the parameter α is the smallest length scale. Fast modes are characterized by momenta and frequencies between Λ' and Λ, where $\Lambda' < \Lambda$.

$$\begin{aligned} \Lambda' < |k| < \Lambda \\ \Lambda' < |\omega_n|/u < \Lambda \end{aligned} \tag{2.47}$$

Slow modes have momenta smaller than Λ' and frequencies ω_n smaller than $\Lambda' u$.

$$\begin{aligned} |k| < \Lambda' \\ |\omega_n|/u < \Lambda' \end{aligned} \tag{2.48}$$

However, it will turn out that a circle in $(k, \omega_n/u)$-space is more suited for calculations than the square defined by Eq. (2.48). Thus momenta and frequencies for slow modes are restricted in the following way:

$$\sqrt{k^2 + (\omega_n/u)^2} < \Lambda' \tag{2.49}$$

For fast modes, there is the following relation:

$$\Lambda' < \sqrt{k^2 + (\omega_n/u)^2} < \Lambda \tag{2.50}$$

In the following we use the disorder averaged action Eq. (2.46). Thus, in this subsection τ denotes the imaginary time and $1/\gamma$ is the transport scattering time.

$$\phi(x,\tau) = \phi^{>}(x,\tau) + \phi^{<}(x,\tau) \tag{2.51}$$

where

$$\begin{aligned} \phi^{>}(x,\tau) &= \frac{1}{\beta}\int \frac{\mathrm{d}k}{2\pi} \sum_{\omega_n} e^{i(kx-\omega_n\tau)}\phi(k,\omega_n)\theta(\Lambda - q)\theta(q-\Lambda') \\ \phi^{<}(x,\tau) &= \frac{1}{\beta}\int \frac{\mathrm{d}k}{2\pi} \sum_{\omega_n} e^{i(kx-\omega_n\tau)}\phi(k,\omega_n)\theta(\Lambda' - q) \\ q &:= \sqrt{k^2 + (\omega_n/u)^2} \end{aligned} \tag{2.52}$$

Using this decomposition one can divide the quadratic action into high energy and low energy components:

$$S_0 = S_0^{>} + S_0^{<} \tag{2.53}$$

where the quadratic part in Fourier space reads:

$$S_0 = \frac{1}{2\pi u K}\frac{1}{\beta}\int \frac{\mathrm{d}k}{2\pi}\sum_{\omega_n}\left(\omega_n^2 + u^2k^2\right)\phi^*(k,\omega_n)\phi(k,\omega_n)$$

and $S_0^{>}$, $S_0^{<}$ on the right hand side of (2.53) can be obtained from restricting the momentum-energy summation in S_0. To derive the RG equations we also have to introduce a cut-off that restricts the double time integral in Eq. (2.46): $u|\tau - \tau'| \geq \alpha$. However, if we simply restrict this integral, then $S_D^{Matsubara}$ will no longer describes elastic impurity scattering. Hence for consistency it is necessary to keep also the part where $u|\tau - \tau'| \leq \alpha$. In that case $\tau \approx \tau'$ and the two fold integration over τ, τ' of the second term simplifies:

$$D_b \iint \mathrm{d}\tau \mathrm{d}\tau' \approx D_b \iint_{u|\tau-\tau'|\geq\alpha} \mathrm{d}\tau\mathrm{d}\tau' + \frac{2\alpha}{u}\cdot D_b \int \mathrm{d}\tau \tag{2.54}$$

The full action of the system is:

$$S = S_0 - \frac{D_b}{(2\pi\alpha)^2}\sum_{a,b=1}^{n}\int \mathrm{d}x \iint_{u|\tau-\tau'|\geq\alpha} \mathrm{d}\tau\mathrm{d}\tau' \cos(2\phi_a(x,\tau) - 2\phi_b(x,\tau'))$$

$$- \frac{2\alpha}{u}\cdot\frac{D_b}{(2\pi\alpha)^2}\sum_{a,b=1}^{n}\int \mathrm{d}x \int \mathrm{d}\tau \cos(2\phi_a(x,\tau) - 2\phi_b(x,\tau+\Delta\tau)) \tag{2.55}$$

Here, $\Delta\tau < \alpha/u$. From now on we drop the replica indices since we only want to expand up to the first order in D_b. At this order the expression is diagonal in replica space and it is not necessary to consider them explicitly. The cosine of the last term in (2.55) can be expanded up to quadratic order in the difference of the fields: $\phi_a(x,\tau) - \phi_b(x,\tau+\Delta\tau)$. Thus that term describes electron-electron back scattering processes [5]. In the Bosonization section we saw that for a spinless Luttinger liquid backscattering, or g_1-processes, cannot be distinguished from g_2- forward scattering processes[12]. Thus the quadratic part of the last term in (2.55) is included into g_2 processes and hence gives rise to a renormalization of g_2 [7]:

$$g_2 \to \bar{g}_2 = g_2 - \frac{2\alpha D_b}{u} = g_2 - u\pi\bar{D}_b \tag{2.56}$$

As a result the last term in (2.55) is taken into account by a renormalized $\bar{K}$ and $\bar{u}$ which are now governed by interactions and disorder. Now we are able to perform the Wilson RG procedure of integrating out the fast modes. Furthermore we take into account disorder effects in a leading order approximation in the disorder strength D_b.

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}[\phi^>,\phi^<]e^{-S_0^> - S_0^<}$$

$$\left[1 - \frac{D_b}{(2\pi\alpha)^2}\int \mathrm{d}x \iint_{u|\tau-\tau'|\geq\alpha} \mathrm{d}\tau\mathrm{d}\tau' \cos(2\phi(x,\tau) - 2\phi(x,\tau'))\right] \tag{2.57}$$

We average over the fast oscillating modes to get an effective action of the slow modes:

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}[\phi^<]e^{-S_0^<}$$

$$\left[1 - \frac{D_b}{(2\pi\alpha)^2}\int \mathrm{d}x \iint_{u|\tau-\tau'|\geq\alpha} \mathrm{d}\tau\mathrm{d}\tau' \cos(2\phi^<(x,\tau) - 2\phi^<(x,\tau'))e^{-2\langle[\phi^>(x,\tau)-\phi^>(x,\tau')]^2\rangle_{S_0^>}}\right] \tag{2.58}$$

and we reexponentiate Eq. (2.58) to get the result of the first loop of the RG:

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}[\phi^<]e^{-S_0^< - \frac{D_b}{(2\pi\alpha)^2}\int \mathrm{d}x \iint_{u|\tau-\tau'|\geq\alpha} \mathrm{d}\tau\mathrm{d}\tau' \cos 2(\phi^<(x,\tau)-\phi^<(x,\tau'))e^{-2\langle[\phi^>(x,\tau)-\phi^>(x,\tau')]^2\rangle_{S_0^>}}} \tag{2.59}$$

[12] see figure (1.4) where the various electron-electron scattering processes are characterized.

The action of (2.59) is similar to the original action (2.55). However, here we only have slow fields, that means now the energy cut-off is smaller: Λ' instead of Λ. In order to compare this action with the original one we have to rescale the system by:

$$\mathrm{d}x = \tfrac{\Lambda}{\Lambda'}\mathrm{d}\tilde{x} \qquad \mathrm{d}\tau = \tfrac{\Lambda}{\Lambda'}\mathrm{d}\tilde{\tau} \tag{2.60}$$

The effective action is similar to the original one but the disorder strength D_b is renormalized:

$$\tilde{D}_b = D_b \cdot \left(\frac{\Lambda}{\Lambda'}\right)^3 e^{-2\langle[\phi^> - \phi'^>]^2\rangle_{S_0^>}} \tag{2.61}$$

The correlation function in the exponent for $\beta \to \infty$, $L \to \infty$ is:

$$-2\langle\left[\phi^>(x,\tau) - \phi^>(x,\tau')\right]^2\rangle_{S_0^>} =$$

$$\begin{aligned}
&= -2\int\frac{\mathrm{d}\omega_n}{2\pi}\int\frac{\mathrm{d}p}{2\pi}[2-2\cos(qx-\omega t)]\frac{\pi u K}{\omega_n^2+u^2p^2}\theta(\Lambda-Q)\theta(Q-\Lambda') \\
&\geq -4\pi K\int\frac{\kappa}{2\pi}\int\frac{d\omega_n}{2\pi}\frac{1}{\omega_n^2+\kappa^2}\theta(\Lambda-Q)\theta(Q-\Lambda') \\
&= -2K\int_{\Lambda'}^{\Lambda}\mathrm{d}q\frac{1}{q} \\
&= -2K\ln\left(\frac{\Lambda}{\Lambda'}\right)
\end{aligned} \tag{2.62}$$

where $Q = \sqrt{p^2 + (\omega_n/u)^2}$.
Thus the renormalized disorder strength (2.61) becomes :

$$\tilde{D}_b(\Lambda') = D_b(\Lambda)\cdot\left(\frac{\Lambda}{\Lambda'}\right)^{3-2K} \tag{2.63}$$

From this expression we are able to derive the flow equation of the disorder strength. We parametrize the cut-off by $\Lambda(l) = \Lambda_0 e^{-l}$ and assume that the other cut-off Λ' differs infinitesimally from Λ. Hence $\Lambda'(l+\mathrm{d}l) = \Lambda_0 e^{-l-\mathrm{d}l}$, and the flow equation reads:

$$\frac{\mathrm{d}\bar{D}_b}{\mathrm{d}l} = (3-2K)\bar{D}_b \tag{2.64}$$

where $\bar{D}_b = \frac{2\alpha}{\pi u^2}D_b$.
The other flow equations are calculated in a similar way, [5] Appendix E1. They are:

$$\frac{\mathrm{d}\bar{K}}{\mathrm{d}l} = -\frac{\bar{K}^2}{2}\bar{D}_b \tag{2.65}$$

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}l} = -\frac{\bar{u}\bar{K}}{2}\bar{D}_b \tag{2.66}$$

These flow equations describe the behaviour of a spinless disordered Luttinger liquid. The corresponding phase diagram is given in figure (2.8).

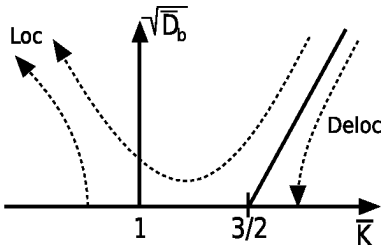


Figure 2.8: Phase diagram and flow of a spinless disordered Luttinger liquid in terms of the renormalized parameter $\bar{D}_b$ and $\bar{K}$. $\bar{D}_b = \frac{2\alpha}{\pi u^2} D_b$. $\bar{K} = \left(\frac{1+y_4/2-\bar{y}_2/2}{1+y_4/2+\bar{y}_2/2}\right)^{1/2}$, where $y_4 = g_4/(\pi v_F)$ and $\bar{y}_2 = \bar{g}_2/(\pi v_F)$

The phase diagram in Fig. (2.8) shows a localization-delocalization transition. However in Fig. (2.8) it looks like initially weak interactions $\bar{K} \sim 1$ become strong under the flow. Of course this is not the case and the reason for a decreasing $\bar{K}$ is the increasing disorder strength $\bar{D}_b$.
In order to get the flow equations in terms of the old parameters we have to insert $\bar{K} = K - K\bar{D}_b/2$; equation (2.65) becomes:

$$\frac{\mathrm{d}K}{\mathrm{d}l} = \frac{3}{2} K(1-K)\bar{D}_b + \mathcal{O}(\bar{D}_b^2) \tag{2.67}$$

Furthermore we are interested in weak interactions $K = 1 - \epsilon$, $\epsilon \ll 1$. The flow equations (2.64) and (2.65) read:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}l} = -3\epsilon \frac{\bar{D}_b}{2} + \mathcal{O}(\bar{D}_b^2) \tag{2.68}$$

$$\frac{\mathrm{d}\bar{D}_b}{\mathrm{d}l} = (1+2\epsilon)\bar{D}_b \tag{2.69}$$

The flow diagram is shown in figure (2.9). Around $K = 1$ where interactions are weak, the system flows torwards stronger and stronger disorder strength.
This flow indicates that the inelastic interactions are reduced by disorder, [5] p. 287. Obviously,

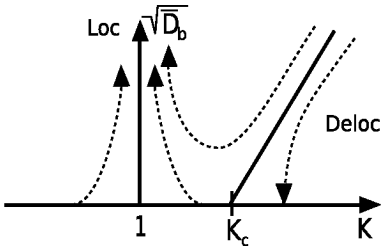


Figure 2.9: Phase diagram in terms of the real interaction constant K and the disorder strength $\bar{D}_b$.

disorder does not produce inelastic interactions, cf. Fig. (2.8). The flow pattern is now in agreement with the physical picture that the interaction effects become less relevant in the localized phase due to the exponentially small overlap of the individual wave functions.
To conclude this section we discuss the transport scattering time of disordered Luttinger liquids.

At higher temperatures, where the thermal length $L_T = u/T$ is smaller than the mean free path l, the impurities are independently renormalized by Friedel oscillations [2]. The single impurity problem was studied in [16], [17]. It was found that the renormalized linear conductance G_{cond} is:

$$G_{cond}(T) = \frac{e^2}{2\pi\hbar}\frac{\mathcal{T}_0(T\cdot\alpha)^{2\epsilon}}{\mathcal{R}_0 + \mathcal{T}_0(T\cdot\alpha)^{2\epsilon}} \tag{2.70}$$

Thus, the scattering time[13] $1/\gamma$ becomes temperature dependent:

$$\frac{1}{\gamma(T)} = \frac{1}{\gamma_0}(T\cdot\alpha)^{2\epsilon} \tag{2.71}$$

where $\epsilon = 1 - K \ll 1$.

[13]Since τ denotes the imaginary time in this subsection, $1/\gamma$ is the scattering time

2.2 Transport in disordered Luttinger liquids

As mentioned in the introduction, the DC conductivity of a non interacting one dimensional disordered system is zero due to Anderson localization. At finite frequencies ω the conductivity increases and is described by the Berezinskii-Mott conductivity [4]. The RG analysis in the last section has shown that for repulsive interactions disorder is relevant and the system flows to the localized strong coupling regime. This is the pinned charge density wave (CDW) phase[14].
In subsection (2.1.2) we have seen that a finite temperature leads to dephasing due to inelastic electron-electron scattering. Dephasing limits the magnitude of quantum interference effects like weak localization. Thus we expect that dephasing effects weaken Anderson localization in one-dimensional systems. Note that the thermal length $L_T = u/T$ and the coherence length L_ϕ are in general different parameters. Thus it is in general not sufficient to use L_T to argue whether localization is strong or not. In order to estimate the influence of dephasing on the transport behaviour we have to evaluate the weak localization (WL) correction [2].
Since we have to use τ for the imaginary time in this chapter, the transport scattering time $\tau_{tr} =: 1/\gamma$ will be denoted by γ in order to avoid confusions.
In the following we analyze the answer obtained by I.V. Gornyi, A.D. Mirlin and D.G. Polyakov (GMP) [2]. They used a functional bosonization approach to calculate the WL correction and the relevant dephasing rate τ_ϕ^{-1}. They conclude, that the transport behaviour at high temperatures[15] is close to the Drude conductivity. All further quantum corrections are small. According to GMP [2], it is possible that one dimensional disordered systems with interactions exhibit a transport behaviour similar to mesoscopic systems in higher dimensions.
In order to compare our full bosonization approach of subsequent sections, we will briefly show the main steps of the calculation of [2], part VII.

2.2.1 Weak localization and dephasing in 1D

We start with the basics of diagrammatic techniques for functional bosonization. A right moving electron which propagates from $1 = \{x_1, \tau_1\}$ to $2 = \{x_2, \tau_2\}$ is described by the following expression, see Eq. (1.68), (1.74):

$$\mathcal{G}_R(1,2) = \langle g_R(x_2 - x_1, \tau_2 - \tau_1; [\varphi]) e^{i[\theta_R(2) - \theta_R(1)]} \rangle_{S_\varphi} \tag{2.72}$$

The corresponding diagrammatic expression is depicted in figure (2.10). The functional bosonization technique separates the free electron propagation from interactions. As noted in the bosonization part, interactions are exactly taken into account in the functional bosonization formulation. Now we include an external backscattering potential $\xi(x_2)$ at position x_2. The corresponding Green's function reads:

$$\mathcal{G}(x_3, x_1, \tau_3 - \tau_1) = \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}x_2 \xi(x_2) g_R(x_2 - x_1, \tau_2 - \tau_1) g_L(x_3 - x_2, \tau_3 - \tau_2) \cdot \\ \cdot \langle e^{i[\theta_R(2) - \theta_R(1) + \theta_L(3) - \theta_L(2)]} \rangle_{S_\varphi} \tag{2.73}$$

Figure (2.11) shows the diagrammatic representation of Eq. (2.73)

[14] [5] chapter 9
[15] i.e. the thermal length L_T is smaller than the mean free path l

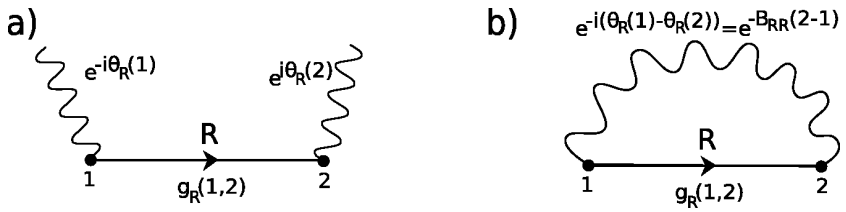


Figure 2.10: Diagrammatic representation of the Green's function of a right mover. The solid line represents the free Green's function $g_R(1,2)$, the wavy line the interaction factors $\exp(i\theta)$. a) Green's function before averaging with respect to S_φ. b) After averaging the total Green's function $\mathcal{G}$ with respect to the interacting system.

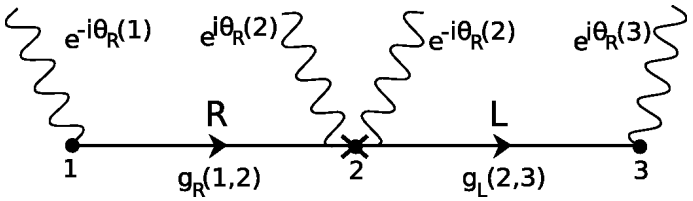


Figure 2.11: Diagrammatic representation of the Green's function $\mathcal{G}(x_3, x_1, \tau_3 - \tau_1)$ in Eq. (2.73)

The $S_\varphi-$ average of the exponential of θ-fields describes all possible pairings of the wavy lines.

$$\langle e^{i[\theta_R(2)-\theta_R(1)+\theta_L(3)-\theta_L(2)]}\rangle_{S_\varphi} = e^{-\mathcal{B}_{RR}(x_2-x_1,\tau_2-\tau_1)+\mathcal{B}_{RL}(x_2-x_1,\tau_2-\tau_1)} \cdot \\ \cdot\, e^{-\mathcal{B}_{LL}(x_3-x_2,\tau_3-\tau_2)-\mathcal{B}_{RL}(x_3-x_2,\tau_3-\tau_2)-\mathcal{B}_{RL}(x_3-x_1,\tau_3-\tau_1)} \tag{2.74}$$

Note that $\mathcal{B}_{RL}(0,0) = 0$.
The correlation functions $\mathcal{B}_{\mu\nu}(x'-x, \tau'-\tau)$; $\mu,\nu = R/L$ will appear very often in this subsection, they read [2]:

$$\mathcal{B}_{\mu\nu}(x,\tau) = \langle[\theta_\mu(0,0) - \theta_\mu(x,\tau)]\theta_\nu(0,0)\rangle_{S_\varphi}$$

$$\mathcal{B}_{R,R/L}(x,\tau) = \frac{1}{\beta}\sum_{\Omega_n}\int \frac{\mathrm{d}q}{2\pi}\frac{e^{iqx-i\Omega_n\tau}-1}{(i\Omega_n - qv_F)(i\Omega_n \mp qv_F)}V_{R\nu}(i\Omega_n, q) \tag{2.75}$$

$\mathcal{B}_{LL}(x,\tau) = \mathcal{B}_{RR}(-x,\tau)$ and $\mathcal{B}_{RL}(x,\tau) = \mathcal{B}_{LR}(x,\tau)$.
This calculation can be extended to arbitrarily many impurities. Consequently, we are able to take into account interactions exactly and disorder perturbatively.
To summarize, the Feynman rules for Luttinger liquids in the presence of an external backscattering potential are:

- a free Green's functions: $g_{R/L}(x,\tau)$ is represented by solid line
- at vertices there are wavy lines that account for interactions
- a backscattering vertex at x' is given by (see Fig. (2.11) and Eq. (2.73)):

$$\xi(x')e^{i[\theta_{R/L}(x')-\theta_{L/R}(x')]}$$

- To calculate the diagram, one has to:

– take into account all possible pairings of the interaction lines

– sum over internal coordinates.

In section (2.1.2) we have seen that quantum corrections to the conductivity (WL) are represented by maximally crossed diagrams.
It can be shown that the two-impurity non-Drude diagrams vanish to leading order in $(\epsilon_F \tau)^{-1}$, see [2]. Consequently, the next possible diagram that contributes to the Cooperon is the three-impurity bubble shown in figure (2.12).

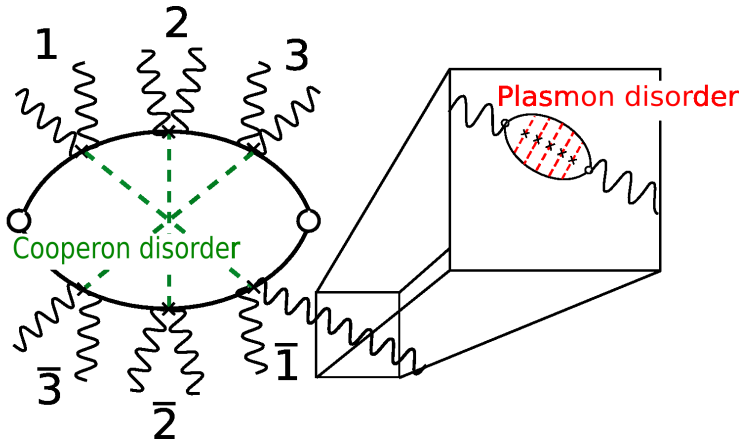


Figure 2.12: Minimal Cooperon. The Cooperon disorder impurity lines (green) couple to the solid electron lines of the Cooperon. The inset shows the disorder renormalization in the interaction propagators. The Plasmon disorder impurity lines (red) indicate the disorder renormalization of the bubbles.

In the regime of strong dephasing $\tau_\phi \gamma \ll 1$ the three impurity Cooperon is the leading contribution to the WL correction. All higher order terms in impurity scattering are subleading corrections in parameters $\tau_\phi \gamma \ll 1$. According to the Feynman rules, each of the six impurity vertices should be dressed by the interaction factors.
Let us analyze the effect of disorder on interactions, namely we have to average the RPA bubbles, see [2]. The assumption is that the Cooperon impurities and the Plasmon impurities are uncorrelated, as indicated in Fig. (2.12). The bubbles are averaged independently, which means that no impurity line connects two different bubbles. The independent averaging of the RPA bubbles and the Cooperon over disorder is justified if the characteristic energy transfer $\omega \sim 1/\tau_\phi \gg \gamma$ (see p.14 of [2]).
The disorder renormalization of the pair bubbles introduces damping in the interaction propagators $V_{\mu\nu}$. Due to Eq. (2.74), damping is also introduced in the interaction factors $\mathcal{B}_{\mu\nu}$ of the Green's functions. The disorder renormalized interaction propagators read (see A21/22 of [2]):

$$V_{RR}(\omega, q) = -\frac{g_2^2}{2\pi v_F} \frac{q^2 v_F^2 - q v_F \omega - i\omega \frac{\gamma}{2}}{q^2 u^2 - \omega^2 - i\omega(\gamma + \delta\gamma)} \tag{2.76}$$

$$V_{RL}(\omega, q) = g_2 \frac{q^2 v_F^2 - \omega^2 - i\omega\left(\gamma + \frac{\delta\gamma}{2}\right)}{q^2 u^2 + \omega^2 - i\omega(\gamma + \delta\gamma)} \tag{2.77}$$

where $\delta\gamma = -\gamma g_2 / 2\pi v_F$. Both propagators, Eq. (2.77) and (2.76), contain dissipative terms proportional to $i\omega\gamma$.

Note that, disorder is introduced at two points, as backscattering vertices of electrons and as a disorder renormalized RPA. This is indicated by Plasmon disorder and Cooperon disorder lines in figure (2.12). There is no rigorous proof that these impurities are independent.
A derivation from first principles, i.e. from a backscattering term in the fermionic action, is highly desirable. Later we use such an approach in the full bosonization framework where disorder is included from the beginning. The disadvantage is that we are not able to formulate convenient mesoscopic diagrams like Diffusons and Cooperons in a straightforward way.

Let's now discuss the weak localization correction $\Delta\sigma_{WL}$. In figure (2.13) all diagrams that contain three impurities are shown. The second and the third diagram together give the same contribution c_3 as the first diagram [2]. Hence we only need to discuss the three-impurity Cooperon diagram.

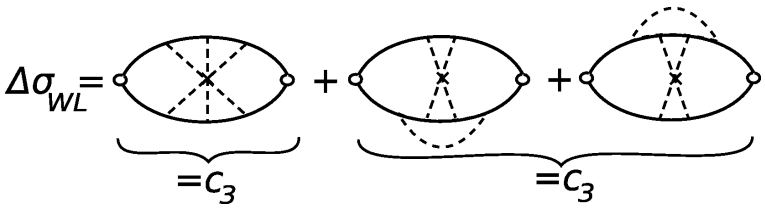


Figure 2.13: Diagrams that contribute to leading order in $\tau_\phi \cdot \gamma$ to the weak localization correction

The interaction factor is responsible for dephasing and the renormalization of impurities:

$$\exp(-S_C) = \langle \exp\left[i(\theta_f - \theta_b)\right] \rangle_{S[\varphi]} \tag{2.78}$$

$\theta_{f,b}$ are the phases accumulated by an electron propagating along the forward and backward paths.

$$\theta_f = \theta_R(1) - \theta_L(1) - \theta_R(2) + \theta_L(2) + \theta_R(3) - \theta_L(3) \tag{2.79}$$
$$\theta_b = \theta_R(\bar{1}) - \theta_L(\bar{1}) - \theta_R(\bar{2}) + \theta_L(\bar{2}) + \theta_R(\bar{3}) - \theta_L(\bar{3}) \tag{2.80}$$

The coordinates $1, 2, \ldots$ denote the space and imaginary time variable, e.g. $1 = \{x_1, \tau_1\}$.
Averaging with respect to S_φ yields all possible pairings between the θ-fields. Therefore, we have a sum of $\mathcal{B}_{\mu\nu}(x_i - x_j, \tau_i - \tau_j)$.
The result of that calculation will be the dephasing action S_C. We skip further calculations and jump to the discussion of the result:

$$\delta\sigma_{WL} = -\frac{1}{4}\sigma_D \left(\frac{\tau_\phi}{\tau}\right)^2 \ln\frac{\tau}{\tau_\phi} \tag{2.81}$$

where

$$\frac{1}{\tau_\phi} = \epsilon\sqrt{\frac{\pi T}{\tau}}$$

The behaviour of the conductivity of one dimensional disordered systems with weak interactions, $\epsilon = 1 - K \ll 1$, is shown in figure (2.14).
The expression of the weak localization correction (2.81) gives rise to an energy scale T_1:

$$T_1 = \frac{1}{\epsilon^2\tau} \tag{2.82}$$

For $T < T_1$ localization effects become strong and the DC conductivity vanishes: $\sigma_{DC} \to 0$. At temperatures $T > T_1/\epsilon$ which are much higher than T_1 the behaviour of the conductivity becomes similar to the conductivity of conventional mesoscopic systems in higher dimensions. In the following chapters we analyze this high temperature regime using an alternative approach to disordered Luttinger liquids.

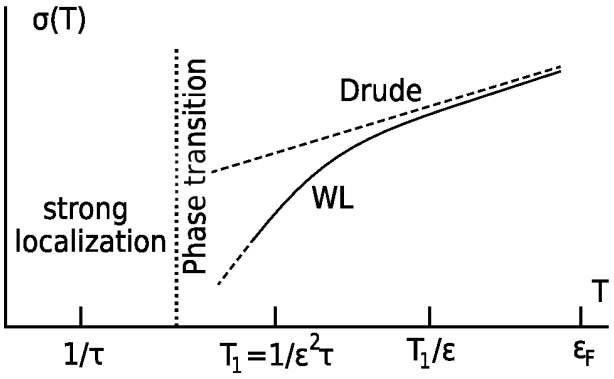


Figure 2.14: Schematic behaviour of the conductivity. The dotted line indicates the Drude conductivity, which is temperature dependent due to the renormalized scattering time. Moreover there is a localization transition in the space of many-body states, see [14], [15], indicated by the vertical dotted line.

Chapter 3

Keldysh Formalism in Disordered Luttinger liquids

In the following chapters a full bosonization approach to disordered Luttinger liquids is presented. In contrast to chapter two, we are going to use the Keldysh technique to calculate disorder averaged quantities. We remind that the advantage of the Keldysh technique in disordered systems is that quantum average and disorder average can be interchanged since the partition function is unity $\mathcal{Z} = 1$.
In section (3.1) we derive the disorder averaged action of disordered Luttinger liquids. The action is used in section (3.2) to calculate the first order correction in the disorder strength D_b to the conductivity. This correction was already obtained in [23] and [32]. Based on this calculation we are able to derive important selection rules which are used in further calculations.

3.1 The disorder averaged action

In section (2.1.3) we discussed the back scattering term of the imaginary time action, Eq. (2.45). The disorder action on the Keldysh contour has the following form:

$$\begin{aligned} S_{dis} &= \int_{\mathcal{C}_K} dt \int dx \left(\xi(x)\bar{\psi}_L(x)\psi_R(x) + \xi^*(x)\bar{\psi}_R(x)\psi_L(x)\right) \\ &= \int_{-\infty}^{\infty} dt \int dx \xi(x)[\bar{\psi}_{L+}\psi_{R+}(x) - \bar{\psi}_{L-}\psi_{R-}(x)] + \\ &\qquad \xi^*(x)[\bar{\psi}_{R+}\psi_{L+}(x) - \bar{\psi}_{R-}\psi_{L-}(x)] \end{aligned} \tag{3.1}$$

The disorder average will generate an effective electron interaction[1].
Using the Debye-Waller relation $\langle e^{iS_{dis}}\rangle_{dis} = e^{-\frac{1}{2}\langle S_{dis}^2\rangle_{dis}} = e^{iS_D}$ one can perform the disorder average:

$$S_D = \frac{D_b}{2i}\int dt dt' dx \left([\bar{\psi}_{L+}\psi_{R+} - \bar{\psi}_{L-}\psi_{R-}](x,t)[\bar{\psi}_{R+}\psi_{L+} - \bar{\psi}_{R-}\psi_{L-}](x,t') + h.c.\right) \tag{3.2}$$

where D_b is the disorder strength.
Now we apply the bosonization identity. Bosonization is a well defined procedure on the Keldysh contour. If a Fermion is on the + line then also the corresponding Boson fields have this subscript and vice versa on the backward time contour. Since the Klein factors $U_{R/L}$ commute with the Hamiltonian $\mathcal{H}$, Eq. (2.43), they have no time dependence and stay constant on the Keldysh contour no matter if we are going forward or backward in time. The bosonization identities read:

$$\psi_{R\pm}(x) = U_{R\pm}\lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi\alpha}}e^{i(k_F-\frac{\pi}{L})x}e^{-i(\phi_\pm(x)-\theta_\pm(x))} \tag{3.3}$$

$$\psi_{L\pm}(x) = U_{L\pm}\lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi\alpha}}e^{-i(k_F+\frac{\pi}{L})x}e^{i(\phi_\pm(x)+\theta_\pm(x))} \tag{3.4}$$

We insert the bosonization identities in Eq. (3.2) and get the effective disorder action similar to the sine-Gordon type but non-local in time.

$$S_D = \frac{D_b}{i(\pi\alpha)^2}\int_{-\infty}^{\infty} dt dt' dx \cos\left[(\phi_{cl}(x,t) - \phi_{cl}(x,t'))\right]\cdot\sin\left(\phi_q(x,t)\right)\cdot\sin\left(\phi_q(x,t')\right) \tag{3.5}$$

Thus we obtained the action that describes disordered Luttinger liquids. It is given by: $S = S_0 + S_D$, where S_0 is given in Eq. (1.157) and S_D in Eq. (3.5).

[1] see chapter 6 of [8]

3.2 First order correction in disorder strength to the clean conductivity

It is not feasible to calculate the retarded Green's function $\langle\phi_{cl}\phi_q^*\rangle_S$ exactly since the effective disorder action S_D, Eq. (3.5), is a product of sine and cosine of the fields. The retarded Green's function, however, is necessary to calculate the conductivity, see Eq. (1.161). Nevertheless, one can calculate the conductivity σ perturbatively assuming weak disorder. Thus, we expand the retarded Green's function in the disorder strength D_b:

$$\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle_{S=S_0+S_D} = \langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle_{S_0} + \langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)iS_D\rangle_{S_0} + \mathcal{O}(D_b^2) \tag{3.6}$$

The leading term of Eq. (3.6) has been evaluated in Eq. (1.162).
The expansion of the Drude conductivity of a non-interacting system in powers of $1/\omega\tau$ has been shown in Eq. (1.137). Here τ denotes the transport scattering time. In the same sense we expand in D_b, calculate the first order correction in disorder strength to the clean conductivity and relate τ and D_b at the end of the calculation by comparing it with the non-interacting limit. We expand the conductivity in D_b:

$$\sigma(\omega)|_{k=0} \approx \sigma_0 + \sigma_1 \tag{3.7}$$

where $\sigma_0 = ie^2uK/\pi\hbar(\omega + i\delta)$ and

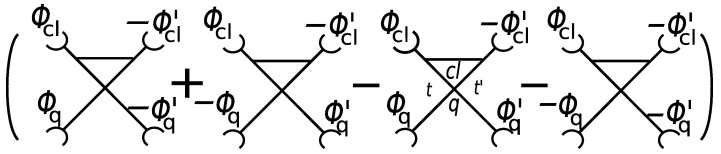
$$\sigma_1(\omega)|_{k=0} = \frac{e^2}{2\pi^2\hbar^2}\int\frac{dq}{2\pi}\int\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)iS_D\rangle_{S_0} \tag{3.8}$$

The disorder action S_D is given in (3.5). However, we rewrite it into a form which is convenient for taking expectation values:

$$\begin{aligned} S_D &= \frac{D_b}{i(2\pi\alpha)^2}\int_{-\infty}^{\infty}\mathrm{d}t\mathrm{d}t'\mathrm{d}x\Big[e^{i\left(\phi_{cl}-\phi'_{cl}+\phi_q-\phi'_q\right)} + e^{i\left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)} \\ &\quad -e^{i\left(\phi_{cl}-\phi'_{cl}+\phi_q+\phi'_q\right)} - e^{i\left(\phi_{cl}-\phi'_{cl}-\phi_q-\phi'_q\right)}\Big] \end{aligned} \tag{3.9}$$

Equation (3.5) has actually eight exponential factors, two for each cosine/sine respectively. But we reduced their number in Eq. (3.9) by taking into account all the complex conjugate terms by a factor of two. To relate the four exponentials of Eq. (3.9) to the box diagrams[2] which will be used in the calculation of Eq. (3.8), we write Eq. (3.9) in a diagrammatic way:

$$S_D = \frac{D_b}{i(2\pi\alpha)^2}\iint\mathrm{d}t\mathrm{d}t'\mathrm{d}x \tag{3.10}$$



Here the diagrammatic representation is only schematical and not an identity since the box diagrams are used to represent correlation functions in this work.
Throughout the text the shorthand notation $\phi = \phi(x,t)$ and $\phi' = \phi(x,t')$ is often used.
Here it is important to note that since $\phi_q \sim \phi_+ - \phi_-$ electric neutrality[3] is fulfilled in all four

[2] see appendix A.3

[3] Since the Boson fields are notoriously divergent, e.g. $\langle exp(i\phi)\rangle = exp(-\langle\phi\phi\rangle/2) = 0$, the expectation value $\langle e^{i\sum_j(A_j\phi_+(r_j)+B_j\phi_-(r_j))}\rangle \neq 0$ if and only if $\sum_j A_j = 0$ and as well for B_j. This is called electric neutrality.

factors of (3.9).
Together with the results from the appendices we have all tools to calculate

$$\langle \phi_{cl}(k,\omega)\phi_q^*(-q,-\nu) iS_D\rangle_{S_0} \tag{3.11}$$

The mixture of fields and exponential of fields in (3.8) can be evaluated by exponentiating the fields at first and then evaluate the expectation value. As an example how to perform this trick explicitly we use the third diagram in Eq. (3.10). At first we exponentiate the two fields $\phi_{cl}(k,\omega)$ and $\phi_q^*(-q,-\nu)$:

$$\begin{aligned}&\langle \phi_{cl}\phi_q^* e^{i\left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)}\rangle_{S_0} = \\ &= \lim_{I_1,I_2\to 0} \partial^2_{I_2,I_1}\langle exp(i[I_1\phi_{cl} - I_2\phi_q^* + \left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)])\rangle\end{aligned} \tag{3.12}$$

and apply the Debye Waller relation before taking the derivatives and the limits. Thus (3.12) becomes:

$$\begin{aligned}-\langle\phi_q^*\left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)\rangle\cdot\langle\phi_{cl}\left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)\rangle e^{-\frac{1}{2}\langle[\ldots]^2\rangle}\\ + \langle\phi_{cl}\phi_q^*\rangle e^{-\frac{1}{2}\langle[\ldots]^2\rangle}\end{aligned} \tag{3.13}$$

with $\langle[\ldots]^2\rangle = \langle[\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q]^2\rangle$.
The second term is a disconnected diagram. It can be shown in fermionic language without interactions that the last term basically corresponds to a clean pair bubble times a closed loop diagram with one impurity line, Fig (3.1). Note that the structure in (3.13) is the same for the other exponentials of iS_D. Only the signs in front of the fields change.

3.2.1 Cancellation of vacuum diagrams

In the introduction about the Keldysh technique it was shown that the partition function $\mathcal{Z}$ is unity and that this property is not altered by perturbation theory. To show that this is the case, at least up to first order in D_b, we expand the partition function: $\mathcal{Z} = 1+\langle iS_D\rangle+\mathcal{O}(D_b^2)$. Since the leading term is already unity all other powers of iS_D have to be zero:

$$\begin{aligned}\langle iS_D\rangle \;=\; &\frac{D_b}{(2\pi\alpha)^2}\iint \mathrm{d}t\mathrm{d}t'\mathrm{d}x\Big(\langle e^{i\left(\phi_{cl}-\phi'_{cl}+\phi_q-\phi'_q\right)}\rangle + \langle e^{i\left(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q\right)}\rangle\\ &-\langle e^{i\left(\phi_{cl}-\phi'_{cl}+\phi_q+\phi'_q\right)}\rangle - \langle e^{i\left(\phi_{cl}-\phi'_{cl}-\phi_q-\phi'_q\right)}\rangle\Big)\end{aligned} \tag{3.14}$$

The results for various correlation functions are given in the appendix. Since all terms have the $\exp(-\langle(\phi_{cl}-\phi'_{cl})^2\rangle)$ in common we can factorize them out and consider the different phase factors alone. These phase factors correspond to correlation functions of the type: $\exp(\langle\phi_{cl}\phi'_q\rangle)$. Thus equation (3.14) is equal to zero.

$$\langle iS_D\rangle_{S_0} \sim e^{i\pi K} + e^{-i\pi K} - e^{-i\pi K\cdot sign(t-t')} - e^{i\pi K\cdot sign(t-t')} = 0 \tag{3.15}$$

The corresponding diagrammatic representation is shown in figure (3.1, 2). The cancelation eliminates the vacuum diagrams which we have encountered in equation (3.13).

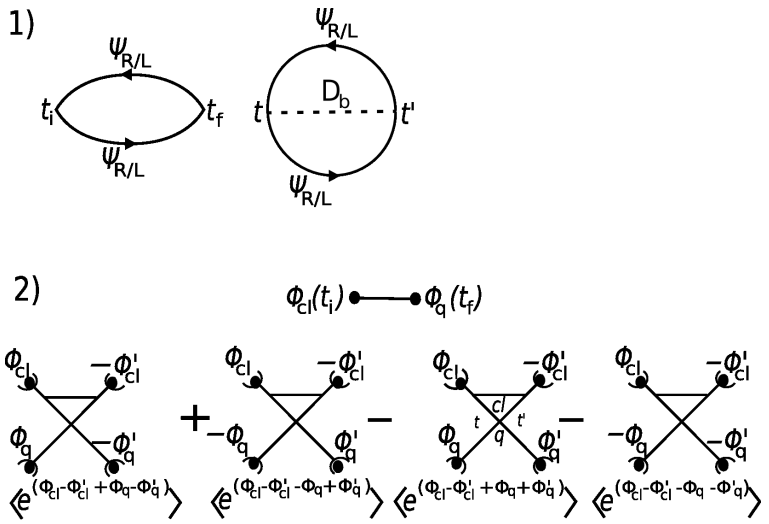


Figure 3.1: 1) Disconnected closed vacuum loop diagram in fermionic language which appears in the first order of perturbation theory $\mathcal{O}(D_b)$ of the conductivity. The closed diagram with the impurity line (dotted line) is disconnected from the pair-bubble which would represent the zeroth order approximation if it was taken alone. The solid lines denote electron Green's functions in Keldysh space. 2) Diagrammatic representation of disconnected vacuum terms in bosonization (see Eq.(3.13)). Here, the $\phi_{cl} - \phi_q$-line above the boxes is not connected to the disorder potential. The $\phi_{cl} - \phi_q$-line corresponds to the bubble in 1) if K=1.

3.2.2 Causality principle in Keldysh calculation

From the theory of free fermions in disordered systems we know that the Drude conductivity has no explicit temperature dependence. However, there are four terms in equation (3.13) that contain a Keldysh Green's function[4]:

$$\begin{array}{ll} \langle\phi_q^*\phi_{cl}\rangle\cdot\langle\phi_{cl}\phi_{cl}\rangle; & \langle\phi_q^*\phi'_{cl}\rangle\cdot\langle\phi_{cl}\phi'_{cl}\rangle; \\ \langle\phi_q^*\phi'_{cl}\rangle\cdot\langle\phi_{cl}\phi_{cl}\rangle; & \langle\phi_q^*\phi_{cl}\rangle\cdot\langle\phi_{cl}\phi'_{cl}\rangle \end{array} \tag{3.16}$$

Since Keldysh Green's functions, eq.(1.127), are a combination of retarded and advanced Green's functions, retardation can be in general violated if such terms are present. However, this would be wrong since the conductivity is calculated by retarded expressions. Moreover, Keldysh Green's functions have a factor $\coth(\omega\beta/2)$ these terms introduce a temperature dependence which does not vanish when the Luttinger parameter K is set to unity[5]. Hence these terms are expected to vanish.
Let us define:

$$\mathcal{K}(t-t') = -\frac{1}{2}\langle\left[\phi_{cl}(x,t) - \phi_{cl}(x,t')\right]^2\rangle_{S_0+S_D} \tag{3.17}$$

[4]each of the other four terms are basically a product of two retarded correlation function:

$$\langle\phi_q^*\phi_{cl}\rangle\langle\phi_{cl}\phi_q\rangle; \langle\phi_q^*\phi'_{cl}\rangle\langle\phi_{cl}\phi'_q\rangle; \langle\phi_q^*\phi'_{cl}\rangle\langle\phi_{cl}\phi_q\rangle; \langle\phi_q^*\phi_q\rangle\langle\phi_{cl}\phi'_q\rangle$$

[5]$K = 1$ corresponds to the limit where no interactions are present

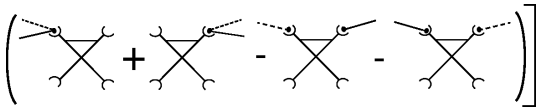
and

$$F(t-t') := \frac{D_b}{(2\pi\alpha)^2} e^{\mathcal{K}(t-t')} \tag{3.18}$$

Furthermore we expand $\langle iS_D \rangle$ and denote the four terms that are not proportional to D^K by R.

$$\langle \phi_{cl}\phi_q^* iS_D\rangle_{\bar{S}} = R + \int dt' \int dt \int dx F(t-t') \cdot \Big[\tag{3.19}$$

$$\left(e^{i\pi K} + e^{-i\pi K} - e^{-i\pi K sign(t-t')} - e^{i\pi K sign(t-t')}\right) \times$$



The four different phase factors in the second line of (3.19) come from the four exponentials of the action. The diagrams correspond to a product of two correlation functions, e.g. $\langle \phi_q^* \phi_{cl}\rangle \cdot \langle \phi_{cl}\phi_{cl}\rangle$. The external fields are represented by a dotted line ($\phi_q^*(-q,-\nu)$) and solid line ($\phi_{cl}(k,\omega)$). They are coupled to one of the four fields of the exponential which is represented by a box. It is easy to see that the second line yields zero and hence terms that are proportional to D^K do not contribute.

This is an important result since it also eliminates the quantum-quantum correlation function

$$\langle \phi_q \phi_q^* iS_D\rangle = 0 \tag{3.20}$$

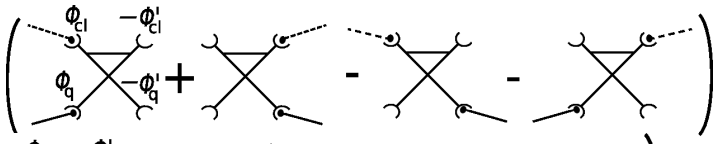
Since two external quantum fields can only couple to the classical fields[6] of the box one can easily verify this by replacing the solid line by a dotted line in the diagrams of Eq. (3.19).
To conclude, we showed that the disconnected diagrams and the diagrams that contain a Keldysh Green's function vanish. Moreover, the cancellation of the terms proportional to Keldysh Green's functions preserves retardation in our calculation.

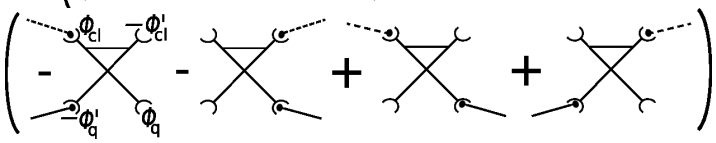
[6] $\langle \phi_q(x,t)\phi_q(x',t')\rangle = 0$

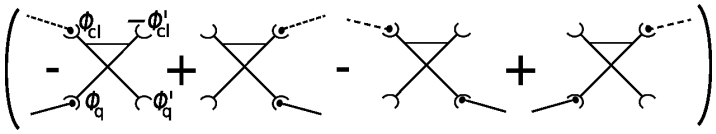
3.2.3 Retarded terms of the conductivity

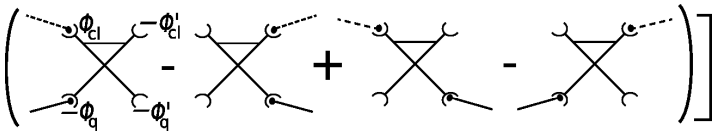
So far we have shown which terms do not give a contribution to the perturbation series. Now let us calculate the remaining parts that will give a contribution to the first order correction in D_b to the clean conductivity. Here, all diagrams denote two retarded Green's functions[7], as indicated in Eq.(3.23).

$$\langle \phi_{cl}(k,\omega)\phi_q^*(-q,-\nu) iS_D\rangle_{\bar{S}} = -\int dt' \int dt \int dx F(t-t')\cdot \Big[$$

$+e^{i\pi K}\times$ 

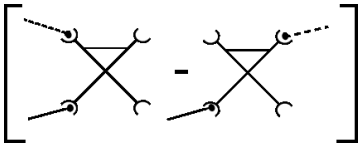
$+e^{-i\pi K}\times$ 

$+e^{-i\pi K sign(t-t')}\times$ 

$+e^{i\pi K sign(t-t')}\times$ 

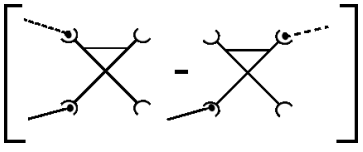
(3.21)

Since $F(t-t') = F(t'-t)$ we can simplify Eq. (3.21) by changing the time integration variables $t \leftrightarrow t'$. Namely, the diagrams of the second row are equivalent to the diagrams in the first row and the diagrams of the fourth row are equivalent to the diagrams in the third row.

$$= -\int dt \int dt' \int dx F(t-t') \sin(\pi K)\, i8\theta(t-t')\times$$ 

(3.22)

where

$$\langle \phi_q(t_i)\phi_{cl}(t)\rangle\langle \phi_{cl}(t_f)\phi_q(t)\rangle - \langle \phi_q(t_i)\phi_{cl}(t')\rangle\langle \phi_{cl}(t_f)\phi_q(t)\rangle =$$ 

(3.23)

[7] The fields which are written on the first box of each row indicate the origin of the signs in front of each box-diagram. The overall minus sign in Eq. (3.21) is due to the derivative $\partial^2_{I_2,I_1}$ taken in Eq.(3.12)
Note that the exponentials with a *sign*-function in the exponent have an extra minus sign in the action S_D

We remind that the conductivity is given by[8]

$$\sigma(\omega, k=0)E_\omega = \frac{e^2}{2\hbar^2\pi^2}\int\frac{dq}{2\pi}\int\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle_{S_0+S_D}E_{-\nu} \tag{3.24}$$

Our goal is to approximate the correlation function in (3.24) up to the first order in the effective disorder action S_D.

$$\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle_{S_0+S_D} \approx \underbrace{\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)\rangle_{S_0}}_{\to\sigma_0} + \underbrace{\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)iS_D\rangle_{S_0}}_{\to\sigma^1} \tag{3.25}$$

The second term on the right hand side was evaluated in Eq. (3.21). Moreover we define:

$$\bar{F}(t-t') = \frac{e^2}{2\hbar^2\pi^2}i\cdot\sin(\pi K)F(t-t') \tag{3.26}$$

The result of equation (3.21) $\times e^2/2\hbar^2\pi^2$ is:

$$\frac{e^2}{2\hbar^2\pi^2}\langle\phi_{cl}(k,\omega)\phi_q^*(-q,-\nu)iS_D\rangle =$$

$$-8\iint dtdt'dx\bar{F}(t-t')\theta(t-t')\cdot(\langle\phi_q^*(-q,-\nu)\phi_{cl}(x,t)\rangle\langle\phi_{cl}(k,\omega)\phi_q(x,t)\rangle)$$

$$+8\iint dtdt'dx\bar{F}(t-t')\theta(t-t')\cdot(\langle\phi_q^*(-q,-\nu)\phi_{cl}(x,t')\rangle\langle\phi_{cl}(k,\omega)\phi_q(x,t)\rangle) \tag{3.27}$$

Furthermore we insert the retarded Green's function in energy-momentum representation[9] into the Fourier transformed current:

$$\tilde{j}(\omega,k) = \sigma(\omega,k)E_\omega =$$

$$= \iint dtdt'dx\int\frac{dq}{2\pi}\int\frac{d\nu}{2\pi}32\cdot\bar{F}(t-t')\theta(t-t')\omega\left[e^{-i(k+q)x}e^{i\omega t}e^{i\nu t} - e^{-i(k+q)x}e^{i\omega t}e^{i\nu t'}\right]$$

$$\times D^R(k,\omega)D^R(-q,-\nu)E_{-\nu} \tag{3.28}$$

We transform to center of mass and relative times:

$$\begin{array}{ll}\mathcal{T} = \frac{t+t'}{2} & \tilde{t} = t-t' \\ t = \mathcal{T}+\frac{\tilde{t}}{2} & t' = \mathcal{T}-\frac{\tilde{t}}{2}\end{array} \tag{3.29}$$

Equation (3.28) becomes

$$\tilde{j}(\omega,k) =$$

$$\int d\mathcal{T}d\tilde{t}dx\int\frac{dq}{2\pi}\int\frac{d\nu}{2\pi}32\bar{F}(\tilde{t})\theta(\tilde{t})\omega e^{i[(\mathcal{T}+\frac{\tilde{t}}{2})(\omega+\nu)]}\left[1-e^{-i\nu\tilde{t}}\right]D^R(k,\omega)D^R(-q,-\nu)E_{-\nu}e^{-i(k+q)x} \tag{3.30}$$

[8] see equation (1.160)

[9] Note that: $\langle\phi_{cl}(k,\omega)\phi_q^*(k,\omega)\rangle = 2iD^R(k,\omega)$

We perform the x -integral and get a δ-function $\int \mathrm{d}x \exp(-i(k+q)x) = 2\pi\delta(k+q)$. Furthermore we integrate over the q-variable and get the condition $q = -k$. By repeating the same procedure with the $\mathcal{T}$- and ν-integration we get the condition $\nu = -\omega$. Equation (3.30) becomes:

$$\tilde{j}(\omega,k) = \int d\tilde{t} 32 \bar{F}(\tilde{t})\theta(\tilde{t})\omega \left(1 - e^{i\omega\tilde{t}}\right) \left(D^R(k,\omega)\right)^2 E_\omega \tag{3.31}$$

By dividing by the electric field E_ω we get the conductivity

$$\sigma_1(\omega)|_{k=0} = \frac{2}{\hbar}\left(\frac{e}{\pi}\right)^2 D_b \frac{i\pi^2 K^2}{(\omega+i\delta)^3\beta^3}\left(\frac{\pi\alpha}{\beta u}\right)^{2K-2} \underbrace{\beta \cdot 2\sin(\pi K)\int_0^\infty dt \frac{1-e^{i\omega t}}{\sinh^{2K}(\frac{\pi}{\beta}t)}}_{=:\mathcal{I}} \tag{3.32}$$

First order in D_b result

In order to discuss equation (3.32) we need to simplify it further. The remaining integral is:

$$\mathcal{I}(K,\omega\beta) = 2\sin(\pi K)\int_0^\infty \mathrm{d}t \frac{1-e^{i\omega\beta t}}{\sinh^{2K}(\pi t)} \tag{3.33}$$

Note that the integral Eq. (3.33) is regular for $K < 1$. Moreover the integrand is exponentially small at large times. This integral can be solved by an analytic continuation of the following expression (See 4.131, 1. and 2. in [20]):

$$\int_0^\infty e^{iax}\sinh^\nu(gx)e^{-bx}dx = \frac{\Gamma(\nu+1)}{2^{\nu+1}g}\cdot\frac{\Gamma\left[\frac{b-\nu g-ia}{2g}\right]}{\Gamma\left[\frac{b+\nu g-ia}{2g}+1\right]} \tag{3.34}$$

The conditions given in [20] are: $\mathcal{R}e\nu > -1, \mathcal{R}eg > 0, |\mathcal{R}e(g\nu)| < \mathcal{R}e\beta$.
The factor $\exp(-bx)$ guarantees convergence of the integral for large times. In Eq. (3.34) we have to assume that $0 < K < 0.5$. Hence the hyperbolic sine is an exponentially small factor at large times and we can analytically continue Eq. (3.34) to $\beta = 0$ and effectively forget about the third condition. In this way we get:

$$-2\frac{\sin(\pi K)}{\beta}\int_0^\infty \frac{e^{i\omega t}}{\sinh^{2K}(\pi T t)}dt = -2^{2K}\Gamma(1-2K)\frac{\sin(\pi K)}{\pi}\frac{\Gamma\left[K - i\frac{\omega}{2\pi T}\right]}{\Gamma\left[1-K-i\frac{\omega}{2\pi T}\right]} \tag{3.35}$$

and

$$2\frac{\sin(\pi K)}{\beta}\int_0^\infty \frac{1}{\sinh^{2K}(\pi T t)}dt = 2^{2K}\frac{\Gamma(1-2K)}{(\Gamma(1-K))^2} \tag{3.36}$$

where we used:

$$\frac{\sin(\pi K)}{\pi} = \frac{1}{\Gamma(K)\Gamma(1-K)}$$

Equation (3.35) and (3.36) hold for $0 < K < 0.5$. If we subtract Eq. (3.35) from (3.36), it will be the result of the integration in Eq. (3.33) which holds for $0 < K < 1$. Due to the prefactor: $\sin(\pi K)$ the answer can be analytically continued to $K = 1$:

$$\mathcal{I}(K,\omega/T) = 2^{2K}\Gamma[1-2K]\left(\frac{1}{\Gamma^2[1-K]} - \frac{\sin(\pi K)}{\pi}\frac{\Gamma\left[K-i\frac{\omega}{2\pi T}\right]}{\Gamma\left[1-K-i\frac{\omega}{2\pi T}\right]}\right) \tag{3.37}$$

This is also the result which was found in [32] and [23] using the Matsubara technique.

We analyze the limit of small $\omega\beta$ and no interactions:

$$\lim_{\omega\beta\to 0}\lim_{K\to 1}\mathcal{I}(K,\omega/T) = -\frac{i\omega\beta}{\pi} \tag{3.38}$$

Thus in the case without interactions the first order correction becomes:

$$\sigma_1(\omega)|_{k=0} = \frac{e^2}{\pi\hbar}\frac{2D_b}{(\omega+i\delta)^2} = -\sigma_0\times\frac{2iD_b}{u\omega} \tag{3.39}$$

where $\sigma_0 = \frac{e^2}{\hbar\pi}\frac{iu}{\omega+i\delta}$.
When this is compared with the expansion in $1/\tau$ of the Drude conductivity we get the following relation for D_b:

$$D_b = \frac{u}{2\tau} \tag{3.40}$$

It is interesting to compare the steps that led to the integral (3.37) in equation (3.32) with the Matsubara formalism. We remind that the effective disorder action in the Matsubara technique is:

$$S^{\alpha_1,\alpha_2}_{D-Matsubara} = -\frac{D_b}{(2\pi\alpha)^2}\int dx\int_0^\beta d\tau d\tau'\cos\left(2[\phi_{\alpha_1}(x,\tau)-\phi_{\alpha_2}(x,\tau')]\right) \tag{3.41}$$

Where τ is the imaginary time ($\tau = it$) and $\alpha_{1,2}$ are the replica indices. Using this action one obtains for the first order correction to the clean conductivity the following expression [32].

$$\sigma_1 = -\frac{2}{\hbar}\left(\frac{e}{\pi}\right)^2 D_b\frac{(\pi K)^2}{\beta\omega_n^3}\left(\frac{\pi\alpha}{\beta u}\right)^{2K-2}\underbrace{\frac{1}{\beta}\int_0^\beta d\tau\frac{1-\cos(\omega_n\tau)}{\sin^{2K}(\pi T\tau)}}_{=:\mathcal{I}} \tag{3.42}$$

To compare (3.42) with (3.32) the integral over the imaginary time τ has to be evaluated by using the Cauchy integral theorem [21]. The integrand of $\mathcal{I}$ in equation (3.42) diverges for $\tau = \beta\cdot n$, $n\in\mathbf{Z}$ on the real line of the complex τ plane. Due to the Cauchy integral theorem we get zero when we integrate over the closed line in Fig. (3.2). For $\exp(i\omega_n\tau)$ we use the contour in the upper half plane, for $\exp(-i\omega_n\tau)$ we use the contour in the lower half plane. At first we go from 0 to β (segment 1: $J_1 = \mathcal{I}$), secondly we go from β to $\beta+i\infty$ (segment 2 J_2), then we go from $\beta+i\infty$ to $i\infty$ (segment 3 J_3) and finally from $i\infty$ to 0. It is easy to see that segment 3 is zero $J_3 = 0$ and segment 1 is the original integral. Hence we can express the integral over τ by two integrals over $\tau = it$, (J_4) and $\tau = it+\beta$, (J_2):

$$\int_0^\beta d\tau\frac{1-\cos(\omega_n\tau)}{\sin^{2K}(\pi T\tau)} = J_1 = -J_2 - J_4 \tag{3.43}$$

In segment 4 $\tau = it$ the denominator of (3.42) becomes
$\sin^{2K}(\pi\tau/\beta) = [i\sinh(\pi t/\beta)]^{2K} = e^{i\pi K}[\sinh(\pi t/\beta)]^{2K}$.
In segment 2 $\tau = it+\beta$ the denominator of (3.42) becomes
$\sin^{2K}(\pi\tau/\beta) = [-i\sinh(\pi t/\beta)]^{2K} = e^{-i\pi K}[\sinh(\pi t/\beta)]^{2K}$.
Hence when J_2 and J_4 are added then $\mathcal{I}$ becomes:

$$\mathcal{I} = \frac{2}{\beta}\sin(\pi K)\int_0^\infty dt\frac{1-e^{\omega_n t}}{\sinh^{2K}(\pi t/\beta)};\qquad i\omega_n\to\omega+i\delta \tag{3.44}$$

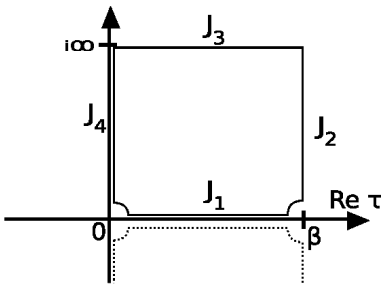


Figure 3.2: Integration contour in the complex τ-plane. The same contour is drawn in the lower half plane as a dotted line. $\mathcal{I} = J_1$.

In the Keldysh method the four exponentials in equation (3.9) have been responsible for the retardation $\theta(t)$ and for the phase factors which led to the $\sin(\pi K)$ factor.

Chapter 4

High-Temperature Regime in One Dimensional Systems

In the previous chapter we used bosonization and the Keldysh formalism to derive the disorder averaged action of disordered Luttinger liquids. As an example, we have calculated the first order correction in D_b to the clean conductivity. The result was already obtained in [23], [32] using replica technique and Matsubara formalism. However, in contrast to the replica technique we have a fixed amount of fields in the Keldysh formalism, namely ϕ_{cl} and ϕ_q. Another advantage of the Keldysh technique is that we do not have to do an analytic continuation in the end to obtain retarded correlation functions. Thus it may be more suited for the following calculations.
In the following section we derive the saddle-point equations for ϕ_{cl} and ϕ_q of the disorder averaged action[1] $S = S_0 + S_D$. We use these equations to derive the semiclassical equation of motion for the retarded Green's function.
In the final section we go beyond the saddle-point approximation and calculate corrections in the small parameters $\epsilon = 1 - K$ and $\omega\beta$.

4.1 Equation of motion derivation of plasmon-dissipation

The goal of this section is to derive the equations of motions for the retarded Green's function of the boson fields of a one dimensional disordered system with interaction. This part was done in collaboration with M. Kiselev [28].
We remind that the retarded Green's function of a clean Luttinger liquid is:

$$D^R(\omega, k) = \frac{1}{2i}\langle \phi_{cl}(k,\omega)\phi_q^*(k,\omega)\rangle_{S_0} = \frac{\pi u K}{(\omega + i\delta)^2 - u^2k^2} \tag{4.1}$$

The corresponding equation of motion, for example is:

$$\left(-\partial_t^2 + u^2\partial_x^2\right) D^R(x,t) = \pi u K \delta(t)\delta(x) \tag{4.2}$$

For a disordered Luttinger liquid we expect an additional self-energy term on the left hand side of (4.2).
As shown in the section about transport quantities, the retarded Green's function (4.1) leads to the conductivity σ_0 of a clean system:

$$\sigma_0(\omega) = \frac{e^2}{\pi\hbar}\frac{iuK}{\omega + i\delta} \tag{4.3}$$

According to [2], a weakly interacting 1D system shows Drude like behaviour in the high-temperature regime. The Drude conductivity is:

$$\sigma_D(\omega) = \frac{e^2}{\hbar\pi}\frac{iuK}{\omega + \frac{i}{\tau}} =: \frac{e^2}{\hbar\pi^2}\cdot i\omega\cdot \bar{D}^R(\omega, k=0) \tag{4.4}$$

τ is the transport scattering time.
From the expression of the Drude conductivity we can guess the form of the retarded Green's function of a dissipative system:

$$\left[\bar{D}^R(\omega,k)\right]^{-1} = \frac{1}{uK\pi}\left(\omega^2 + i\frac{\omega}{\tau} - u^2k^2\right) = \left[D^R(\omega,k)\right]^{-1} + i\frac{\omega}{uK\pi\tau} \tag{4.5}$$

[1] S_0 is given in Eq. (1.157) and S_D in Eq. (3.5)

Consequently, the Green's function of the disordered system $\bar{D}^R$ has an additional $i\omega/\tau$ term on the left hand side of the equations of motion (4.2) compared to the clean Green's function D^R.
In the next section we show that $\bar{D}^R$ can be calculated from a quadratic action. This allows one to analyze corrections to the Drude behaviour.
In the following we derive the equations of motion and analyze necessary conditions for a dissipative term of the form $i\omega/\tau$. We have already derived the action of disordered Luttinger liquids in the full bosonization approach:

$$S = S_0 + S_D \tag{4.6}$$

where the action of the clean system is S_0:

$$S_0 = \frac{1}{4}\iint d(x,t)\iint d(x',t')\,(\phi_{cl},\phi_q)_{x',t'}\begin{pmatrix} 0 & D_A^{-1} \\ D_R^{-1} & (D^{-1})_K \end{pmatrix}\begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}_{x,t} \tag{4.7}$$

and the disorder part S_D, which we obtained from averaging over the random backscattering potential, is:

$$S_D = \frac{\gamma}{i}\int_{-\infty}^{\infty} dtdt'dx\cos\left[(\phi_{cl}(x,t)-\phi_{cl}(x,t'))\right]\cdot\sin\left(\phi_q(x,t)\right)\cdot\sin\left(\phi_q(x,t')\right) \tag{4.8}$$

where $\gamma = \frac{D_b}{(\pi\alpha)^2}$.
The saddle point equations for the fields that extremize the action S are:

$$\frac{\delta S}{\delta\phi_q} = 0 \Rightarrow$$

$$\frac{1}{2\pi uK}\left(-\partial_t^2 + u^2\partial_x^2\right)\phi_{cl} + \frac{2\gamma}{i}\int \mathrm{d}t''\cos(\phi_{cl}-\phi_{cl}'')\sin\phi_q''\cos\phi_q + \frac{1}{2}D_K^{-1}\phi_q = 0 \tag{4.9}$$

$$\frac{\delta S}{\delta\phi_{cl}} = 0 \Rightarrow$$

$$\frac{1}{2\pi uK}\left(-\partial_t^2 + u^2\partial_x^2\right)\phi_q - \frac{2\gamma}{i}\int \mathrm{d}t''\sin(\phi_{cl}-\phi_{cl}'')\sin\phi_q\sin\phi_q'' = 0 \tag{4.10}$$

To derive an equation for the retarded Green's function, we use the expectation value of Eq.(4.9) multiplied by ϕ_q. Thus $D_K^{-1}\phi_q$ in (4.9) vanishes since quantum fields are not correlated. Hence we will not consider it anymore. We define the following operator $\hat{L}_0$ and its Fourier transform $\hat{L}_0^{\omega,q}$:

$$\hat{L}_0 = -\partial_t^2 + u^2\partial_x^2 \qquad \hat{L}_0^{\omega,q} = \omega^2 - u^2q^2 \tag{4.11}$$

Thus the saddle point equations of a disordered Luttinger liquid have the following non-linear form:

$$\frac{1}{4\pi uK}\hat{L}_0\phi_{cl} = -\frac{\gamma}{i}\int \mathrm{d}t'\cos(\phi_{cl}-\phi_{cl}')\sin\phi_q'\cos\phi_q \tag{4.12}$$

$$\frac{1}{4\pi uK}\hat{L}_0\phi_q = \frac{\gamma}{i}\int \mathrm{d}t'\sin(\phi_{cl}-\phi_{cl}')\sin\phi_q'\sin\phi_q \tag{4.13}$$

The retarded Green's function in terms of bigger and lesser Green's functions was given in the introduction of the Keldysh technique:

$$D^R(x,x',t-t') = \theta(t-t')\left(G^>(x,x',t-t') - G^<(x,x',t-t')\right) \tag{4.14}$$

We remind, the bigger and lesser Green's functions are given by correlation functions of Boson fields on the Keldysh contour $\phi_\pm$

$$\langle\phi_+(x,t)\phi_-(x',t')\rangle = iG^<(x,x',t,t') \tag{4.15}$$
$$\langle\phi_-(x,t)\phi_+(x',t')\rangle = iG^>(x,x',t,t') \tag{4.16}$$

Note that there is the following relation between ϕ_{cl}, ϕ_q and ϕ_+, ϕ_-:

$$\begin{aligned} \phi_{cl} &= \phi_+ + \phi_- \qquad & \phi_+ &= \tfrac{1}{2}(\phi_{cl} + \phi_q) \\ \phi_q &= \phi_+ - \phi_- \qquad & \phi_- &= \tfrac{1}{2}(\phi_{cl} - \phi_q) \end{aligned} \tag{4.17}$$

In order to derive the equation of motions for the retarded Green's function we need to take a second order derivative with respect to space x and time t.

$$\partial_x^2 D^R(x,x',t-t') = i\theta(t-t')\left(\langle\partial_x^2\phi_-(x,t)\phi_+(x',t')\rangle - \langle\partial_x^2\phi_+(x,t)\phi_-(x',t')\rangle\right) \tag{4.18}$$

The derivative with respect to the time is:

$$\begin{aligned} \partial_t D^R(x,x',t-t') &= \delta(t-t')\left(G^>(x,x',t-t') - G^<(x,x',t-t')\right) \\ &+ i\theta(t-t')\left(\langle\partial_t\phi_-(x,t)\phi_+(x',t')\rangle - \langle\partial_t\phi_+(x,t)\phi_-(x',t')\rangle\right) \end{aligned} \tag{4.19}$$

The first term vanishes since the difference $G^> - G^<$ is for $t = t'$ basically the commutator $[\phi(t), \phi(t)]$. This commutator is zero at coinciding times , see [11] p.17 Eq. (49). The second order derivative with respect to time is:

$$\begin{aligned} \partial_t^2 D^R(x,x',t-t') &= i\delta(t-t')\left(\langle\partial_t\phi_-(x,t)\phi_+(x',t')\rangle - \langle\partial_t\phi_+(x,t)\phi_-(x',t')\rangle\right) \\ &+ i\theta(t-t')\left(\langle\partial_t^2\phi_-(x,t)\phi_+(x',t')\rangle - \langle\partial_t^2\phi_+(x,t)\phi_-(x',t')\rangle\right) \end{aligned} \tag{4.20}$$

Summing up (4.18) and (4.20) we obtain the equation of motion for the retarded Green's function:

$$\begin{aligned} \hat{L}_0 D^R(x-x',t-t') &= i\theta(t-t')\left(\langle\hat{L}_0\phi_+(t)\phi_-(t')\rangle - \langle\hat{L}_0\phi_-(t)\phi_+(t')\rangle\right) \\ &+ \pi u K\delta(t-t')\delta(x-x') \end{aligned} \tag{4.21}$$

Where we used $[\partial_z\phi(z), \phi(z')] \sim \delta(z-z'), z = it + ix$ (see [11] p.17 Eq. (47)) in the following relation:

$$\pi u K\delta(t-t')\delta(x-x') = -i\delta(t-t')\left(\langle\partial_t\phi_+(t)\phi_-(t')\rangle - \langle\partial_t\phi_-(t)\phi_+(t')\rangle\right) \tag{4.22}$$

The operator relations $\hat{L}_0\phi_+$, $\hat{L}_0\phi_-$ can be found from the saddle point equations (4.9) and (4.10) by applying the rotation in Keldysh space (4.17). Inserting the saddle point equations and expressing everything in ϕ_{cl}, ϕ_q fields, we obtain the following equation of motion:

$$\hat{L}_0 D^R(x-x',t-t') = \pi u K\delta(t-t')\delta(x-x') - \theta(t-t')2\pi u K(\Sigma^{(1)} + \Sigma^{(2)}) \tag{4.23}$$

where $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are:

$$\Sigma^{(1)} := -\int \mathrm{d}t'' \langle \cos\phi_q(x,t)\sin\phi_q(x,t'')\cos(\phi_{cl}(x,t)-\phi_{cl}(x,t''))\phi_q(x',t')\rangle_{S_0} \quad (4.24)$$

$$\Sigma^{(2)} := \int \mathrm{d}t'' \langle \sin\phi_q(x,t)\sin\phi_q(x,t'')\cos(\phi_{cl}(x,t)-\phi_{cl}(x,t''))\phi_{cl}(x',t')\rangle_{S_0} \quad (4.25)$$

These two terms describe the self energy of the retarded Green's function. In the following we rewrite it as $\hat{L}_\tau \cdot D^R$:

$$\left(\hat{L}_0 + \hat{L}_\tau\right) D^R = \pi u K \delta(t-t')\delta(x-x') \quad (4.26)$$

We will focus on the imaginary part of $L_\tau = 2\pi u K \gamma \theta(t-t')(\Sigma^{(1)}+\Sigma^{(2)})$ because it describes damping.

Self-energy of the retarded Green's function

The self energy terms Eq. (4.24) and (4.25) can be treated in a similar way as it has been done in the calculation of the first order correction to the Drude conductivity.
We rewrite the sine and cosines in $\Sigma^{(1)}$ in terms of exponentials:

$$\Sigma^{(1)} := \frac{i}{4}\int \mathrm{d}t'' \sum_{a,a''=\pm 1} a'' \cdot \langle e^{i(a\phi_q + a''\phi_q'' + \phi_{cl} - \phi_{cl})}\phi_q'\rangle \quad (4.27)$$

We raise the ϕ_q' into the exponent and evaluate the expectation value by using the Debye-Waller relation.

$$\Sigma^{(1)} := i\int \mathrm{d}t'' \sum_{a,a''=\pm 1} a''\left(D^R(x-x',t-t') - D^R(x-x',t''-t')\right)\times$$
$$\times \exp\left(-\frac{1}{2}\langle[\phi_{cl}-\phi_{cl}''+a\cdot\phi_q + a''\cdot\phi_q'']^2\rangle\right) \quad (4.28)$$

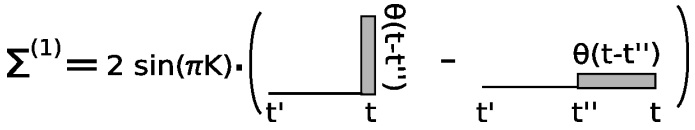


Figure 4.1: Retarded time structure of $\Sigma^{(1)}$, Eq. (4.29). The solid line represents the retarded Green's function. The shaded box represents the $\exp(-\langle(\phi_{cl}-\phi_{cl}'')^2\rangle/2)$ correlation function. [5]

The exponential of the correlation function is discussed in appendix A. Eq. (4.28) can be written as:

$$\Sigma^{(1)} := 2\sin(\pi K)\int \mathrm{d}t''\left(D^R(x-x',t-t') - D^R(x-x',t''-t')\right)\theta(t-t'')\times$$
$$\times \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}}{\sinh^{2K}\left(\frac{\pi}{\beta}(t-t'')\right)} \quad (4.29)$$

The time structure of Eq. (4.29) is shown in figure (4.1). The first term in Eq. (4.29) in brackets contains a $\theta(t-t')$ function, which is due to the retarded Green's function. The second term contains a $\theta(t''-t')$ Green's function. Thus, the time structure of both retarded Green's functions is compatible with the overall θ-function of $\hat{L}_\tau$: $\theta(t-t')$.
Repeating the same manipulations on $\Sigma^{(2)}$ we obtain:

$$\Sigma^{(2)} := 2\sin(\pi K)\int \mathrm{d}t''\left(-D^R(x'-x,t'-t)\theta(t-t'') - D^R(x'-x,t'-t'')\theta(t''-t)\right)\times$$

$$\times \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}}{\sinh^{2K}\left(\frac{\pi}{\beta}(t-t'')\right)} \tag{4.30}$$

The time structure is shown in figure (4.2). $\Sigma^{(2)}$ is not compatible with the overall θ-function of $\hat{L}_\tau$: $\theta(t-t')$, since the first retarded Green's function in Eq. (4.30) is proportional to $\theta(t'-t)$ and the time structure of the second term is: $t < t'' < t'$. In contrast to $\Sigma^{(1)}$, this part contributes only to the advanced Green's function whereas $\Sigma^{(1)}$ contributes only to the retarded Green's function.

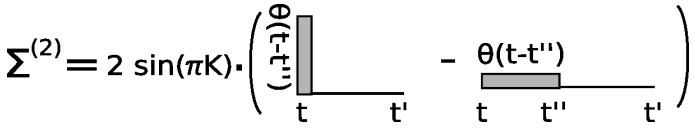


Figure 4.2: Retarded time structure of $\Sigma^{(2)}$ equation (4.30). The solid line represents the retarded Green's function The shaded box represents the $\exp(-\langle(\phi_{cl}-\phi''_{cl})^2\rangle/2)$ correlation function. [5]

Finally, the equations of motion are reduced to:

$$\begin{aligned}\hat{L}_0 D^R(x-x',t-t') &= \pi u K\delta(t-t')\delta(x-x') - \hat{L}_\tau D^R(x-x',t-t') \\ &= \pi u K\delta(t-t')\delta(x-x') - i\theta(t-t')2\pi u K\gamma\Sigma^{(1)}\end{aligned} \tag{4.31}$$

The Fourier transformation of the left hand side is straightforward and yields:

$$\hat{L}_0 D^R(x-x',t-t') = \int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}q}{2\pi}e^{i[q(x-x')-\omega(t-t')]}(\omega^2-u^2q^2)\tilde{D}^R(\omega,q) \tag{4.32}$$

The Fourier transform of the first term on the right hand side of Eq. (4.31) is also trivial, it yields: $\pi u K$. Before we Fourier transform the second term on the right hand side of Eq. (4.31) we define:

$$\Sigma_1^{(1)} = i4\pi u K\gamma\sin(\pi K)D^R(t-t',x-x')\int\mathrm{d}t''\theta(t-t'')\frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}}{\sinh^{2K}\left(\frac{\pi}{\beta}(t-t'')\right)} \tag{4.33}$$

$$\Sigma_2^{(1)} = -i4\pi u K\gamma\sin(\pi K)\int\mathrm{d}t''D^R(t''-t',x-x')\theta(t-t'')\frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}}{\sinh^{2K}\left(\frac{\pi}{\beta}(t-t'')\right)} \tag{4.34}$$

$$f(\omega) := \int_0^\infty \mathrm{d}t \frac{e^{i\omega t}\left(\frac{\pi\alpha}{\beta u}\right)^{2K}}{\sinh^{2K}\left(\frac{\pi}{\beta}(t-t'')\right)} \tag{4.35}$$

Note that: $\hat{L}_\tau D^R(x-x',t-t') = i\theta(t-t')2\pi uK\gamma\Sigma^{(1)} = \Sigma_1^{(1)} + \Sigma_2^{(1)}$.
We Fourier transform $\Sigma_1^{(1)}$ and rewrite (4.33) as follows:

$$\begin{aligned}\Sigma_1^{(1)} &= i4\pi uK\gamma\sin(\pi K)\int \mathrm{d}t'' \iint \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}q}{2\pi} e^{i[q(x-x')-\omega(t-t')]}\tilde{D}^R(\omega,q)\int \frac{\mathrm{d}\nu}{2\pi}e^{-i\nu(t-t'')}f(\nu) \\ &= -4\pi uK\gamma\sin(\pi K)\iint \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}q}{2\pi} e^{i[q(x-x')-\omega(t-t')]}\tilde{D}^R(\omega,q)f(0)\end{aligned} \tag{4.36}$$

The Fourier transform of $\Sigma_2^{(1)}$ is:

$$\Sigma_2^{(1)} = 4\pi uK\gamma\sin(\pi K)\iint \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}q}{2\pi} e^{i[q(x-x')-\omega(t-t')]}\tilde{D}^R(\omega,q)f(\omega) \tag{4.37}$$

Thus we obtained the Fourier transform of the self energy part $\hat{L}_\tau$:

$$\mathcal{F}_{Fourier}(\hat{L}_\tau) = 4\pi uK\frac{D_b}{(\pi\alpha)^2}\sin(\pi K)\cdot[f(\omega)-f(0)] \tag{4.38}$$

Where we inserted the definition of $\gamma = D_b/(\pi\alpha)^2$. We evaluated the function $f(\omega)$ of (4.35) already in the last chapter (3.35). The result is:

$$f(\omega) := \left(\frac{\pi\alpha}{\beta u}\right)^{2K}\cdot\beta\cdot 2^{2K-1}\frac{\Gamma[1-2K]}{\pi}\frac{\Gamma\left[K-i\frac{\omega\beta}{2\pi}\right]}{\Gamma\left[1-K-i\frac{\omega\beta}{2\pi}\right]} \tag{4.39}$$

Thus equation (4.38) becomes:

$$\mathcal{F}_{Fourier}(\hat{L}_\tau) = \frac{8KD_b\sin(\pi K)}{u\beta}2^{2K-2}\left(\frac{\pi\alpha}{\beta u}\right)^{2K-2}\left[\frac{\Gamma\left[K-i\frac{\omega\beta}{2\pi}\right]}{\Gamma\left[1-K-i\frac{\omega\beta}{2\pi}\right]} - \frac{\Gamma[K]}{\Gamma[1-K]}\right]\Gamma[1-2K] \tag{4.40}$$

For weak interactions $\epsilon = 1-K \ll 1$ and high temperatures $\beta\to 0$ we obtain[2]:

$$\mathcal{F}_{Fourier}(\hat{L}_\tau) = \frac{i\omega}{\tau} \tag{4.41}$$

where we used: $D_b = u/2\tau$.
Finally we can formulate the Fourier transformed equation of motion in this parameter regime:

$$\left(\omega^2 - u^2q^2 + \frac{i\omega}{\tau}\right)D^R(\omega,q) = \pi uK \tag{4.42}$$

Thus the retarded Green's function has the expected form:

$$\Rightarrow \bar{D}^R(\omega,q) = \frac{\pi uK}{\omega^2 - u^2q^2 + \frac{i\omega}{\tau}} \tag{4.43}$$

[2]See section (3.2.3) for the limit.

Equation (4.43) is rather the saddle-point approximation of the retarded Green's function since we obtained the equation of motion by using the saddle-point equations of the ϕ_{cl}- and ϕ_q-fields. The fields which fulfill the saddle-point equation should minimize the action. Thus it is reasonable to assume that the retarded Green's function in (4.43) is the most relevant contribution to the exact Green's function.

4.2 Corrections to the Drude conductivity beyond saddle-point approximation

In section (2.2) we have remarked that it is not sufficient to analyze the transport behaviour of disordered Luttinger liquids as done in [2]. The essential step, namely, to prove that including disorder for interaction propagators and electrons can be done independently, was missing in previous studies. We calculate corrections beyond the saddle point approximation from the last section to fill this gap.
In the last section we have obtained the retarded Green's function with the self energy of a disordered Luttinger liquid in the saddle-point approximation, Eq. (4.43). As shown in Eq. (4.4), the retarded Green's function (4.43) immediately yields the Drude conductivity. In the following we analyze the corrections to the Drude conductivity in the leading orders of $\epsilon = 1 - K$, $\omega\beta$.

Let us construct a quadratic action which corresponds to the Green's function (4.43). In order to establish a connection to the clean Luttinger liquid action S_0, we introduce a trial action ΔS such that $\bar{S} = S_0 + \Delta S$.
A possible guess for the trial action ΔS is:

$$\Delta S \sim \int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\phi_q^*(k,\omega) i\frac{\omega}{uK\pi\tau}\phi_{cl}(k,\omega) \tag{4.44}$$

Since $\bar{D}^A(\omega,k) = [\bar{D}^R(\omega,k)]^*$ we have to add the complex conjugate of (4.44).
Hence a reasonable choice for the trial action ΔS is:

$$\Delta S = \iint \frac{dk}{2\pi}\frac{d\omega}{2\pi}\left(\phi_{cl}^*, \phi_q^*\right)_{k,\omega} \begin{pmatrix} 0 & -\frac{i\omega}{4uK\pi\tau} \\ \frac{i\omega}{4uK\pi\tau} & \Delta D_K^{-1} \end{pmatrix} \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}_{k,\omega} \tag{4.45}$$

It is not necessary to find the Keldysh component in the lower right of the matrix. Due to the fluctuation-dissipation theorem we do know the full Keldysh Green's function in equilibrium:

$$\bar{D}^K(\omega,k) = \coth\left(\frac{\omega\beta}{2}\right)\left[\bar{D}^R(\omega,k) - \bar{D}^A(\omega,k)\right] \tag{4.46}$$

Henceforth, Green's functions are taken with respect to the action $\bar{S} = S_0 + \Delta S$.
Let us write an identity and calculate correlation functions using ΔS as the quadratic action.

$$S = S_0 + \Delta S + S_D - \Delta S = \bar{S} + (S_D - \Delta S) \tag{4.47}$$

The action $\bar{S}$ immediately yields the Drude conductivity.
In order to discuss transport behaviour it is necessary to consider the conductivity.

$$\sigma(\omega,k) = \frac{e^2}{2\hbar\pi^2}\int \frac{dq}{2\pi}\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\rangle_{S_0+S_D} \tag{4.48}$$

Using (4.47), the correlator in (4.48) can also be written in terms of $\bar{S}$.

$$\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\rangle_{S_0+S_D} = \int \mathcal{D}[\phi_{cl},\phi_q]\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)e^{i(\bar{S}+S_D-\Delta S)} \tag{4.49}$$

We are going to explore the deviation of the Drude conductivity on a perturbative level in $S_D - \Delta S$. Since D_b is related to τ by $D_b = u/2\tau$, see Eq. (3.40), an expansion of $\exp(S_D - \Delta S)$ in $S_D - \Delta S$ corresponds to an expansion in the small parameter $1/\tau$.

$$\langle \phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\rangle_{\bar{S}} = \int \mathcal{D}[\phi_{cl},\phi_q]\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)e^{i(S_0+\Delta S)} \cdot \left(1 + i(S_D - \Delta S) + \mathcal{O}\left((S_D - \Delta S)^2\right)\right) \tag{4.50}$$

Assuming that the leading term is $S_D - \Delta S$ we have to proof that:

$$\Delta\sigma(\omega) := \frac{e^2}{2\hbar\pi^2}\int \frac{dq}{2\pi}\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)i(S_D - \Delta S)\rangle_{\bar{S}} \tag{4.51}$$

is small in our range of paramters.
Let us define:

$$\Delta\sigma =: \Delta\sigma^D + \Delta\sigma^\Delta \tag{4.52}$$

where

$$\Delta\sigma^D(\omega) = \frac{e^2}{2\hbar\pi^2}\iint \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\nu}{2\pi}\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)iS_D\rangle_{\bar{S}} \tag{4.53}$$

and

$$\Delta\sigma^\Delta = \frac{e^2}{2\hbar\pi^2}\int \frac{dq}{2\pi}\int \frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)(-i\Delta S)\rangle_{\bar{S}} \tag{4.54}$$

Let us start with $\Delta\sigma^D(\omega)$. The disorder action S_D is given in Eq. (3.5). Since the correlation function in Eq. (4.53) has the same structure as Eq. (3.8) we can repeat most of the steps done in section (3.2). The only difference is that correlation functions are now taken with respect to $\bar{S}$ and not S_0. The following selection rules, established in chapter 3, simplify our calculation:

- the quantum fields are not correlated: $\langle\phi_q\phi_q\rangle = 0$
- the sum of terms that are proportional to $\langle\phi_{cl}\phi_{cl}\rangle$ is zero, which reflects causality of the retarded propagators.
- terms that are proportional to a product of two retarded Green's functions and thus preserve causality remain.

All calculations of the Green's functions with respect to the dissipative action $\bar{S}$ are shifted to appendix B.
To simplify the notation in further calculations, we define:

$$F(t-t') := \frac{D_b}{(2\pi\alpha)^2}e^{\mathcal{K}(t-t')} \tag{4.55}$$

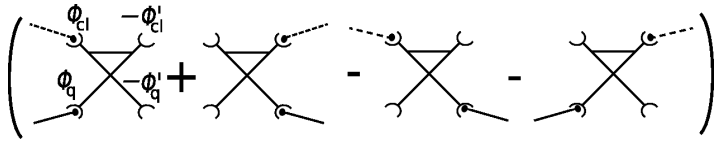
where

$$\mathcal{K}(t-t') = -\frac{1}{2}\langle\left[\phi_{cl}(x,t) - \phi_{cl}(x,t')\right]^2\rangle_{\bar{S}} \tag{4.56}$$

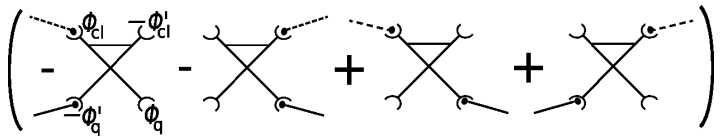
We calculate the correlation function in Eq. (4.53):

$$\langle\phi_{cl}\phi_q^* i S_D\rangle_{\bar{S}} = -\int dt' \int dt \int dx F(t-t')\cdot\Big[\tag{4.57}$$

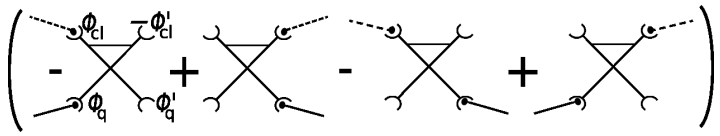
$$+e^{i\pi K I_0\left(\frac{t-t'}{2\tau}\right)\left[\theta(t-t')e^{-\frac{t-t'}{2\tau}}+\theta(t'-t)e^{-\frac{t'-t}{2\tau}}\right]}\times$$



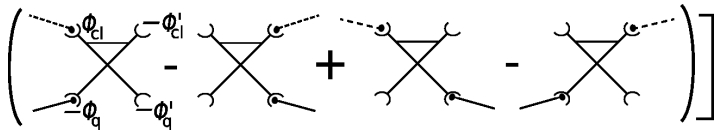
$$+e^{-i\pi K I_0\left(\frac{t-t'}{2\tau}\right)\left[\theta(t-t')e^{-\frac{t-t'}{2\tau}}+\theta(t'-t)e^{-\frac{t'-t}{2\tau}}\right]}\times$$



$$+e^{-i\pi K I_0\left(\frac{t-t'}{2\tau}\right)\left[\theta(t-t')e^{-\frac{t-t'}{2\tau}}-\theta(t'-t)e^{-\frac{t'-t}{2\tau}}\right]}\times$$



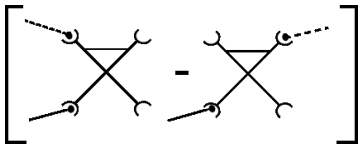
$$+e^{i\pi K I_0\left(\frac{t-t'}{2\tau}\right)\left[\theta(t-t')e^{-\frac{t-t'}{2\tau}}-\theta(t'-t)e^{-\frac{t'-t}{2\tau}}\right]}\times$$



Here $I_0(x) = BesselI[0,x]$ is the modified

Bessel function of the first kind[3]. Since $F(t-t') = F(t'-t)$ we can change the time variables $t \leftrightarrow t'$ in Eq. (4.57). Thus, we get:

$$= -\int dt \int dt' \int dx F(t-t') \sin\left(\pi K e^{-\frac{t-t'}{2\tau}} I_0\left(-\frac{t-t'}{2\tau}\right)\right) i8\times$$



The result of the S_D-part is:

$$\Rightarrow \int \frac{dq}{2\pi}\int \frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q) i S_D\rangle_{\bar{S}} =$$

$$= \omega\frac{8iD_b}{(\pi\alpha)^2}\left[\bar{D}^R(\omega,k)\right]^2\int_0^\infty dt\left(1-e^{i\omega t}\right)\sin\left(\pi K e^{-t/(2\tau)} I_0\left(\frac{t}{2\tau}\right)\right)\cdot e^{\mathcal{K}(t)} \tag{4.58}$$

Since $I_0(0) = 1$, we reproduce the $\sin(\pi K)$ factor of our former result, Eq. (3.32) in the limit of vanishing dissipation $1/\tau = 0$. Since[4] $D_b = u/2\tau$ we obtain the following correction to the conductivity:

$$\Delta\sigma^D = s(\omega,\tau,K)\cdot\frac{iu^2}{(\pi\alpha)^2}\int_0^\infty dt\left(1-e^{i\omega t}\right)\sin\left(\pi K e^{-t/(2\tau)} I_0\left(\frac{t}{2\tau}\right)\right)\cdot e^{\mathcal{K}(t)} \tag{4.59}$$

[3] See http://functions.wolfram.com

[4] Note that D_b is the *bare* disorder strength.

where

$$s(\omega,\tau,K) := \frac{2e^2\omega}{\hbar\pi^2\tau u}\left[\bar{D}^R(\omega,k=0)\right]^2 \tag{4.60}$$

Now we also have to treat the ΔS, Eq. (4.54):

$$\int\frac{dq}{2\pi}\int\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)(-i\Delta S)\rangle_{\bar{S}} \tag{4.61}$$

Furthermore we insert ΔS given in (4.45):

$$\iint\frac{dq}{2\pi}\frac{d\nu}{2\pi}\frac{dk'}{2\pi}\frac{d\omega'}{2\pi}\frac{\omega'\cdot\omega}{4uK\pi\tau}\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\left(\phi_q^*(\omega',k')\phi_{cl}(\omega',k')-\phi_{cl}^*(\omega',k')\phi_q(\omega',k')\right)\rangle_{\bar{S}} \tag{4.62}$$

To evaluate the two 4-point correlation functions we have to use Wick's theorem:

$$\langle\overbrace{\phi_{cl}(\omega,k)\overbrace{\phi_q^*(-\nu,-q)\overbrace{\phi_q^*(\omega',k')}\phi_{cl}(\omega',k')}}\rangle_{\bar{S}}$$

$$=\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\rangle\langle\phi_q^*(\omega',k')\phi_{cl}(\omega',k')\rangle+\langle\phi_{cl}(\omega,k)\phi_q^*(\omega',k')\rangle\langle\phi_q^*(-\nu,-q)\phi_{cl}(\omega',k')\rangle \tag{4.63}$$

The same procedure has to be applied to the second term:

$$\langle\overbrace{\phi_{cl}(\omega,k)\overbrace{\phi_q^*(-\nu,-q)\overbrace{\phi_{cl}^*(\omega',k')}\phi_q(\omega',k')}}\rangle_{\bar{S}}$$

$$=\langle\phi_{cl}(\omega,k)\phi_q^*(-\nu,-q)\rangle\langle\phi_{cl}^*(\omega',k')\phi_q(\omega',k')\rangle+\langle\phi_{cl}(\omega,k)\phi_q(\omega',k')\rangle\langle\phi_q^*(-\nu,-q)\phi_{cl}^*(\omega',k')\rangle \tag{4.64}$$

Note that we do not need to think about the Keldysh component from the lower left of the matrix in (4.45) since it introduces two quantum fields ϕ_q in Eq. (4.61). Thus, such a term would vanish after averaging with the other quantum field ϕ_q^*, in (4.62), since Wick's theorem yields a correlator: $\langle\phi_q\phi_q\rangle=0$ in that case.
The first term of (4.63) cancels the first term of (4.64) when we insert equations (4.63) and (4.64) in (4.62). In other words the lower Wick contractions of (4.63) and (4.64) cancel each other. Inserting Eq. (4.63) and (4.64) into Eq. (4.54) and performing the q,ν- integrals followed by the k',ω' integration, we obtain:

$$\Delta\sigma^{\Delta} = -\frac{e^2}{\hbar\pi^2}\frac{\omega^2}{uK\pi\tau}\cdot\left[\bar{D}^R(\omega,k)\right]^2 \tag{4.65}$$

Now we can insert $\Delta\sigma^D$, equation (4.59), and $\Delta\sigma^{\Delta}$ (4.65), into Eq. (4.51) and obtain an explicit expression for the deviation from the Drude conductivity:

$$\Delta\sigma = \frac{e^2}{2\hbar\pi^2}\iint\frac{dq}{2\pi}\frac{d\nu}{2\pi}\omega\langle\phi_{cl}(\omega,k)\phi_q^*(\omega,k)i\left(S_D-\Delta S\right)\rangle_{\bar{S}}$$

$$=s(\omega,\tau,K)\left(\frac{iu^2}{(\pi\alpha)^2}\int_0^{\infty}\mathrm{d}t(1-e^{i\omega t})\sin\left(\pi Ke^{-t/2\tau}I_0\left(\frac{t}{2\tau}\right)\right)e^{\mathcal{K}(t)}-\frac{\omega}{2K\pi}\right) \tag{4.66}$$

where $\mathcal{K}(t)$ is:

$$\mathcal{K}(t) = -4uK\pi\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\coth\left(\frac{\omega\beta}{2}\right)\omega\frac{\frac{1}{\tau}\left(1-\cos(\omega t)\right)}{(\omega^2-u^2k^2)^2+\frac{\omega^2}{\tau^2}} \tag{4.67}$$

We simplify the integral by defining new frequency and momentum variables:

$$\kappa = \beta u k \quad \Omega = \omega \cdot \beta \tag{4.68}$$

Then equation (4.67) becomes:

$$\mathcal{K}(t) = -4K\pi \int \frac{d\kappa}{2\pi} \int \frac{d\Omega}{2\pi} \coth\left(\frac{\Omega}{2}\right) [1 - \cos(\Omega t/\beta)] \frac{1}{\Omega} \cdot \frac{\frac{\beta}{\tau}}{(\Omega - \frac{\kappa^2}{\Omega})^2 + \frac{\beta^2}{\tau^2}} \tag{4.69}$$

We take the limit $\beta/\tau \to 0$. Thus the leading term of the integrand reduces to a delta function:

$$\frac{\beta/\tau}{x^2 + (\beta/\tau)^2} \to \pi\delta(x) \tag{4.70}$$

For the moment we only consider the zeroth order term and assume that the first order term in β/τ is regular. Thus, Eq.(4.69) reduces to:

$$\mathcal{K}(t) = -K \log\left[\left(\frac{\beta u}{\pi\alpha}\right)^2 \sinh^2\left(\frac{\pi}{\beta}t\right)\right] \tag{4.71}$$

When Eq. (4.71) is inserted in the S_D-part, Eq. (4.58), we obtain:

$$\Delta\sigma^D = \frac{e^2}{2\hbar\pi^2} \int \frac{dq}{2\pi} \int \frac{d\nu}{2\pi} \omega \langle \phi_{cl}\phi_q^* i S_D \rangle_{\bar{S}} =$$

$$= s(\omega,\tau,K) i \left(\frac{\pi\alpha}{\beta u}\right)^{2K-2} \underbrace{\frac{1}{\beta^2} \int_0^\infty dt \left(1 - e^{i\omega t}\right) \frac{\sin\left(\pi K e^{-t/(2\tau)} I_0\left(\frac{t}{2\tau}\right)\right)}{\sinh^{2K}(\frac{\pi}{\beta}t)}}_{\mathcal{J}(\omega,\beta,\tau)} \tag{4.72}$$

It is convenient to scale the time-variable $t \to t' = t/\beta$:

$$\mathcal{J}(\omega,\beta/\tau,K) = \frac{1}{\beta} \int_0^\infty dt' \left(1 - e^{i\omega\beta t'}\right) \frac{\sin\left(\pi K e^{-\frac{t'}{2}\frac{\beta}{\tau}} I_0\left(\frac{t'}{2}\frac{\beta}{\tau}\right)\right)}{\sinh^{2K}(\pi t')} \tag{4.73}$$

In this way β and τ are paired such that we obtain the small parameter $\frac{\beta}{\tau}$. In the limit $\frac{\beta}{\tau} \to 0$ we obtain:

$$\begin{aligned} \mathcal{J}(\omega,\beta/\tau \to 0, 1-\epsilon) &= \frac{\sin(\pi K)}{\beta} \int_0^\infty dt' \frac{1 - e^{i\omega\beta t'}}{\sinh^{2K}(\pi t')} \\ &= \frac{1}{\beta} 2^{1-2\epsilon} \Gamma[1-2K] \left(\frac{1}{\Gamma^2[1-K]} - \frac{\sin(\pi K)}{\pi} \frac{\Gamma\left[K - i\frac{\omega\beta}{2\pi}\right]}{\Gamma\left[1 - K - i\frac{\omega\beta}{2\pi}\right]}\right) \end{aligned} \tag{4.74}$$

Corrections in ϵ (weak interactions) and $\omega\beta$ (high temperature):

Note that, the result of $\mathcal{J}$ was evaluated in the last chapter, Eq. (3.37). Let us put $K = 1-\epsilon$ with $0 < \epsilon \ll 1$.

$$\mathcal{J}(\omega, \beta/\tau \to 0, 1-\epsilon) = \frac{1}{\beta} 2^{1-2\epsilon} \Gamma[2\epsilon - 1] \left(\frac{1}{\Gamma^2[\epsilon]} - \frac{\sin(\pi(1-\epsilon))}{\pi} \frac{\Gamma\left[1-\epsilon-i\frac{\omega}{2\pi T}\right]}{\Gamma\left[\epsilon - i\frac{\omega}{2\pi T}\right]} \right) \tag{4.75}$$

Thus, Eq. (4.72) becomes:

$$\Delta\sigma^D = \frac{2e^2\omega}{\hbar\pi^2\tau u} \left[\bar{D}^R(\omega, k=0)\right]^2 i \left(\frac{\pi\alpha}{\beta u}\right)^{-2\epsilon} \mathcal{J}(\omega, \beta/\tau \to 0, 1-\epsilon) \tag{4.76}$$

The factor $(\pi\alpha/\beta u)^{-2\epsilon}$ represents the impurity renormalization, Eq. (2.71), of τ which we did not take into account by using the bare disorder strength D_b. Henceforth we absorb this factor into τ and use the renormalized scattering time.
We analyze $\mathcal{J}$ in the limit $\beta\omega \ll 1$, $\epsilon \ll 1$.

$$\mathcal{J}(\omega, \beta/\tau \to 0, 1-\epsilon) = \frac{1}{\beta} \left(-i\frac{\omega\beta}{2\pi} + i(\ln(2) - 2) \cdot \epsilon\omega\beta - \frac{\pi^2}{3} \cdot \epsilon(\omega\beta)^2 + \mathcal{O}\left((\omega\beta)^3, \omega\beta\epsilon^2\right) \right) \tag{4.77}$$

Furthermore, we expand Eq. (4.65) for $\epsilon \ll 1$:

$$\Delta\sigma^\Delta = -s(\omega, \tau, K) \cdot \frac{\omega}{2\pi} \left(1 + \epsilon + \mathcal{O}(\epsilon^2)\right) \tag{4.78}$$

After inserting Eq. (4.77) into Eq.(4.72) we sum up Eq. (4.78) and Eq. (4.72):

$$\Delta\sigma(\omega) = \frac{e^2}{\hbar\tau} \frac{u}{\left(\omega + \frac{i}{\tau}\right)^2} \cdot \left(\frac{1}{2\pi}[1 - \ln(2)] \cdot \epsilon - \frac{\pi^2}{3} \cdot \epsilon \cdot \omega\beta + \mathcal{O}\left((\omega\beta)^2, \epsilon^2\right) \right) \tag{4.79}$$

Corrections in β/τ: weak interactions, high temperature and weak disorder:

So far we have not calculated all corrections in β/τ. Nevertheless, we can discuss their influence on the result, Eq. (4.79).
We scale the time variable in Eq. (4.72): $t \to t' = t/\beta$:

$$\Delta\sigma^D = s(\omega, \tau, K) i \left(\frac{\pi\alpha}{\beta u}\right)^{2K-2} \frac{1}{\beta} \int_0^\infty dt' \left(1 - e^{i\omega\beta t'}\right) \frac{\sin\left(\pi K e^{-t'\beta/(2\tau)} I_0\left(\frac{t'\beta}{2\tau}\right)\right)}{\sinh^{2K}(\pi t')} \tag{4.80}$$

Eq. (4.80) contains two terms which we are going to expand in β/τ. The first one is:

$$\sin\left(\pi K e^{-t'\beta/2\tau} Bsl I_0\left(\frac{t'\beta}{2\tau}\right)\right) \approx \sin(\pi K) - \cos(\pi K)\pi K \frac{t'}{2} \cdot \left(\frac{\beta}{\tau}\right) + \mathcal{O}\left((\beta/\tau)^2\right) \tag{4.81}$$

The first term in Eq. (4.81) has been evaluated above, see Eq. (4.79). The second term in Eq. (4.81) yields the following β/τ correction term:

$$\Delta\sigma^{\tau,1} := -s(\omega, \tau, K) \cdot K \cdot \frac{\beta}{\tau} i \cos(\pi K) \frac{\pi}{2} \left(\frac{\pi\alpha}{\beta u}\right)^{2K-2} \frac{1}{\beta} \int_0^\infty \frac{1 - e^{i\omega\beta t}}{\sinh^{2K}(\pi t)} \cdot t dt \tag{4.82}$$

This correction is calculated in appendix B.2, it gives rise to the following correction terms:

$$\Delta\sigma^{\tau,1} = \frac{e^2}{\hbar\tau}\frac{u}{\left(\omega+\frac{i}{\tau}\right)^2}\cdot\left[-\frac{\pi}{3}\frac{1}{\omega\tau}+c\cdot\epsilon\frac{\beta}{\tau}+\mathcal{O}\left(\frac{\epsilon(\omega\beta)^2}{\tau},\frac{\epsilon^2\omega\beta}{\tau}\right)\right] \tag{4.83}$$

where c is a constant[5].
The other correction in β/τ is due to $\mathcal{K}(t)$, Eq. (4.69):

$$\begin{aligned} e^{\mathcal{K}(\beta t)} &= e^{K\mathcal{K}^{(0)}(t)+K\cdot\frac{\beta}{\tau}\mathcal{K}^{(1)}(t)+\mathcal{O}\left((\beta/\tau)^2\right)} \\ &= e^{K\mathcal{K}^{(0)}(t)}\cdot\left(1+\left(\frac{\beta}{\tau}\right)K\cdot\mathcal{K}^{(1)}(t)+\mathcal{O}\left((\beta/\tau)^2\right)\right) \end{aligned} \tag{4.84}$$

where

$$\mathcal{K}^{(0)}(t) := -\log\left[\left(\frac{\beta u}{\pi\alpha}\right)^2\sinh^2(\pi t)\right] \tag{4.85}$$

$$\mathcal{K}^{(1)}(t) = \frac{\partial}{\partial(\beta/\tau)}\mathcal{K}(\beta t)\Big|_{\beta/\tau=0} \tag{4.86}$$

$\mathcal{K}^{(1)}(t)$ has not yet been calculated. It contains the first derivative of the delta function, Eq. (4.70).
To conclude, the corrections to the Drude conductivity, Eq. (4.79) and Eq. (4.83), are small in the high temperature regime of a weakly interacting disordered Luttinger liquid. The remaining corrections in β/τ, Eq. (4.84), will be considered elsewhere.

[5] $c = \frac{\pi}{3}(2\ln(2)-2)+\frac{6}{\pi}\zeta(3)$, $\zeta(3) = 1.202$

Chapter 5

Conclusion

Conclusion & Outlook

We have studied transport of interacting spin-polarized electrons in a disordered one- dimensional system using full bosonization and the Keldysh formalism. We have developed a diagrammatic technique in Keldysh formalism for this system using as an example the first order correction to the clean conductivity in disorder strength for arbitrary repulsive interactions and $\omega \neq 0$ (ω is the frequency of the external field). The perturbation theory in disorder strength can be extended to any order with an increasing complexity of combinatorics.
Furthermore, we have derived the semiclassical equation of motion for the retarded Green's function using a saddle point approximation and analyzed it in the high temperature regime. We have shown that the retarded Green's function which corresponds to the saddle point approximation yields the Drude conductivity for weak interactions and high temperatures.
Finally, we have calculated interaction corrections to the Drude conductivity at high temperatures. We found that these corrections are small in agreement with the conclusions drawn in [2].
So far, we have not calculated corrections to the Drude conductivity in β/τ (β is the inverse temperature, τ is the transport time). However, these corrections would be interesting since they would allow one to calculate interaction-induced corrections to the Berezinskii-Mott conductivity [4].
For further work in this field one could extend the perturbation theory to the third order in $1/\tau$, to establish direct connections to the minimal Cooperon diagram which was considered in [2]. Furthermore, an analysis of spin-effects or additional degrees of freedom like pseudo spin, due to the two Dirac cones in the dispersion relation of carbon nanotubes would be interesting. Since the Keldysh technique provides a framework for studying out-of-equilibrium systems one could calculate the nonlinear response to an external bias cf. [24].

Appendix A

Calculation of the first order correction to the Drude conductivity

A.1 Keldysh correlation functions

In this section we calculate the correlation functions: D^R retarded, D^A advanced and D^K Keldysh. We start from the action that describes a one dimensional bosonized system[1].

$$S_{\Pi,\phi} = \int_{-\infty}^{\infty} dt \int dx \left[\Pi(x,t)\partial_t\phi(x,t) - \frac{1}{2\pi}\left(uK(\pi\Pi(x,t))^2 + \frac{u}{K}(\partial_x\phi(x,t))^2 \right) \right] \tag{A.1}$$

We transform the time integral to an integral on the Keldysh contour $\mathcal{C}_K$. We introduce two species of fields on forward and backward contours and we denote them by + on the forward contour and by - on the backward contour. The first term in (A.1) with coupled Π and ϕ is due to the Legendre transform of the Hamiltonian to the Lagrangian:

$$\int_{\mathcal{C}_K} dt \Pi(x,t)\partial_t\phi(x,t) \to \int_{-\infty}^{\infty} dt \frac{1}{2} \left(\Pi_{cl}\partial_t\phi_q + \Pi_q\partial_t\phi_{cl} \right) \tag{A.2}$$

We use the following transformation rules for the fields:

$$\phi_{cl} = \phi_+ + \phi_- \qquad \phi_+ = \tfrac{1}{2}(\phi_{cl} + \phi_q)$$

$$\phi_q = \phi_+ - \phi_- \qquad \phi_- = \tfrac{1}{2}(\phi_{cl} - \phi_q)$$

The full action (A.1) on the Keldysh contour becomes:

$$\frac{i}{\hbar}S_{\Pi,\phi} = -\int_{-\infty}^{\infty} dt \int dx \left(\Pi_{cl}, \Pi_q\right) \begin{pmatrix} 0 & \frac{i}{2\pi\hbar}\frac{Ku\pi^2}{2} \\ \frac{i}{2\pi\hbar}\frac{Ku\pi^2}{2} & 0 \end{pmatrix} \begin{pmatrix} \Pi_{cl} \\ \Pi_q \end{pmatrix}$$

$$+ \int_{-\infty}^{\infty} \int dt dx \frac{i}{2\hbar} \left(\partial_t\phi_q, \partial_t\phi_{cl}\right) \begin{pmatrix} \Pi_{cl} \\ \Pi_q \end{pmatrix}$$

[1]The fields have the following units: $[\phi] = \sqrt{Js}$ and $[\Pi] = \sqrt{Js}/m$

$$-\int_{-\infty}^{\infty}\int dtdx\frac{iu}{\hbar 4\pi K}\partial_x\phi_{cl}\partial_x\phi_q \tag{A.3}$$

By integrating out the Π fields[2] we get an action which depends on ϕ fields.

$$\int \mathcal{D}[\Pi_{cl},\Pi_q]e^{iS_{\Pi,\phi}/\hbar}=e^{iS_\phi/\hbar} \tag{A.4}$$

with

$$\frac{i}{\hbar}S_\phi=\frac{i}{\hbar}\frac{1}{4}\iint d(x,t)\iint d(x',t')\,(\phi_{cl},\phi_q)_{x',t'}\begin{pmatrix}0 & D_A^{-1}\\ D_R^{-1} & (D^{-1})_K\end{pmatrix}\begin{pmatrix}\phi_{cl}\\ \phi_q\end{pmatrix}_{x,t} \tag{A.5}$$

The Keldysh component is needed for regularization, see the short presentation of the Keldysh technique, section (1.3). All three components of the Green's function read.

$$D_{R/A}^{-1}=\frac{-\partial_t^2+u^2\partial_x^2}{uK\pi}\delta(x-x')\delta(t-t') \tag{A.6}$$

$$(D^{-1})_K=D_R^{-1}\circ F-F\circ D_A^{-1} \tag{A.7}$$

We remind that F is hermitian in time, space and Keldysh indices.
Eq. (A.5) yields the following correlation functions:

$$\langle\phi_{cl}(x,t)\phi_q(x',t')\rangle = i\cdot 2\cdot\hbar D^R(x,x';t,t') \tag{A.8}$$

$$\langle\phi_q(x,t)\phi_{cl}(x',t')\rangle = i\cdot 2\cdot\hbar D^A(x,x';t,t') \tag{A.9}$$

$$\langle\phi_{cl}(x,t)\phi_{cl}(x',t')\rangle = i\cdot 2\cdot\hbar D^K(x,x';t,t') \tag{A.10}$$

Retarded and advanced Green's function in the energy/momentum representation read:

$$\hbar D^R(k,\omega)=\hbar\frac{\pi Ku}{(\omega+i\delta)^2-u^2k^2} \tag{A.11}$$

$$\hbar D^A(k,\omega)=\hbar\frac{\pi Ku}{(\omega-i\delta)^2-u^2k^2} \tag{A.12}$$

One may notice that the relations in (A.8) - (A.10) are twice as big as in the usual convention of the Keldysh rotation [12]. This is due to the different definition of the ϕ_{cl} and ϕ_q with a factor of $1/2$ in front instead of $1/\sqrt{2}$. It is important that this does not influence physical observables.

A.2 Exponentials of correlation functions I

In section (3.2) we need to evaluate exponentials of correlation functions like

$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')-\phi_q(x,t)+\phi_q(x,t')]^2\rangle_{S_0}} \tag{A.13}$$

The correlation function $\langle\phi_q\phi_q\rangle$ is zero due to a fundamental relation in Keldysh space [12]. So the exponent is

$$-\frac{1}{2}\langle\left[\phi_{cl}(x,t)-\phi_{cl}(x,t')\right]^2\rangle+\langle\left[\phi_{cl}(x,t)-\phi_{cl}(x,t')\right]\left[\phi_q(x,t)-\phi_q(x,t')\right]\rangle \tag{A.14}$$

[2] The integration routine is a functional extension of the finite dimensional formula:
$\int d^N\vec{x}e^{\vec{x}^T\cdot A\vec{x}+\vec{b}^T\vec{x}}=\sqrt{\pi^N}det(A)e^{\frac{1}{4}\vec{b}^T\cdot A^{-1}\vec{b}}$

Evaluation of the Keldysh Green's function

We start with the first term.

$$\begin{aligned}\bar{K}(t) &:= -\frac{1}{2}\langle\left[\phi_{cl}(x,t)-\phi_{cl}(x,t')\right]^2\rangle \\ &= -\frac{1}{2}\int\frac{dk}{2\pi}\frac{d\omega}{2\pi}2\left[1-\cos[\omega(t-t')]\right]i2D^K(k,\omega) = \\ &= -\int\frac{dk}{2\pi}\frac{d\omega}{2\pi}\left[1-\cos[\omega(t-t')]\right]\coth\left(\frac{\omega}{2T}\right)\left(\frac{2\pi iuK}{(\omega+i\delta)^2-u^2k^2}-\frac{2\pi iuK}{(\omega-i\delta)^2-u^2k^2}\right)\end{aligned} \tag{A.15}$$

The ω - integral can be simplified with the Dirac relation.

$$\begin{aligned}\frac{1}{x\pm i\delta} &= \mathcal{P}\frac{1}{x}\mp i\pi\delta(x) \\ \frac{1}{x-i\delta}-\frac{1}{x+i\delta} &= 2i\pi\delta(x)\end{aligned} \tag{A.16}$$

Hence the Keldysh Green's can be written as

$$D^K(q,\omega)=\coth\left(\frac{\omega}{2T}\right)\frac{\pi uK}{2uk}\left(-2i\pi\delta(\omega-uk)+2i\pi\delta(\omega+uk)\right) \tag{A.17}$$

If we insert this into equation (A.15) we get

$$\bar{K}(t)=-\int_{-\infty}^{\infty}dk\frac{K}{k}\coth\left(\frac{uk}{2T}\right)\left(1-\cos[uk(t-t')]\right) \tag{A.18}$$

This integral is ill-defined until we introduce a proper convergence factor $\exp(-\alpha|k|)$. This momentum cut-off corresponds to a finite bandwidth.
Moreover we split $\coth\left(\frac{x}{2}\right)=2f_B(x)+1$. Since the integrand is even under the change of sign from $k\to -k$ we simply take twice the integral from 0 to ∞ and hence (A.18) becomes:

$$\bar{K}(t)=-4K\int_0^{\infty}dk\frac{e^{-\alpha k}f_B\left(\frac{uk}{T}\right)\left(1-\cos[uk(t-t')]\right)}{k}-2K\int_0^{\infty}\frac{dk}{k}e^{-\alpha k}\left(1-\cos[uk(t-t')]\right) \tag{A.19}$$

This integral can be found, for example in appendix C of [5]. I will just state the result.

$$\bar{K}(t)=-K\log\left[\left(\frac{\beta u}{\pi\alpha}\right)^2\sinh^2\left(\frac{\pi}{\beta}(t-t')\right)\right] \tag{A.20}$$

Evaluation of the retarded Green's function

The remaining part of (A.14) is:

$$\langle[\phi_{cl}-\phi'_{cl}][\phi_q-\phi'_q]\rangle \tag{A.21}$$

It includes the retarded Green's function at coinciding times:

$$\langle\phi_{cl}(x,t)\phi_q(x,t)\rangle=2i\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\frac{\pi uK}{(\omega+i\delta)^2-u^2k^2} \tag{A.22}$$

Since the two poles are in the lower half plane we immediately see that the ω integral is zero. On the other hand, if we would do the k integral first it is important to include a convergence factor $exp(-\alpha|k|)$ before integrating. With that convergence factor the expression is equal to zero.

$$\begin{aligned}\langle\phi_{cl}(x,t)\phi_q(x,t)\rangle &= -2i\int\frac{d\omega}{2\pi}\int\frac{dz}{2\pi u}\frac{\pi uKe^{-\alpha|z|/u}}{(z-\omega-i\delta)(z+\omega+i\delta)} \\ &= \pi K\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-\alpha|z|/u}}{\omega+i\delta}=0\end{aligned} \tag{A.23}$$

Practically spoken, all retarded and advanced Green's functions are zero when they are taken at the same time. For the other retarded Green's function we get:

$$iD^R(t,t') = i\frac{K\pi}{2}\theta(t-t') \tag{A.24}$$

Moreover, the advanced Green's function can be easily obtained from the relation $D^R(t,t') = D^A(t',t)$.
Note that, the retarded Green's function at different times $t \neq 0, t' = 0$ and different positions $x \neq 0, x' = 0$ is:

$$-i\langle\phi_{cl}(x,t)\phi_q(0,0)\rangle = 2D^R(t,x) = \theta(t)\pi K\left(\theta(x+ut)-\theta(x-ut)\right) \tag{A.25}$$

Finally, we are able to evalute the second term of Eq. (A.14)

$$\langle\left[\phi_{cl}(x,t)-\phi_{cl}(x,t')\right]\left[\phi_q(x,t)-\phi_q(x,t')\right]\rangle = -i\pi K(\theta(t-t')+\theta(t'-t)) \tag{A.26}$$

The two θ - functions in equation (A.26), $-i\pi K(\theta(t-t')+\theta(t'-t))$, yield unity in the exponent. In total (A.26) is equal to $i\pi K$.
However the disorder action contains also terms such as:

$$e^{i(\phi_{cl}(x,t)-\phi_{cl}(x,t')+\phi_q(x,t)+\phi_q(x,t'))} \tag{A.27}$$

Eq. (A.27) gives rise to: $-i\pi K(\theta(t-t')-\theta(t'-t))$ in the exponent. The difference in two theta functions yields:

$$\theta(t-t')-\theta(t'-t) = sign(t-t')$$

Exponentials of Correlation Functions, Results

The final result of the four exponentials of correlators is

$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')-\phi_q(x,t)+\phi_q(x,t')]^2\rangle} = \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}e^{-i\pi K}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} \tag{A.28}$$

$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')+\phi_q(x,t)-\phi_q(x,t')]^2\rangle} = \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}e^{i\pi K}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} \tag{A.29}$$

$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')+\phi_q(x,t)+\phi_q(x,t')]^2\rangle} = \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}e^{-i\pi K\cdot sign(t-t')}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} \tag{A.30}$$

$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')-\phi_q(x,t)-\phi_q(x,t')]^2\rangle} = \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}e^{i\pi K\cdot sign(t-t')}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} \tag{A.31}$$

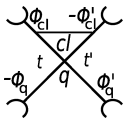
A.3 Diagrammatic representation

A key concept in quantum field theory are Feynman diagrams which allows one to keep track of perturbation series on an intuitive level. Unfortunately, the kind of Feynman diagrams usually encountered in quantum field theory, where lines represent particles and holes which can be connected due to interactions, cannot be employed here. At first this is due to the cosine and sine structure of the fields in the disorder part of the action. They do not provide a simple vertex structure. Nevertheless, a diagrammatic representation can be helpful to make calculations more transparent.

We find it convenient to represent each exponential appearing in the effective disorder action, equation (3.9), by a box. As we have seen in appendix (A.2) the correlation function of the exponentials read:

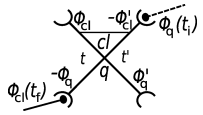
$$e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')\pm\phi_q(x,t)\pm\phi_q(x,t')]^2\rangle_{S_0}} = \frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}\mathcal{N}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} \tag{A.32}$$

where $\mathcal{N}$ is a phase factors: $e^{\pm i\pi K}$ or $e^{\pm i\pi K\cdot sign(t-t')}$.
We remind that $\langle\phi_q\phi_q'\rangle$ and $D^{R/A}(t,t)$ are always zero. Hence there are no lines connecting two ϕ_q-fields or fields at coinciding times:

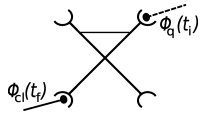
$\langle e^{i(\phi_{cl}-\phi'_{cl}-\phi_q+\phi'_q)}\rangle_{S_0} =$ 

(A.33)

In proceeding from equation (3.12) to (3.13) we exponentiated the fields in order to calculate the correlation functions of fields times exponential of fields. In that way, we have coupled the fields from the exponents of the effective disorder action to the external fields $\phi_{cl}(x_i,t_i)$ and $\phi_q(x_f,t_f)$. The diagrammatic representation takes these couplings into account by a connection of the external lines to one of the four possible slots at the fields from the box:

$$\langle\phi_{cl}\phi_q\rangle\langle\phi_q\phi'_{cl}\rangle\frac{\left(\frac{\pi\alpha}{\beta u}\right)^{2K}e^{-i\pi K}}{|\sinh(\frac{\pi}{\beta}(t-t'))|^{2K}} =$$ 

Sometimes it is more convenient to exclude the factor $e^{-\frac{1}{2}\langle[\phi_{cl}(x,t)-\phi_{cl}(x,t')\pm\phi_q(x,t)\pm\phi_q(x,t')]^2\rangle}$:

$\langle\phi_{cl}(x_f,t_f)\phi_q(x,t)\rangle\langle\phi_q(x_i,t_i)\phi_{cl}(x,t')\rangle =$ 

It is mentioned in the text whether the first or the second convention is used for diagrams.

Note that in the second order of perturbation theory we do not have four but eight fields in the exponent and integrals over four different times. The box is certainly no longer sufficient and we would draw an octagon. However, some of the selection rules and the procedure of connecting the external lines are still present.

Appendix B

Dissipative Action & Corrections to the Drude conductivity

B.1 Exponentials of correlation functions II

In this section we will evaluate expectation values of S_D, see Eq. (3.5) with respect to the dissipative action $\bar{S}$.

Evaluation of the Keldysh Green's function

$$\mathcal{K}(t-t') = -\frac{1}{2}\langle\left[\phi_{cl}(x,t) - \phi_{cl}(x,t')\right]^2\rangle_{\bar{S}} \tag{B.1}$$

Based on the Fourier transformed Keldysh Green's function we can evaluate this correlation function.

$$\mathcal{K}(t-t') = -\frac{1}{2}\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}2\cdot\left[1-\cos[\omega(t-t')]\right]i\cdot 2\bar{D}^K(k,\omega) \tag{B.2}$$

The Keldysh Green's function in equilibrium is:

$$\bar{D}^K(k,\omega) = \coth\left(\frac{\omega\beta}{2}\right)\cdot\left[\bar{D}^R(\omega,k) - \bar{D}^A(\omega,k)\right] \tag{B.3}$$

But now the propagators are evaluated for a disordered system with dissipation:

$$\bar{D}^R(\omega,k) = \frac{\pi uK}{\omega^2 + i\frac{\omega}{\tau} - u^2k^2} \tag{B.4}$$

Substituting (B.1) into (B.1) we obtain:

$$\mathcal{K}(t-t') = -4\pi uK\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\coth\left(\frac{\omega\beta}{2}\right)\left[1-\cos(\omega(t-t'))\right]\frac{\frac{\omega}{\tau}}{(\omega^2-u^2k^2)^2+\frac{\omega^2}{\tau^2}} \tag{B.5}$$

Evaluation of the retarded Green's function

For the sake of simplicity we consider the retarded Green's function at different times:

$$\bar{D}^R(t,t') = -\frac{i}{2}\langle\phi_{cl}(x,t)\phi_q(x,t')\rangle_{\bar{S}} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{-i\omega(t-t')}\bar{D}^R(\omega,k) \tag{B.6}$$

We insert the expression for $\bar{D}^R(\omega,k)$:

$$\bar{D}^R(t,t') = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\pi uK}{\omega^2 + i\frac{\omega}{\tau} - u^2k^2}e^{-i\omega(t-t')} \tag{B.7}$$

The poles of the integrand have negative imaginary parts:

$$\omega_{1,2} = -\frac{i}{2\tau} \pm \frac{1}{2}\sqrt{4u^2k^2 - \frac{1}{\tau^2}} \tag{B.8}$$

If $t' > t$ then $\bar{D}^R = 0$ as expected. Thus we have to close the contour in the lower half-plane. This introduces a condition $\mathcal{I}m[\omega] < 0$ which means that $t - t' > 0$.

$$\bar{D}^R(t,t') = iuK\theta(t-t')e^{-\frac{t-t'}{2\tau}}\cdot\int_{-\infty}^{\infty}dk\frac{\sin\left(\frac{1}{2}\sqrt{4u^2k^2 - \frac{1}{\tau^2}}\cdot(t-t')\right)}{\sqrt{4u^2k^2 - \frac{1}{\tau}}} \tag{B.9}$$

We introduce the following notation:

$$\begin{aligned} a &:= \tfrac{t-t'}{2\tau} > 0 \\ x &:= uk\cdot(t-t') \end{aligned} \tag{B.10}$$

$$\bar{D}^R(t,t') = iK\theta(t-t')e^{-\frac{t-t'}{2\tau}}\cdot\int_0^{\infty}dx\frac{\sin(\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}} \tag{B.11}$$

The calculation of the time representation of the retarded Green's function is reduced to the integral in (B.11)

$$\int_0^{\infty}dx\frac{\sin(\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}} = \int_0^a\frac{\sinh(\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}dx + \int_a^{\infty}\frac{\sin(\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}}dx$$

We substitute $y'^2 = a^2 - x^2$ and $y^2 = x^2 - a^2$:

$$= \int_0^a\frac{\sinh(y')}{\sqrt{a^2-y'^2}}dy' + \int_a^{\infty}\frac{\sin(y)}{\sqrt{a^2+y^2}}dy = \frac{\pi}{2}I_0(a) \tag{B.12}$$

see [20]. Thus, we have found the retarded Green's function in the time representation:

$$\bar{D}^R(t,t') = i\frac{\pi K}{2}\theta(t-t')e^{-\frac{t-t'}{2\tau}}I_0\left(\frac{t-t'}{2\tau}\right) \tag{B.13}$$

In the limit of vanishing dissipation: $\mathcal{I}m\Sigma = \frac{1}{\tau} \to 0$, equation (B.13) is equal to the retarded Green's function of the clean system.

Moreover, in Eq. (B.13) one can see that the correlation vanishes for large time differences $t - t' \gg \tau$ due to the finite scattering rate $\frac{1}{\tau}$.

Evaluation of the retarded Green's function at coinciding times

$$\bar{D}^R(t,t') = -\frac{i}{2}\langle\phi_{cl}(x,t)\phi_q(x,t')\rangle_{\bar{S}} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\bar{D}^R(\omega,k) \tag{B.14}$$

The poles $\omega_{1,2} = -\frac{i}{2\tau} \pm \sqrt{u^2k^2 - \frac{1}{2\tau^2}}$ are always in the lower complex ω half-plane. One can close the ω-integral in the upper half-plane and prove that the whole expression is zero. Note that, if we do the k-integral at first we will have to include a convergence factor $e^{-\alpha k}$. Consequently, we will treat the retarded and advanced Green's function as being zero at coinciding times.

B.2 Evaluation of the β/τ correction $\Delta\sigma^{\tau,1}$

We calculate corrections in β/τ which we obtained in Eq. (4.82), section (4.2).

$$\Delta\sigma^{\tau,1} := -s(\omega,\tau,K)\cdot K\cdot\frac{\beta}{\tau}\cos(\pi K)\frac{\pi}{2}i\left(\frac{\pi\alpha}{\beta u}\right)^{2K-2}\frac{1}{\beta}\int_0^{\infty}\frac{1-e^{i\omega\beta t}}{\sinh^{2K}(\pi t)}\cdot t\mathrm{d}t \tag{B.15}$$

where $s(\omega,\tau,K) := \frac{2e^2\omega}{\hbar\pi^2\tau u}\left[\bar{D}^R(\omega,k=0)\right]^2$. Let us calculate the integral in Eq. (B.15).

$$\mathcal{I}^2(\omega\beta,K) := \int_0^{\infty}dt\frac{1-e^{i\omega\beta t}}{\sinh^{2K}(\pi t)}\cdot t \tag{B.16}$$

Eq. (B.16) is regular for $t=0$ and $K<1$. Moreover, the integrand is exponentially small for large t and $0<K$.
Eq. (B.16) can be evaluated in a similar way as in the calculation of the first order correction to the clean conductivity (3.34) - (3.37).
In section (3.2) we obtained Eq. (3.37) from an analytic continuation ($b\to 0$) of the following integral [20]:

$$\int_0^{\infty}\left(1-e^{iax}\right)\sinh^{\nu}(\pi x)e^{-bx}\mathrm{d}x = f(\nu,0,b) - f(\nu,a,b) \tag{B.17}$$

where

$$f(\nu,a,b) = \frac{\Gamma(\nu+1)}{2^{\nu+1}\pi}\cdot\frac{\Gamma\left[\frac{b-\nu\pi-ia}{2\pi}\right]}{\Gamma\left[\frac{b+\nu\pi-ia}{2\pi}+1\right]} \tag{B.18}$$

Based on Eq. (B.17), we are going to derive the result of the integration in Eq. (B.16). We take the derivative with respect to b in Eq. (B.17):

$$\left(-\frac{\partial}{\partial b}\right)\int_0^{\infty}\left(1-e^{iax}\right)\sinh^{\nu}(\pi x)e^{-bx}dx = \int_0^{\infty}x\cdot\left(1-e^{iax}\right)\sinh^{\nu}(\pi x)e^{-bx}dx \tag{B.19}$$

This is equal to $\mathcal{I}^{(2)}$ for $a=\omega\beta/2\pi$, $\nu=-2K$ and $b=0$. Thus, we obtain the following expression after taking the derivative of Eq.(B.18) with respect to b:

$$\begin{aligned} F(\nu,a,b) &:= \left(-\frac{\partial}{\partial b}\right)f(\nu,a,b) \\ &= \frac{\Gamma(\nu+1)}{2^{\nu+2}\pi^2}\frac{\Gamma\left(\frac{b-\nu\pi-ia}{2\pi}\right)\psi\left(\frac{b-\nu\pi-ia}{2\pi}\right)-\Gamma\left(\frac{b-\nu\pi-ia}{2\pi}\right)\psi\left(\frac{b+\nu\pi-ia}{2\pi}+1\right)}{\Gamma\left(\frac{b+\nu\pi-ia}{2\pi}+1\right)} \end{aligned} \tag{B.20}$$

We obtain the result of Eq. (B.16) by taking the derivative of the right hand side of Eq. (B.17) with respect to b and analytically continue it: $b \to 0$. The analytic continuation was checked numerically.
Thus we obtain:

$$\mathcal{I}^2(\omega\beta, K) = F(\epsilon, \omega\beta) - F(\epsilon, 0) \tag{B.21}$$

where

$$F(\epsilon,\omega) := \frac{\Gamma(2\epsilon-1)}{2^{2\epsilon}\pi^2}\frac{\Gamma\left[1-\epsilon-i\frac{\omega\beta}{2\pi}\right]}{\Gamma\left[\epsilon-i\frac{\omega\beta}{2\pi}\right]}\left(\psi\left(1-\epsilon-i\frac{\omega\beta}{2\pi}\right)-\psi\left(\epsilon-i\frac{\omega\beta}{2\pi}\right)\right) \tag{B.22}$$

We expand $\mathcal{I}^{(2)}$ in $\epsilon = 1 - K \ll 1$ and $\omega\beta \ll 1$:

$$\mathcal{I}^2(\omega\beta, 1-\epsilon) = \frac{i}{3} - \frac{i}{3}\left[(2\ln(2)-2) + \frac{6}{\pi^2}\zeta(3)\right]\cdot\epsilon\omega\beta + \mathcal{O}\left(\epsilon\cdot(\omega\beta)^2, (\omega\beta)^2\epsilon\right) \tag{B.23}$$

where $\zeta(3) = 1.202$.
We insert the expansion, Eq. (B.23), into Eq.(B.15) and obtain[1]:

$$\Delta\sigma^{\tau,1} = \frac{e^2}{\hbar\tau}\frac{u}{\left(\omega+\frac{i}{\tau}\right)^2}\cdot\left[-\frac{\pi}{3}\frac{1}{\omega\tau} + c\cdot\epsilon\frac{\beta}{\tau} + \mathcal{O}\left(\frac{\epsilon(\omega\beta)^2}{\tau}, \frac{\epsilon^2\omega\beta}{\tau}\right)\right] \tag{B.24}$$

where $c = \frac{\pi}{3}(2\ln(2)-2) + \frac{6}{\pi}\zeta(3)$, $\zeta(3) = 1.202$.

[1]Note that the factor $(\beta u/\pi\alpha)^{2\epsilon}$ is absorbed into the bare disorder strength $D_b = u/2\tau$ of the disorder action S_D. All other τ-factors are renormalized. The renormalization of τ is given by Eq.(2.71).

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Mit der Abgabe der Masterarbeit versichere ich, dass ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.