Bose Fermi Kondo Model

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Motivation

Over the last decades impurity problems have drawn a lot of attention and theoretical methods have been developed in order to solve those problems. One of these impurity models is the Kondo model, where the electrons of the conduction band interact with a only spin dependent magnetic impurity. The low energy physics is governed by the quenching of the spin degenerated impurity through the conduction band electrons. There exist numerous experiments concerning the Kondo model, in most cases well understood on the theoretical side .
The Bose-Fermi Kondo model (BFKM), which will be discussed in great detail in this thesis, is a rather recent impurity model, but nevertheless, it can be considered as a sort of generalisation of the ordinary Kondo model, as it additionally couples a bosonic dissipative bath to the impurity. This new type of bath tries, on the one side, to decohere the spin, on the other side the fermionic bath, known from the Kondo model, tries to quench the spin, resulting in a competition of the two different baths depending on the coupling strength of the respective baths to the impurity spin. This competition gives rise to a phase transition between the Kondo phase, where the Kondo coupling dominates, and the Bose phase, with the coupling of the spin to the bosonic bath prevailing. On the contrary to the Kondo model, a lot of experimental outcomes of the BFKM are not understood by theoretical means. The additional complexity due to bosonic bath makes it very difficult to seize the full problem.
However, as an extension to the Kondo model, the Bose-Fermi Kondo model has another origin, which comes from the mapping of a translation invariant lattice model, the so called Kondo lattice, on an impurity model, via extended dynamical mean field theory (EDMFT). It is widely believed that these Kondo lattices describe certain types of heavy fermion metals. Assuming that the translation invariant lattice of our concern consists of an on-site as well as an inter-site interaction of the lattice impurities such a mapping will result into the Bose-Fermi-Kondo model, describing the local physics. The EDMFT is a very well established theory and one can think of it as a kind of extended quantum version to the usual mean field theory.
The impurity model may be solved easier, where one has reliable analytic methods such as poor man's scaling, renormalisation groups and, a rather recent one, the flow equation method, and via the mapping, information of the lattice model can be gained. The usefulness of the aforementioned connection lies in its ability to be a possible description of a quantum phase transition of such a Kondo lattice, as in the very same model, a com-

petition between the on-site interaction and the inter-site interaction is present. Such a quantum phase transition takes place at zero temperature, hence it can not be driven by thermal fluctuations.
The first attempts to understand the quantum phase transition have been made by using Hertz-Millis theory. This theory considers only long wavelength fluctuations and do not account for any local fluctuations created by the on-side interactions. It turned out that the Hertz-Millis theory is indeed an adequate description for weakly coupled systems, however it fails to produce the right experimental data when strongly coupled systems are considered. In the case of strong coupling the most promising candidate is the EDMFT, which also accounts for local fluctuation, mainly displayed by the local theory, the BFKM. Those local fluctuations give rise to a breakdown of Fermi liquid theory at the quantum critical point.
The strong coupling nature of the model requires methods that go beyond perturbation theory, such as NRG on the numerical side and the flow equation method on the analytical side. Actually, there are two types of BFKM one can consider, on the one hand the Ising type model, the spin couples to the bosonic bath in a specific direction, and on the other hand the isotropic one, the spin couples to the bosonic baths equally in all three directions. The Ising BFKM is accessible through numerical methods and we will also study this model, but more as a check of the capability of the flow equation method concerning its applicability on the model. After doing so we proceed on to the isotropic case, which attracts our attention, as no numerical solution is present, due to the three emerging bosonic baths.
We will reproduce the already known results from RG-methods on the approximate ω-dependence of the spin correlation function, and furthermore, we obtain a full solution of the general spin correlation function, by solving the gained flow equations, which has not been done so far. That means our calculations are not restricted to consider the zero temperature case only and therefore we can check an assumed correspondence of the BFKM at the critical point with a conformal field theory. This is so important, because the assumption has been made in order to give an expression for the temperature dependent dynamical spin susceptibility, since the conformal field theory has been the only way to give an expression at all. So far there is no justification of such an correspondence in terms of microscopical physics. However, as the Kondo lattice is concerned the very same local dynamical spin susceptibility is needed in order to solve the self-consitency equations and thus make statements on the nature of the quantum phase transition. Obviously, one would not like to base such important statements on mere unproven assumptions.

Outline

In chapter 1 an introduction to the different types of quantum phase transition as well as the two main theoretical descriptions the Hertz-Millis theory and the EDMFT, is provided.

Moreover, the known results of the local isotropic BFKM are presented.

The flow equations for the Ising BFKM are derived in chapter 2. The flow of the couplings are the same as in the Kondo model, except the fact that the perpendicular coupling is shifted by a constant, given by the dissipative strength of the bosonic bath.

Chapter 3 presents the differential equations of the couplings of the isotropic BFKM. We reproduce the ω-dependence of the spin correlation function at zero temperature,using the flow equation method, further on we calculate the non-zero temperature dependence of the dynamical spin susceptibility and compare it to a particular correlation function of a conformal field theory, since there has been recent conjecture, claiming a connection between the BFKM at the quantum critical point and a certain conformal field theory.

The last chapter provides an outlook for future work. It includes a schematical guideline how to get an expression for the $\mathcal{T}$-matrix of the isotropic BFKM and a consideration of the time evolution of the spin operator. By solving the Heisenberg equation of motions with the help of the flow equation method, in order to avoid secular terms.

Chapter 1

Introduction

1.1 The Models

We consider two models, one is the Kondo model the other one, the Bose Fermi Kondo model, which can be regarded as an extention of the first one. Both of them are worthy to be studied for their own sake, but our main attention is drawn to the later one. Nevertheless, it is possible to deduce them via DMFT (dynamical mean field theory) respectively EDMFT (extended dynamical mean field theory) from a lattice model the so called Kondo lattice. Historically, the Kondo model was regarded as an impurity system to model magnetic behaviour of metals. In the next section will present its relation to the Anderson impurity model. The Bose Fermi Kondo model, though shows some similarities to the conventional Kondo model, it was originally deduced via EDMFT. In the section on the Bose Fermi Kondo model we will give a kind of sketch how EDMFT works.

1.1.1 The Kondo Model

The s-d model originally introduced by Zener [1] (also known as Kondo model because of Kondo who made major advantages in understanding the problem [2]) has been of great interest over the last decades. In addition to DMFT one can also deduce the Kondo model by a so called Schrieffer Wolff [3] transformation from the Anderson model. This canonical transformation also connects the respective coupling constants of the two different models and hence explains the antiferromagnetic dominating contribution. The most general form of a single site Anderson model is given by

$$\begin{aligned} H_A &= \sum_{\sigma} \epsilon_d n_{d\sigma} + U n_{d\uparrow} n_{d\downarrow} \\ &\quad + \sum_{k\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{k\sigma} (V_k c^{\dagger}_{d\sigma} c_{k\sigma} + V^*_k c^{\dagger}_{k\sigma} c_{d\sigma}), \end{aligned} \tag{1.1}$$

where d stands for the impurity level more presicely the orbital, which can at maximum contain two electrons with opposite spin. V_k and V_k^* denote the interaction between the impurity and conduction electron and U the on site interaction. After applying the Schrieffer

Wolff transformation, the Anderson model is mapped onto the Kondo model

$$H_K = \sum_{p\sigma} \epsilon_p c^\dagger_{p\sigma} c_{p\sigma} + \sum_{pq} J(p,q)\mathbf{s}_{p,q} \cdot \mathbf{S} \tag{1.2}$$

with $\mathbf{s}_{p,q} = \sum_{\alpha\beta} c^\dagger_{p\alpha} \frac{\sigma_{\alpha\beta}}{2} c_{q\beta}$ and σ denotes the Pauli matrizes, in which the conduction spin operator is expressed in terms of fermionic creation and annihilation operators respectively. The connection between the two models is reflected by the relation of the coupling constants

$$J(p,q) = V^*_p V_q \left(\frac{1}{U + \epsilon_d - \epsilon_q} + \frac{1}{(\epsilon_p - \epsilon_d)} \right). \tag{1.3}$$

The Kondo model can be viewed as an impuritity model, the electrons of the conduction band scatter at a spin dependend impurity on the contrary to the Anderson model the charge fluctuations on the impurity orbital are frozen. We take the impurity to have a Spin $\frac{1}{2}$ dependence. The great intrest of studying such impurity models is to model magnetic properties of metals. Experimentally, it was a long known fact that the usual description of the electrical resistivity $R(T)$ in terms of phonon-electron interaction does not hold in impurity metals. It was observed that the resistivity has got a minimum at a certain temperature value instead of a monotonic decreasing behaviour as the temperature decreases. Phonon-electron interaction alone would not cause such a dependence. Kondo could explain the minimum of the electrical resistivety by means of third order perurbation theory in the coupling constant J, which gives raise to a logarithmic dependence $\ln(T)$, fitting the experimental data quite well. So the resistivity is of the following form

$$R(T) = aT^5 + c_{imp}R_0 - c_{imp}R_1 ln(\frac{k_B T}{D}), \tag{1.4}$$

D is the band width of the conduction electrons, c_{imp} the impurity concentration and a, R_1 and R_0 are some material constants. Although Kondo could show the right behaviour, the result can not be applicable over the full temperature range. As $T \to 0$ the logarithmus will diverge and the perturbation thoery breaks down. Finding a solution also valid in the low temperature region is the famous Kondo problem. Actually, if one rewrites the spin operator in a SU(2) Eqs.(1.2) invariant form it is possible to apply field theoretical methods to the fermionic operators. The result was a devergence of e.g. the impurity susceptibility at non zero temperature, given by

$$\chi_{imp}(T) \sim \left(1 - \frac{2J\rho_0}{1 + 2J\rho_0 ln(\frac{k_B T}{D})} \right), \tag{1.5}$$

hence this expression in terms of perturbation theory is just valid down to

$$k_B T_K \sim D e^{-\frac{1}{2}J\rho_0}, \tag{1.6}$$

T_K marks this non-zero temperature, also known as the Kondo temperature. Notice, the divergence only occures in the antiferromagnetic case $J > 0$. Many attempts have been

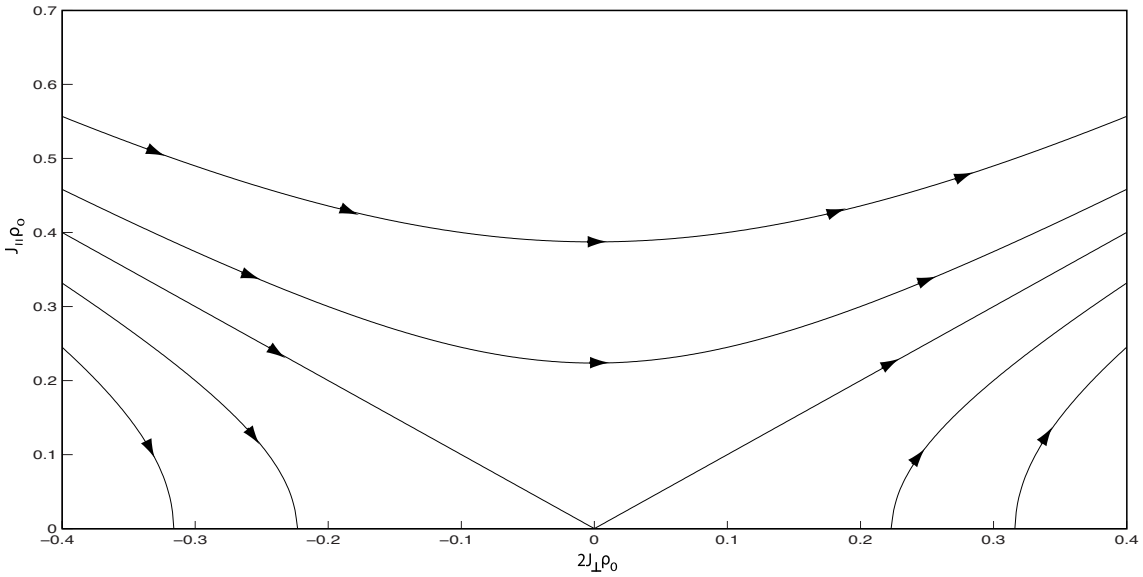

Figure 1.1: The flow of the coupling constants in the poor man's scaling approach, taken from [3]

made to solve that problem, one of the most interesting is that of Anderson. Within a series of papers [4, 5, 6], Anderson *et. al.* have developed a new method called poor man's scaling. The method can be regarded as direct antecessor of Wilson's renormalisation group theory. They applied their ideas to the Ising type model

$$H = \sum_{pq} J_+ S^+ c^\dagger_{p\downarrow} c_{q\uparrow} + J_- S^+ c^\dagger_{p\uparrow} c_{q\downarrow} + J_z S^z (c^\dagger_{p\uparrow} c_{q\uparrow} - c^\dagger_{p\downarrow} c_{q\downarrow}), \tag{1.7}$$

which can be easily derived from the isotropic Kondo model. J_+ and J_- generate the spin flip. From now on we set them equal $J_+ = J_- = J_\perp$. The poor man's scaling follows the philosophy that the high energy excitations or rather the states around the upper and lower band edge are cut off, at the same time the band width D gets reduced by a small amount $\pm|\delta D|$ and one is left with states in the range of $0 < |\epsilon_p| < D - |\delta D|$. Moreover, one has to demand form invariance of the new Hamiltonian in comparison to the old one. It is obvious that the form invariance can solely be achieved by rescaling the coupling constants and thus the new couplings differ from the old ones by depending on the reduced band width $|\delta D|$. In other words the couplings can be rewritten as

$$J_\perp \to J_\perp + \delta J_\perp \quad J_z \to J_z + \delta J_z, \tag{1.8}$$

where both $\delta J_\perp$ and δJ_z depend on δD. By considering the physics close to the Fermi surface we obtain to differential equations

$$\frac{dJ_\perp}{dlnD} = -2\rho_0 J_z J_\perp \quad \frac{dJ_z}{dlnD} = -2\rho_0 J_\perp^2, \tag{1.9}$$

ρ_0 is the constant density of states. After dividing and integration one ends up with

$$\boxed{J_z^2 - J_\perp^2 = const}. \tag{1.10}$$

In Fig.1.1 the divergence of the coupling constant in the antiferromagnetic reginon is shown. Therefore the interaction between the magnetic impurity and the electrons of the conduction band becomes arbitary high, eventually a conduction electron will be trapped by the impurity. A bound state is formed giving rise to an increase in the density of states in the vicinity of the impurity (Kondo effect). A lot of further efforts have been put in to solve this problem by using methods like Fermi liquid [3] and renormalisation group [7]. It was Andrei (1980) [8] and Wiegman(1980) [9] by applying the Bethe ansatz [10] who solved the model in great detail, at least in one dimension. In fact, Andrei and Wiegmann were not the first who could give a solution to the low temperature physics, it was Wilson [7] with his numerical renormalisation group, but their main achievment was the derivation of an analytic solution. For a general introduction to Kondo physics as such, which encoperates almost all approaches, the reader is adviced to [3].

1.2 Bose Fermi Kondo Model

1.2.1 Motivation

A Quantum phase transition [11] occures e.g. in heavy fermion metals at zero temperature, inspite of the classical phase transition at non-zero temperature. Although, it is not experimentally accessible, the behaviour of $T = 0$ temperature systems are of great interest, since some of them, namely the heavy electron systems, show so called quantum critically. Generally, quantum criticality provides a mechanism for the breakdown of Fermi-liquids, which emerges in high temperature superconductor and heavy fermion metals. Tracing down the temperature scale within the critical region one eventually ends up in a quantum critical point, which seperates to different ground states of the system. There, one expects non-Fermi-liquid behaviour, best detected through transport and thermodynamic measurements near the QCP. A lot of materials exhibit magnetic quantum critical point (QCP). Such a QCP also effects the non zero temperature of the sample, leading to a so called quantum critical matter. Within this new type of phase certain unexpected properties of measurable quantities arise, which help to classify the universality class of the QCP. The quantum critical point is expected to go along with a second order phase transition (continuous phase transition). In classical system the thermal fluctutions, which are the driving forces behind a phase transition, frezze out at zero temperature, hence there is no possibility for any kind of classical phase transition. However, a quantum system is principally able to undergo a phase transition, due to the emergence of a novel force that come into the game, the uncertainty principle. Actually, at zero temperature no kinetic energy would be present so the momentum of a specific paritcle would be known and in addition its position. The uncertian principle causes so called quantum fluctuations in order to anticipate that. These quantum fluctuations, if sufficiently strong, give rise to

the possibility of a quantum phase transition. A QPT is governed by the variation of a non thermal parameter, for instance pressure or magnetic field. The system has different ground states depending on the phase it has occupied. Besides, that it is not possible to observe such a quantum phase transition at least directly, it is very well justified to study such systems, since it has also an effect on the non-zero temperature physics. Observable quantities such as correlation function or electrical resistivity behave differentely in the quantum critical regime, as they would within any specific phase.

In the last few years, there have been lots of experiments with intreresting results in heavy fermion physics as well as in high temperature superconductivity [1], for instance $La_{2-x}CuO_4$ which is a insulator for a x less than 0.05 and high temperature superconductivity is found for x greater than 0.05, where x denotes density of holes relative to the insulation state with one electron per site [12]. Inelastic neutron scattering [13, 14] on $CeCu_{6-x}Au_x$ that undergoes a transition from a paramagnetic metal to an antiferromagnetic metal by changing the concentration x, was one of the major key experiments. At the critical value $x_c \approx 0.1$ the phase transition occurs and one expects a QCP. This can also be performed by varying external parameters such as pressure e.g. $CeCu_5Au$ [15] or magnetic field in order to alter competing coupling constants [16].

The following results will be the hallmarks on testing the validity of a quantum critical theory and mainly come from inelastic neutron scattering on $CeCu_{6-x}Au_x$;

1. Fractional exponent α of the frequency dependent dynamical spin susceptibility as well as the temperature dependent dynamical spin susceptibility.
2. The susceptibility exhibits a $\frac{\omega}{T}$ scaling.
3. The fractional exponent does not occur only at the ordering wavevector Q, but over the entire Brillouin zone.

Probably the most interesting novel feature, which was detected, is the non-Fermi liquid behaviour near the quantum critical point, where new kinds of excitations are expected to appear. Roughly speaking, on the theoretical side there are two major ideas, an extention of the standard second order phase transition introduced by Hertz and the emergent of new critical exitations creating a completely new kind of criticality. The two major theoretical ideas are the well known Hertz-Millis theory [17, 18] or the novel theory named local quantum criticality [19, 20]. The two theories produce different results on the value of e.g. susceptibility, hence the above stated experimental criterions decide the validity of the proposed theories[2].

[1] Although the second one leads to new quantum phase such as unconventional superconductivity which will have most likely practical applications, once the quantum criticality is fully undersand, we are mainly interested heavy fermion physics as it occurs in the Bose-Fermi-Kondo model.

[2] There are experiments on quantum critical metals that produce different results, which can be described by the Hertz-Millis theory, only. However, in this thesis we are interested in the local typ of quantum criticality, so we exclusively deal with this one.

Historically, it was believed that only the long wavelength fluctuations, namely the paramagnons, are considered to contribute to the critical modes, driving the quantum phase transition in the low-energy regime. In this picture the low-energy excitations are represented by the paramagnons. There is an order parameter characterising the crossover between the antiferromagnetic phase specified by a staggered magnetic field and a paramagnetic phase, both phases are metallic in their nature. In order to give an adequate description one has to extend the classical Φ^4-theory by including quantum fluctuations, but experiments forced a new kinds of idea, since the extended Φ^4-theory or Hertz-Milles theory can not produce the correct results, as it is shown in the next section. The local quantum criticality was, among others, one of these new ideas and at this stage is the most promising one. To sum up, the Hertz-Millis works in weak coupling systems sufficiently well, however, in strong coupling systems it fails.

In this thesis we just discuss the local criticallity as a promising candidate to explain the new critical phenomena.

1.2.2 Kondo Lattice

A Kondo lattice is a heavy fermion system built up of strongly correlated f-electrons and conduction band electrons. These f-electrons usually originated in rare-earth metals have a on-site Coulomb repulsion much stronger then the kinitic energies that is why they form a so called localised magnetic moments. The energy scale for the onset of this formation is in most cases given by the room temperature, where below the formation takes place and therefore the Kondo lattice can be regarded as a appropriate model to describe heavy fermion physics, but only below this termperature threshold.

Generally, it is assumed that the quantum magnetic phase transition is driven by the competition between the Kondo physics and the RKKY [3] interaction, which are expressed in the following Hamiltionian

$$\boxed{H = \sum_{ij\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + \sum_{i} J_K \mathbf{S}_i \cdot \mathbf{s}_{c,i} + \sum_{ij} \frac{I_{ij}}{2} \mathbf{S}_i \cdot \mathbf{S}_j}, \tag{1.11}$$

J_K is the Kondo coupling, governing the interaction strength between the local moments and the free conduction electrons, I_{ij} is the coupling between the local moments. The RKKY interaction is, instead of the local on site Kondo interaction, a purely non-local intersite interaction and the last parameter, the tight-binding determines the kinetic energy, namely the dispersion relation ϵ_k. It can be said that the dimensionality of the lattice and other features of the lattice, e.g. the kind of lattice (e.g.square lattice) are contained in the specific value of t_{ij}. $\mathbf{S}_i$ are the local spin operators sitting on site i, for this work the local

[3] RKKY stands for Ruderman-Kittel-Kasuya-Yoshida.

moment spin is exclusively taken to be $\frac{1}{2}$, though the above Hamiltionian is quite generic, viz the spin operator could in principal take higher values (e.g. $\frac{3}{2}$)and $s_{c,i}$ are the spins of the conduction electrons. The two coupling parameters determine the only relevant energy scales setting the problem, with T_K as the Kondo temperature that has to be fallen below in order to obtain a Kondo screening. The conduction band density of states is given by

$$\rho_0(\epsilon) = \sum_p \delta(\epsilon - \epsilon_p), \tag{1.12}$$

whereas the 'RKKY density of states' is

$$\rho_I(\epsilon) = \sum_k \delta(\epsilon - I_q), \tag{1.13}$$

for further analysis this density of states will mainly govern the physical characteristics of the lattice system. To ensure the existence of a stable paramagnetic solution, at least in a finite dimensional system, the Fourier-transformed I_q of I_{ij} is suppose to be non-zero solely over a finite region.

By merely considering the ordinary Kondo lattice there is no coupling between the local moments just a coupling between the local moment and the conduction electrons via the Kondo-coupling. Later, an additional interaction was thought to be important, namely the RKKY-interaction of the local moments. Donaich [21] could show that in a simplyfied model [4] a second order quantum phase transition occures. On the one site the quantum mechanical ground state is given by an insulating paramagnet (the Kondo coupling dominates over the RKKY-couling) on the other site an insulating antiferromganet (the RKKY-coupling is dominating). Nowadays, Donaich's system is considerd to by too easy in order to describe a real system, the fact of a quantum phase transition survived and has become a subject of great physical research. Before the emerge of Quantum criticality as a completely new type fo phase transition, one usualy explained the Kondo lattice as a strongly correlated system in terms of Landau quasi-paricles a so called Fermi-liquid. In the Kondo lattice the formation of the Kondo singlet leads to a Kondo resonance [5] that creates quasi-particles, therefore one can explain the system in terms of these quasi-particles, if their excitations are not too far away from the Fermi-surface and the localised f-electrons become delocalised contributing to the Fermi-volume. It is easier to think of this Kondo resconance in terms of a single impurity system rather than considering the whole lattice, because the important physics can be grasp by this picture and basically each local moment of the lattice contribute to the Kondo resonace.
The next two sections discuss the theoretical approaches, however, the Herts-Millis theory is not of direct concern in this thesis, it has to be mentioned as it serves a very natural understanding of the quantum criticality.

[4] There a 1-D system, where the conduction electrons are replaced by a lattice of coupled -localised spins, is regarded.

[5] The f-electron density of states displays a peak in the vicinity of the Fermi-energy.

1.2.3 Hertz-Millis Theory

Before considering the Hertz-Millis theory a few statements have to be pointed out to bring some basic features of statistical and quantum mechanics back to mind. This follows closely the review of Sondhi *et.al.*[22]. The partition function for a given Hamiltonian H reads

$$Z(\beta) = Sp(e^{-H\beta}) = \sum_j \langle j|e^{-H\beta}|j\rangle, \tag{1.14}$$

from which one can extract all physical quantities of interest. It is kind of obvious to see that the operator $e^{-\beta H}$ is the same as the time evolution $e^{-\frac{iHt}{\hbar}}$, if one identifies $\beta = \frac{1}{k_b T} = \frac{it}{\hbar}$. This rewriting of the partition function leads to the following depiction

$$Z(\beta) = \sum_j \langle j|e^{-\beta H}|j\rangle = \sum_j \langle j|e^{-\frac{iHt}{\hbar}}|j\rangle, \tag{1.15}$$

the part after the second equation sign can be interpreted as the transition probability after an imaginary time step. The crucial point is to realise that it is actually the same, if one considers the thermodynamical behaviour at a certain temperature and the transition amplitudes of a quantum mechanical system after its time evolution, where the magnitude of the time interval is governed by the temperature value. In analogy to the derivation of the path integral formulation it is convinient to consider N infinitesimal time steps $\delta\tau$, where τ denotes the imaginary time it, such that $N\tau = \hbar\beta$. Formally, one obtains

$$e^{\beta H} = e^{-(\frac{1}{\hbar})N\delta\tau H} = [e^{-(\frac{1}{\hbar})\delta\tau H}]^N, \tag{1.16}$$

the second step is just valid up to $(\delta\tau)^2$. Eqs.(1.14) involves already a sum (integral) over d space dimensions, by inserting the sequence into Eqs.(1.14), one can interpret the action of time as a further spatial dimension

$$\begin{aligned} Z(\beta) &= \sum_j \langle j|[e^{-(\frac{1}{\hbar})\delta\tau H}]^N|j\rangle \\ &= \sum_j \sum_{j_1, j_2 \cdots j_N} \langle j|e^{-(\frac{1}{\hbar})\delta\tau H}|j_1\rangle\langle j_1|\cdots|j_N\rangle\langle j_N|e^{-(\frac{1}{\hbar})\delta\tau H}|j\rangle. \end{aligned} \tag{1.17}$$

This can be understood in terms of a transfer matrix, the imaginary time can be imagined as an additional spatial dimension, ending up with effectively $d+1$ dimension, d represents the proper spatial dimensions. One has to be tentative, because of the finiteness of the new dimension, since time is restricted by the temperature through $\hbar\beta$. Alternatively, this can also be seen in terms of a functional integral, which exhibits a closer relation to the Ginzburg Landau theory

$$Z(\beta) = \sum_{\text{space-time}} e^{-S[\phi]}, \tag{1.18}$$

where the action is

$$S[\phi] = \int_0^{\beta\hbar} d\tau \int_\infty^\infty d^d x L[\phi(x,\tau)]. \tag{1.19}$$

By considering the system at $T = 0$ the extra dimension extents to infinity and can be treated equally to the space dimensions, leading to the conclusion that a d dimensional quantum system is so to say equal to a $d + 1$ dimensional classical system. In extension of the classical not only static fluctuations but also dynamical fluctuations contribute likewise, hence one assumes that these fluctuations of the order parameter are both in space and (imaginary) time. Generally, the number of extra dimensions is given by the dynamical exponent z and can of course deviate from one. Of course it is possible to recover classical dynamics form the high temperature limit, by taking this limit the imaginary time interval can then be choosen arbitrary small. So, for example if the time intervall is shorter than the system inherent frequence, and the typical time histories solely consist of static configurations without any change if different time slices are regarded. What happens is, the dynamics drops out and one is left over with the bare Boltzmann wight only.

In classical phase transition the Landau theory tells us that a second order phase transiton takes place if some order parameter [6] of the system changes abruptly its value. The order parameter fluctuation can generaly be expressed in terms of fields, leading to a ϕ^4 field theory. In terms of physics, the spatial fluctuations of the order parameter characterises the critical fluctuations, their size is usually denoted by the correlation length ξ and diverge as the critical point is reached. The Hertz-Millis theory comprises the QCP in terms of critical fluctuations of the magnetic order parameter, which are just long wavelength in nature. The generated paramagnons were believed to take account of the non- Fermi-liquid behaviour modifying the quantum critical physics. Despite the classical order parameter, which varies merely in space the additional quantum effects provoke a time dependence[7] of the order parameter (the paramagnons), as it was motivated above. Electronic excitations, due to the Kondo resonance, are totally left out, only the long wavelength contribute to the critical modes, which means that at the QCP scattering of the singular electrons by the paramagnons dominate. However, no importance is attributed to the process of Kondo screening. The main effect of the present electrons is, to give rise to extra decay channels for the magnetic fluctuations, leading to a damping (electron hole decay).

The ϕ^4 term in the quantum action represents the non-linear coupling of the paramagnons, additonally, the critical theory has an effective dimension of $d_{eff} = d + z$, z is the dynamical exponent and reflects the critical fluctuation in imaginary time direction. Actually, z displays the generalisation of the QCP concept. The order parameter is typically described in terms of spin-density-waves (SDW), a spontaneous spatial modulation of the spins of the charge carrier (here electrons). The spin density wave refers to the notion of paramagnons, the quantiezed version of waves, their critical behaviour governes the QCP. Despite all the similarities between the classical and quantum mechanical describtion of the criticality there are also so called non matching aspects, like scaling. Classically, a phase transition is

[6] For simplicity one can think of a magnetic field as the order paramter such as it is used, for instance to seperate a pramagnetic phase with zero magnetisation and an antiferromagnetic phase where the order parameter takes a non-zero value.

[7] Actually it is a imaginary time dependence.

given by the non-analyticity of the free energy at the critical point entailing critical scaling behaviour of specific thermodynamical quantities, for instance the magnetic suszeptibility. However quantum mechanically, one, generally, needs two order parameter instead of one, as it is in classical theory, in order to depict the transition, one is a non-thermal parameter here denoted by δ the other one is the temperature T. Varying both parameters individualy when they are close to the critical point different physics occurs, for example taking the limit $\delta \longrightarrow \delta_c$ at $T = 0$ the correlation length is going to increase untill its divergence on the other hand if $\delta = \delta_c$ and reducing the temperature the size of the time droplets are going to increase as $\frac{1}{k_B T}$, remember the previously made identification of time with the inverse temperature T.

In our case of antiferromagnetic metals the value of z is equal to two and therefore a upper critical dimension for the spatial part of $d = 2$ is obtained, similar to the classical Landau theory. For dimension $d \geq 2$, the critical theory becomes Gaussian and the resulting fixed point is non-interacting and its physical properties are quite simple e.g. the dynamical spin susceptibility shows a linear frequency dependence. On the contrary, if $d < 2$, an interacting fixed is obtianed. So it is clear that the Hertz picture would be insufficient in the case $d \geq 2$ and new critical modes are needed at the quantum critical point, as it contradicts the experimental facts, at least in two or more spatial dimensions. This new modes are originated in the Kondo resonance and the thereon resulting electronic excitations, moreover, they have to be treated on equal footing to the magnetic excitations. The critical modes reveal themselves as the destruction of the Kondo resonance, actually, the Kondo singlet is broken up through the transition at zero temperature. There are several attempts to include that new critical fluctuations into a quantum critical theory.

The section on Hertz-Millis theory will be finished by a short summary of alternative canditates for a quantum critical theory.

1. Local quantum criticality

   In the framework of "local" quantum criticality the Kondo effect is destroyed because local moments are coupled not only to the conduction band but also to the fluctuations of the other local moments. These magnetic fluctuations, whose spectrum turns critically at the QCP, act as a source of dissipation and decohere the Kondo effect. In the local formulation of the problem this effect will be modeled by an extra bosonic bath. A local theory has at least one bosonic and one fermionic external bath. In the next section, the microscopic nature of phase transition will be covered. Not only the Kondo and the magnetic interaction, have to be treated on equal footing, but also their dynamical interplay, in order to model the kind of fixed point of interest. The most promising approach is EDMFT , independently proposed by Smith and Si [23] and Sengupta [24], as a description of the QCP in a Kondo lattice.

2. Spin charge seperation

Another idea is the seperation of spin and charge of the electron in the quantum critical regime[8] by Coleman *et. al.* [25]. Basically, the quantum critically is compared with a black hole horizon, once a electron goes beyond the horizon on the paramagnetic side it appears at once on the other side throught the criticality. Persumably, the phase diagram gives the striking argument, its V-shaped diagram shows a quantum critical regime with a horizont seperating the quantum criticaly from the ordinary phase and the critcal matter eventually emerge in the $T = 0$ singularity.

The last note on this section is that it might be possible to cure the problems of the ordinary ϕ^4 theory and consequently saving a field theoretical describtion, by including new terms into the action, such as $\mathcal{S} = \mathcal{S}_{SDW} + \mathcal{S}_{loc} + \mathcal{S}_{mix}$. The terms $\mathcal{S}_{loc}$ (coupling amoung local modes) and $\mathcal{S}_{mix}$ (coupling between local and non-local modes) have to be constructed such that the local fluctuations are expressed in non-linear coupling [9].

1.2.4 The Extended Dynamical Mean Field Theory

The Dynamical mean field theory was encountered by Georges *et.al.* [26] and Meztner and Vollhardt [27] and is a sort of generalization of the classical Weiss mean field theory, in which all fluctuations are negelected[10]. One of the main differences of the classical and quantum mechanical approach is, that the classical effective model[11] turns out to be a single 'particle' problem, whereat the effective quantum mechanical system is still a many body problem. The DMFT comprises local fluctuations such as on-site temporal quantum fluctuations but excludes inter-site non local fluctuations, these were then considered in the Extended Dynamical Mean Field Theory approach. The EDMFT extension of the ordinary dynamical mean field theory is imposed by the new RKKY term, which gives rise to the inter-site quantum fluctuations. Quite often it is futile to hope for a general solution of a microscopic problem, in this sense EDMFT is no exception, one assumes a negligible **q**-dependence of single electron quantities such as self energy $\Sigma(\omega)$. The EDMFT self consistency equations can be obtained in various different ways

1. The "cavity" method [28]
2. The diagrammatic method[23] ,
3. The functional formalism [29].

[8] To our knowledge this concept is so far just of qualitative in its nature.

[9] At the time of this diploma thesis there was no field theory known, that incorporates both critical modes the electronic and the magnetic ones.

[10] In the language of quantum mechanics one would refer to the Hartree-Fock approximation, and basically fluctuations can be incorporated by making a random phase approximation around the static and uniform saddle point. But RPA is a perturbative method and e.g. local quantum fluctuations must be treated non-perturbatively.

[11] It is easiest to think of an one dimensional Ising system.

It should be mentioned that the cavity method is borrowed from ordinary statistical mechanics and therefore is the most 'intuitive' one. In the following the method is outlined only schematically. The idea is to consider an infinite dimensional lattice and to perform EDMFT within this approximation. Obviously, the idea has also been borrowed form classical theory, because classical mean field theory only produces exact results if the dimension d is taken to be infinite. For a finite d EDMFT provides just an approximative solution to the system(it is also valid to consider the number of neighbours z), for instance in a cubic lattice $z = 6$ or a face-centred cubic lattice $z = 12$. In principal $\frac{1}{z}$ acts as a control parameter and its value governs the accuracy of the approximation, generally it holds, the smaller $\frac{1}{z}$ is the better is the approximation, hence EDMFT gets exact in the limit $d \to \infty$. Basically, one rescales $t_{\langle ij\rangle} \to \frac{t_0}{\sqrt{d}}$ and $I_{\langle ij\rangle} \to \frac{I_0}{\sqrt{d}}$ ensuring the finiteness of the overall kinetic and 'potential' energy. Up to zero-th order DMFT is obtained, the inter-site effects completely drop out and one is left with the local part only. First order calculations also take inter-site effects into account resulting in novel results expressed by a self-consistent impurity Hamiltonian. Note, there are some subtle difficulties, such as double counting. Even, if one takes $I_{ij} = 0$ in Eqs.(1.11) non local interaction between nearest neighbours are going to occur. Say, one electron from the conduction band interacts with the impurity and picks up a factor of Jt_{ij}, if this electron interacts then with the next impurity, it picks up another factor of Jt_{ij}, resulting in $(Jt_{ij})^2$. An effective RKKY interaction between local moments is created, while in the $d \to \infty$ limit this term is of higher order and does not contribute, in the case where d, respectively z stays finite it does contribute and thus changes the mean field equation.

EDMFT equations

It is convenient to write the partition function of Eqs(1.11) as a functional over Grassman variables

$$Z = \int \prod_i D(c^\dagger_{i.\sigma}, c_{i.\sigma}) e^{-S[c^\dagger_{i.\sigma}, c_{i.\sigma}]} \tag{1.20}$$

and the action is given by

$$\begin{aligned} S[c^\dagger_{i.\sigma}, c_{i.\sigma}] \;=\; & \int_0^\beta d\tau \left[\sum_{i\sigma} c^\dagger_{i.\sigma}(\tau) \frac{\partial}{\partial\tau} c_{i.\sigma}(\tau) + \sum_{ij\sigma} t_{ij} c^\dagger_{i\sigma}(\tau) c_{j\sigma}(\tau) \right. \\ & \left. + \sum_{ij} \frac{I_{ij}}{2} \mathbf{S}_i(\tau) \cdot \mathbf{S}_j(\tau) + \sum_i J_k \mathbf{S}_i(\tau) \cdot \mathbf{s}_{c,i}(\tau) \right]. \end{aligned} \tag{1.21}$$

By the aforementioned analogy to the classical method all fermions are traced out except one specific site o[12]to obtain an effective description

$$\boxed{\frac{1}{Z_{\text{eff}}} e^{-S_{\text{eff}}[c^\dagger_{o\sigma}, c_{o\sigma}]} \equiv \frac{1}{Z} \int \prod_{i\neq o,\sigma} D(c^\dagger_{i.\sigma}, c_{i.\sigma}) e^{-S}}. \tag{1.22}$$

[12]The site can be chosen arbitrarily

The original action S is split into three different parts, $S = S^{(0)} + S_0 + S_\Delta$, where S_0 denotes the local part S_Δ the inter-site interaction between the site and the rest of the system and $S^{(0)}$ is the lattice action in the presence of the cavity

$$S_0 = \int_0^\beta d\tau \left[\sum_\sigma c^\dagger_{0\sigma}(\tau)(\frac{\partial}{\partial\tau} + t_{00})c_{0\sigma}(\tau) + J_K \mathbf{S}_0(\tau)\cdot\mathbf{s}_{c,0}(\tau)\right] \tag{1.23}$$

and

$$\begin{aligned} S_\Delta &= \int_0^\beta d\tau \left[\sum_{i\sigma} t_{i0}c^\dagger_{i\sigma}(\tau)c_{0\sigma}(\tau) + t_{0i}c^\dagger_{0\sigma}(\tau)c_{i\sigma}(\tau) \right. \\ &\quad \left. + \frac{1}{2}(I_{i0}+I_{0i})\mathbf{S}_i(\tau)\cdot\mathbf{S}_0(\tau)\right] \end{aligned} \tag{1.24}$$

and

$$\begin{aligned} S^{(0)} &= \int_0^\beta d\tau \left[\sum_{i\neq 0\sigma} c^\dagger_{i\sigma}(\tau)\frac{\partial}{\partial\tau}c_{i\sigma}(\tau) + \sum_{i,j\neq 0\sigma} t_{ij}c^\dagger_{i\sigma}(\tau)c_{j\sigma}(\tau) \right. \\ &\quad \left. \sum_{i\neq 0} J_k \mathbf{S}_i(\tau)\cdot\mathbf{s}_{c,i}(\tau)\right]. \end{aligned} \tag{1.25}$$

In order to make use of Eqs.(1.22) it is necessary to expand the S_Δ-part in the parition function

$$\begin{aligned} Z &= \int D[c^\dagger_{0\sigma}, c_{0\sigma}]e^{-S_0}\int\prod_{i\neq 0} D[c^\dagger_{i\sigma}, c_{i\sigma}]e^{-S^{(0)}-\int_0^\beta d\tau S_\Delta(\tau)} \\ &= \int D[c^\dagger_{0\sigma}, c_{0\sigma}]e^{-S_0}\int\prod_{i\neq 0} D[c^\dagger_{i\sigma}, c_{i\sigma}]e^{-S^{(0)}} \\ &\quad \times\left(1 - \int_0^\beta d\tau S_\Delta(\tau) + \frac{1}{2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2 T_\tau S_\Delta(\tau_1)S_\Delta(\tau_2) + \cdots\right) \\ &= \int D[c^\dagger_{0\sigma}, c_{0\sigma}]e^{-S_0}Z^{(0)}\left(1 - \int_0^\beta d\tau\langle S_\Delta(\tau)\rangle^{(0)}\right. \\ &\quad \left. + \frac{1}{2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\langle T_\tau S_\Delta(\tau_1)S_\Delta(\tau_2)\rangle^{(0)} + \cdots\right). \end{aligned} \tag{1.26}$$

Note, the $\langle S_\Delta(\tau)\rangle^{(0)}$-term vanishes, because $\langle\mathbf{S}_i(\tau)\rangle^{(0)}$ is zero (the $i = o$ term is excluded in the $\langle\rangle^{(0)}$ averaging). The same argument holds for the fermionic term. From $\langle T_\tau S_\Delta(\tau_1)S_\Delta(\tau_2)\rangle^{(0)}$ one gets two one point correlation functions, a purely fermionic one and a purely bosonic one. Now, an effective action has to be constructed, thereby the

linked cluster theorem[13] is used

$$\begin{aligned} S_{\text{eff}} &= S_0 + \sum_{n=1}^{\infty} \sum_{i_1,..j_n} \int t_{i_1,0}...t_{0,j_n} c^{\dagger}_{0\sigma}(\tau_{i_1})...c^{\dagger}_{0\sigma}(\tau_{i_n}) c_{0\sigma}(\tau_{j_1})...c_{0\sigma}(\tau_{i_n}) \\ &\quad \times G^{(0)}_{i_1...j_n}(\tau_{i_1}...\tau_{i_n}, \tau_{j_1}...\tau_{j_n}) + \text{const}, \end{aligned} \tag{1.27}$$

where $G^{(0)}_{i_1...j_n}$ is the n-point connected Green's function of the cavity Hamiltonian. The expression for the effective action reduces drastically in the limit of infinite dimensions. In the following the scaling argument is just applied to the fermionic case, though the same arguments are valid for the RKKY term. If the limit, d to infinity, is taken, only the first expansions term survives, because t_{ij} scales as $t_{ij} = \frac{t}{\sqrt{d}}$, the one particle Green's function G_{ij} goes as $\frac{1}{d^{\frac{|i-j|}{2}}}$, because it is proportional to $t^{|i-j|}$. The sum over i and j gives a further factor of d^2, so and if only nearest neigbhour interaction is regarded, this cancels out and one ends up with the first term being of order of 1. By the same reasoning all higher terms are, at least, proportional to $\frac{1}{d}$, so the scale to zero in the large dimension limit. A more explicit discussion was provided by Georges *et.al.*. [26]. The EDMFT equations of a Kondo lattice can be enunciated in an effective impurity action

$$\begin{aligned} \mathcal{S}_{eff} &= \mathcal{S}_{top} + \int_0^{\beta} d\tau \quad J_K \mathbf{S} \cdot \mathbf{s}_c \\ &\quad - \int_0^{\beta} d\tau \int_0^{\beta} d\tau' \sum_{\sigma} c^{\dagger}_{\sigma}(\tau) \mathcal{G}_0^{-1}(\tau - \tau') c_{\sigma}(\tau) \\ &\quad - \frac{1}{2} \int_0^{\beta} d\tau \int_0^{\beta} d\tau' \sum_{\sigma} \mathbf{S}(\tau) \cdot \chi_0^{-1}(\tau - \tau') \mathbf{S}(\tau), \end{aligned} \tag{1.28}$$

where $\mathcal{S}_{top}$ is the so called Berry phase action of the local moments and $\mathcal{G}_0^{-1}$ and χ_0^{-1} are the Weiss fields. $\mathcal{G}_0^{-1}$ encodes the local physics, the on-site quantum fluctuation and χ_0^{-1} represents the non-local nature of the system. In the path integral formalism the Berry phase takes account for the Kondo singlet formation, as an intrinsic quantum mechanical effect. The last two terms were obtained via the application of the linked cluster theorem as it was explained above and the second term refers to the S_0. One can think of these Weiss fields in terms of the familiar Weiss field which is known from the ordinary Ising model as a field which expresses the collective affection of the surrounding degrees of freedom on a single site. Being aware of the fact that in classical theory the Weiss field is just a number in the quantum case , though it is a function depending on imaginary time. The second Weiss field χ_0^{-1} is a nouveau feature of EDMFT, since it depicts the long range properties of the system. However, the other Weiss field has already been present in the usual DMFT description, catching the local physics.

Alternatively, the aforementioned effective action can also be deduced from a self-consitent

[13] At this point, $\frac{1}{d}$ provides an adequate expansion parameter of the perturbation series.

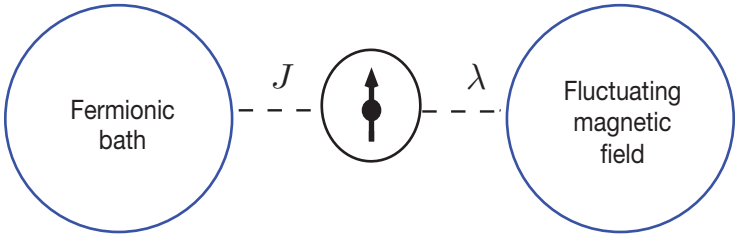

Figure 1.2: The emerged local theory, after applying the EDMFT on the Kondo lattice, is charcterised by the coupling of the local moment to a fermionic bath with the strength J and to a bosonic bath via λ. The picture is taken form Si *et.al.* [19].

impurity model, known as the Bose-Fermi-Kondo model

$$\begin{aligned} H_{imp} &= \sum_{p\sigma} E_p c^\dagger_{p\sigma} c_{p\sigma} + J_K \mathbf{S}\cdot\mathbf{s}_c \\ &\quad + \sum_k \omega_k \mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}_k + \sum_k \lambda \mathbf{S}\cdot(\mathbf{\Phi}^\dagger_{-k} + \mathbf{\Phi}_k). \end{aligned} \tag{1.29}$$

The local impurity couples to fermionic degrees of freedom, representing the local part, and to bosonic degrees of freedom, expressing the non-local part. The RKKY interaction emerge as a bosonic bath, whereas the fermionic bath is induced by the conduction electrons, see Fig.1.2. Actually, the vector bosons give rise to magnetic fluctuations and the electronic degrees of freedom to local fluctuation. Both fluctuation are expected to contribute equally to the critical spectrum at the QCP to generate a phase transition, when going from the Kondo dominated domain to the magnetic domain and vice versa. The parameters E_p, ω_k, and λ have to be ascertained self-consistently from the EDMFT equation. By integrating out the bosonic and fermionic degrees of freedom in Eq.(1.29) one ends up with Eq.(1.28), preconditioned the following identifications have been made

$$\chi_0^{-1} = -\sum_k \lambda^2 \frac{2\omega_k}{(i\nu_m)^2 - \omega_k^2} \tag{1.30}$$

and

$$\mathcal{G}_0^{-1}(iw_n) = \sum_p \frac{1}{iw_n - E_p}, \tag{1.31}$$

ν_m and w_n are the bosonic, respectively fermionic Matsubara frequencies.

Now, it follows a short sketch of the self-consitent procedure, the way how this principally works follows closely [16];

1. Guess a trial form for $\mathcal{G}_0^{-1}$ and χ_0^{-1}. Practically, one would made a intelligent guess for fermionic and bosonic density of states, which are themselves connected to the particular Weiss field.

2. From the solution to the Bose-Fermi-Kondo model one can extract the impurity correlation functions, which readily can be identified with the local correlation function of the lattice model

$$\chi^a_{loc}(\tau) \equiv \langle T_\tau S^a(\tau) S^a(0)\rangle \tag{1.32}$$

and

$$G_{loc}(\tau) \equiv -\langle T_\tau c_\sigma(\tau) c^\dagger_\sigma(0)\rangle, \tag{1.33}$$

where $a = x, y, z$. In addition the spin self energy and the conduction-electron self energy can be obtained from the Bethe-Salter respectively the Dyson equation

$$M_{loc}(i\nu_m) = \chi_0^{-1}(i\nu_m) + \frac{1}{\chi_{loc}(i\nu_m)} \tag{1.34}$$

and

$$\Sigma_{loc}(iw_n) = \mathcal{G}_0^{-1}(iw_n) - \frac{1}{G_{loc}(iw_n)}. \tag{1.35}$$

3. Imposing the self-consitency condition that reflects physically the translation invariance of the lattice, leads to

$$\chi_{loc}(\omega) = \sum_{\mathbf{k}} \chi(\mathbf{k}, \omega) \tag{1.36}$$

and

$$G_{loc}(\omega) = \sum_{\mathbf{p}} G(\mathbf{p}, \omega). \tag{1.37}$$

The lattice Green's function and the lattice spin susceptibility on the right hand side of the above equations read

$$\chi(\mathbf{k}, \omega) = \frac{1}{M_{loc}(\omega) + I_{\mathbf{q}}}, \tag{1.38}$$

and

$$G(\mathbf{p}, \omega) = \frac{1}{\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{loc}(\omega)}, \tag{1.39}$$

and μ is the chemical potential.

Once one went through the whole recipe it is possible to adjust the free parameters in the Weiss fields respectively density of states in order to get the self-consitent equation worked out. After that, the procedure has to be iterated over and over again, means making an even more intelligent guess for the Weiss fields inserting into step one and so on and so forth, since convergence is reached. Eventually, one ends up with the right lattice correlation function or at least a very good approximation of it.

The Occurrence of a Quantum Phase Transition in the EDMFT Framework

It was shown in [16] that EDMFT not necessarily leads to a local critical point, in fact just if the magnetic fluctuations are two dimensional. Three dimensional fluctuations induce a Gaussian fixed point, because the critical exponents exhibit no fractional behaviour. A general criterion was proposed by Si *et. al.* [30], there a 'new' energy scale was introduced E^*_{loc} in order to decide whether a local critical point occurs or not. It is defined such that the local spin self energy $M_{loc}(\omega)$ is regular below E^*_{loc}, likewise one can define E^*_{loc} as the energy scale below which the Kondo singlet is still present. At a fixed temperature one can upon increasing the value δ one goes from one "phase" to the other by crossing $E^*_{\rm loc}$. On the left side one is in the small Fermi-surface phase, the Kondo screening is suppressed and the behaviour of the local moment is mainly governed by the RKKY interaction, right of this energy separation the Kondo resonance is present and the local moments become delocalised and contribute to the Fermi-surface, hence one obtains an enhancement of the Fermi-surface. By cooling the system down to zero temperature one gets two different phases separated by the aforementioned energy scale. Note that at zero temperature $E^*_{\rm loc}$ marks a proper quantum phase transition, however, at non-zero tepmerature just a crossover takes place. Now, there are two different cases, what kind of phase transition appears, determined by $E^*_{\rm loc}$ and δ [14].

Recall, the tuning parameter of the lattice model is given by $\delta = \frac{I_{RKKY}}{J_K}$ and δ_c denotes the critical value, where the phase transition takes place. If $\delta < \delta_c$ the system resides at the paramagnetic side, but if $\delta > \delta_c$ the antiferromagnetic phase is present and δ_c is specified to be that parameter value by which the lattice spin susceptibility diverges. It is well known from classical phase transition, a divergent magnetic susceptibility indicates a phase transition. Although, varying δ alters the local Kondo physics and apparently affects the impurity model, due to the EDMFT connection. As it will be shown in the next section the impurity model undergoes phase transition similarly to the lattice model. The microscopic phase transition arises due to the competition of the relevant energy scales λ and J_K[15], so it is convenient to define δ_{loc}, which is the ratio of the microscopic coupling parameters(e.g. $\frac{\lambda}{J_K}$). With the help of these two parameters one is finally in the position to decide, whether a local quantum critical point is occurring. The local parameters are influenced not only by the attempt of Kondo physics to quench to local moment by the surrounding fermionic bath, but also by the bosonic bath trying to polarise the local moment in order to prevent them from total screening and align them in a certain direction. This reasoning shows, at a particular value $\delta_{loc} = \delta^c_{loc}$, the critical point of the microscopic system is reached, just as E^*_{loc} goes to zero at that point. Furthermore, the lattice system also reaches its critical point at $\delta = \delta^c$ indicated by the smallest value of δ when the magnetic susceptibility diverges. Let's say, $\delta^c < \delta^c_{loc}$ that means the magnetic fluctuations become critical before the onset of local physics being significant in terms of

[14] Actually, there is a third one, but firstly it has not been found experimentally and secondly it neither has got a Kondo screening nor is magnetically ordered.

[15] Sometimes the Kondo temperature T_K itself is taken to be the relevant energy scale.

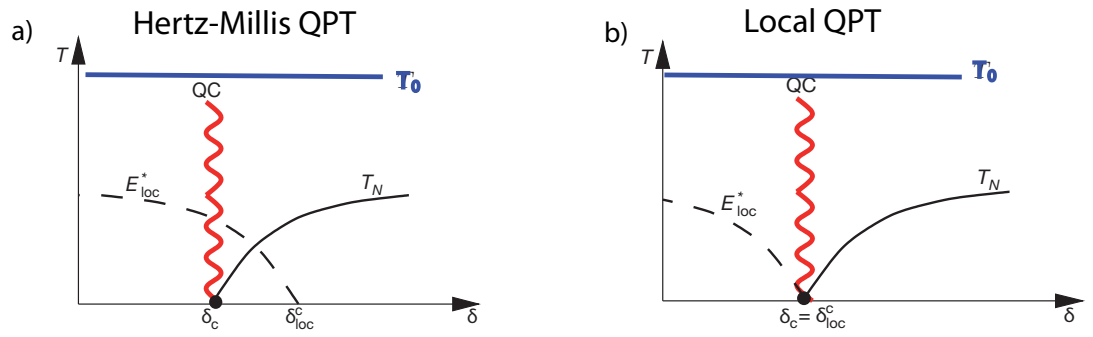

Figure 1.3: The diagrams for the two different types of quantum phase transition in Kondo lattice. The external tuning parameter is $\delta = \frac{I_{\mathrm{RKKY}}}{J_K}$, δ_c is the critical value of the lattice model and $\delta_{\mathrm{loc}} = \frac{\lambda}{J_K}$, with δ^c_{loc} as the critical value of the corresponding local theory. Below the energy scale E^*_{loc} the local Kondo phase is present and therefore the susceptibility has got Pauli form. a) shows the conventional quantum phase transition explained by the Hertz-Millis theory, the onset of the magnetic quantum phase transition sets in before the local theory gets critical, hence the local critical fluctuations do not contribute to the critical theory. b) the magnetic fluctuations get critical at the very same point as the local fluctuations, both fluctuations are going to contribute. In both pictures there also exists the transition of the paramagnetic phase to the antiferromagnetic phase separated by a finite transition temperature T_N. Taken from Si *et.al.* [19].

local Kondo physics and a conventional quantum phase transition takes place, describable by the Hertz-Mill theory. On the other hand, if $\delta^c = \delta^c_{loc}$ the local critical aid to critical spectrum and one eventually spots a local critical quantum point, where new physics is anticipated to emerge. This is visualised in Fig. 1.3.

The fully screened local moments, screened by the surrounding conduction band electron, is a necessary condition for the Kondo resonance to be built up, hence the local spin susceptibility should better has Pauli form, implying a finite value at zero temperature and frequency. In a translations invariant system [16] the local spin susceptibility is equal to the average of the wavevector-dependent dynamical spin susceptibility. Notice, the similarity to the self-consistency condition, there the assumption, that the local spin susceptibility is the same as the susceptibility of the single impurity model, was made. Let Q be the antiferromagnetic ordering wave vector, then the susceptibility $\chi(Q, \omega, T)$ is divergent at the antiferromagnetic QCP, though the average susceptibility could either be divergent or stays finite. As mentioned above a finite average susceptibility implies an unmodified Kondo resonance, while its divergence leads to a destruction of the Kondo singlet. This

[16] The Kondo lattice is considered to be translations invariant in the bulk, otherwise one has to account for the right choice of boundary conditions.

also gives rise to two different types of QCP, in addition to the Grüneisen ratio, which will be discussed in the final section of this chapter, whereas the ratio is more an experimentally accessible criterion and the average susceptibility a theoretical one. In terms of Fermi liquid theory a divergent average susceptibility would imply, no Kondo resonance is present at the QCP, accordingly the non-Fermi-liquid theory must be treated equally to the magnetic ordering. The electronic excitations become critical at the very same point in the phase diagram as the long range magnetically critical fluctuations.

For the sake of completeness apart from the EDMFT method there exists a further method called cluster DMFT, including information about underlying lattice structure into the local features of the effective theory. For an readable introduction we refer to the paper of Ferrero *et. al.* [31]. Without any local features incorporate into the EDMFT, only the long wavelength will contribute to the magnetic phase transition and this is exactly one would expect from classic critical theory. Motivated by the classical theory one equally assumes in quantum system the independence of the phase transition of the underlying microscopic structure as the correlation length tends to infinity.

The solution of impurity model

Several attempts have been made to solve the impurity problem. Among them there are both analytic and numerical ones, whereupon numerical methods merely can deal with the Ising Bose-Fermi-Kondo model. M.Glossop and K. Ingersent [32] could prove the occurrence of a quantum phase transition of such a model, using NRG. The main difficulty in the numerical effort is to handle the baths, one fermionic bath is doable and an extent can be made fairly easily, however, the bosonic bath is rather challenging, as an infinite occupation of a certain energy level is feasible. So far just one bosonic bath could be included into the numerical calculations. To seize the impurity problem by means of analytic methods, conventional perturbative renormalisation group theory [33, 24] had been applied as well as a non-perturbative large N expansion by Zhu *et.al.* [34], N denotes the number of spin channel indices[17].

Collectively, they assumed a constant density of states of fermions within the conduction band width and a power law like behaviour for the bosonic density of states

$$\lambda^2 \sum_k \delta(\omega - \omega_k) = 2\alpha\omega^s \quad \text{for} \quad 0 \leq \omega < \Lambda, \tag{1.40}$$

with s and α to be determined self-consistently. The parameter s has to be chosen for the impurity model by hand. There are three different types of bosonic baths;

i) $s < 1$ sub-ohmic bath

[17] Although, in this thesis we are only interested in the $N = 2$ case, the large N results are the same as for the $N = 2$ case, so it is believed that right critical behaviour can be captured within this model.

ii) $s = 1$ ohmic bath

iii) $s > 1$ super-ohmic bath

and Λ is the cut-off parameter displaying the upper potentially occupied frequency.

Within the RG method an expansion in the anomalous dimension $\varepsilon = 1 - s$ has been performed Sengupta [24] applied the ε-expansion on the Ising BFKM and both Zárand and Demler [33]and Si *et.al.*[16] on the isotropic BFKM, respectively. Zárand and Demler discussed several models, among others the Bose-Kondo model with a completely turned off fermionic coupling, the Ising Bose-Fermi-Kondo model, and the SU(2) symmetric Bose-Fermi-Kondo model. In Fig. 1.4 one can see the flow of the couplings in the isotropic BFKM, two stable fixed points and one unstable fixed point are perceived. The flow of the couplings is based on the scaling equations of the coupling constants

$$\boxed{\frac{d\xi}{dln\Lambda^{-1}} = \xi(\xi - 2g)}, \tag{1.41}$$

$$\boxed{\frac{dg}{dln\Lambda^{-1}} = (\varepsilon g - 4g^4)}. \tag{1.42}$$

Note, Zárand and Demler used a different notation for the coupling constants, by rewriting $g = 2\tilde{g}$, where $\tilde{g}$ is the bosonic coupling they used, the original scaling equations are obtained. Two quite important results have been achieved, firstly they showed that the critical point namely the unstable fixed point is accessible in a perturbative way and secondly the asymptotic developing of the local spin correlation function at this critical point was calculated to be

$$\chi^{\mathrm{SU(2)}}_{\alpha\beta}(\tau) = \langle T S_\alpha(\tau) S_\beta(0)\rangle = \delta_{\alpha\beta}\chi(\tau) \sim \frac{\delta_{\alpha\beta}}{\tau^\varepsilon}, \tag{1.43}$$

τ denotes the imaginary time. After performing a Fourier transformation the frequency dependent spin susceptibility is

$$\boxed{\chi_{\mathrm{loc}}(\omega) \sim \frac{1}{\omega^{1-\varepsilon}}}, \tag{1.44}$$

which is a very important result. This result is just valid at zero temperature At non-zero temperature it is very difficult to derive an expression for the susceptibility for an interacting non gaussian fixed point. One idea is to assume a correspondence between the BFKM at the critical point and a certain boundary conformal field theory, motivated by the matching of the susceptibility of the models at zero temperature. Based on this assumption Aronson *et. al.* [35] gave an expression for the non zero temperature dynamical susceptibility

$$\chi_\alpha(\tau, T) = C\left(\frac{T}{sin(\pi T\tau)}\right)^\varepsilon, \tag{1.45}$$

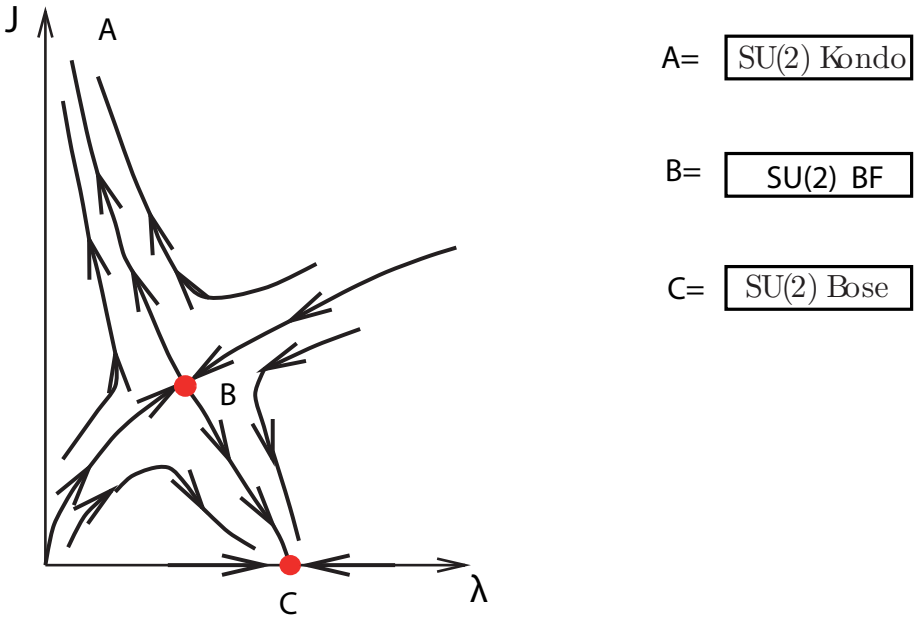

Figure 1.4: The renormalisation group flow of the coupling fermionic coupling J and the bosonic coupling λ. The diagram exhibits three fixed points, two stable ones namely SU(2)-Kondo and the SU(2)-Bose, which is the Kondo and the purely bosonic phase, respectively, and one unstable fixed point the SU(2) BF. This unstable fixed point can be interpreted as the quantum phase transition point. The picture is taken from Si *et.al.*[16].

α indicates the spatial direction and C is a constant. Again, after the Fourier transformation the frequency dependent finite temperature susceptibility is

$$\chi_\alpha(\omega, T) \sim \frac{1}{T^{1-\varepsilon}} \frac{\Gamma(\frac{1}{2} - \frac{\varepsilon}{2})\Gamma(\frac{1}{2} - \frac{i\omega}{2\pi T})}{\Gamma(\frac{\varepsilon}{2})\Gamma(\frac{1}{2} - \frac{\varepsilon}{2} - \frac{i\omega}{2\pi T})} \tag{1.46}$$

and Γ is the well known gamma function. This expression also shows the desired $\frac{\omega}{T}$-scaling. Zaŕand and Demler [33] also could deduce

$$\varepsilon_{\text{QCP}}^{\text{SU(2)}} \equiv 1 \tag{1.47}$$

for the EDMFT self-consitent condition to hold. The deviation of the exponent from one is a clue for a non-Fermi-liquid and leads to the desired scaling behaviour like the $\frac{\omega}{T}$-scaling. Spin-glasses [36, 37] exhibit the same characteristics in spin dynamics and are expected to belong to the same universality class, so one can study them as well, in order to gain insights to the nature of a quantum phase transition.

In the Ising BFKM similar results has been obtained, the main and definitely one of the most important difference is, the non-accessibility of the unstable fixed point via perturbation theory, although the right devolution of the spin susceptibility can be received

by Ward identities, and in the z direction the susceptibility at zero temperature is same as stated above. Nevertheless, an exact solution of the Ising BFKM can be obtained analytically, so far only in the ohmic case, means ε is set equal to zero, and a special choice of the dissipative parameter α. In this case the Ising BFKM can be mapped onto a Caldeira-Leggett model [38]. The flow of the coupling constants of the Ising BFKM, in the ohmic case, is the same, at least qualitatively, as the flow of couplings in the anisotropic Kondo model, the only difference is that the perpendicular fermionic coupling of the Ising BFKM is shifted by a constant, which value is determined by the strength of the coupling of the bosonic bath to the impurity spin, see Li *et. al.* [38].

The section is finished by a few final remarks on the numerical methods and their obtained results. In terms of numerics no solution of the isotropic BFKM is available yet, though in the Ising BFKM there are several attempts, such as [39, 40, 32], moreover, these numerical methods serve a non-perturbative method of solving the impurity problem. Zhu *et.al* [40] applied the Quantum Monte Carlo method and could produce the correct curves for the E^*_{loc} against $\frac{\lambda}{J_K}$ and for the $\frac{\omega}{T}$-scaling. There they could also show the interaction nature of the unstable fixed point by plotting the single particle spectrum, hence the destruction of the Kondo resonance.
Within the NRG framework, in the case $s \leq 1$ two stable fixed points, the Kondo phase and the Bose phase, have been found as well as one unstable fixed point with non zero bosonic and fermionic couplings. The ohmic critical point is Kosterlitz-Thouless like, while in the range of $0 < s < 1$ the static temperature dependent local susceptibility at the critical point is given by

$$\chi^o_{\mathrm{loc}}(T) \sim \frac{1}{T^s}, \tag{1.48}$$

which matches with the result received by an ε-expansion [41] and in the $s = 1$ case it reduces to the well known Curie law, see [38]. Similar to the isotropic BFKM the imaginary part of the dynamical local susceptibility $\chi''_{loc}(\omega)$ at the critical point, in the Ising type case, has the form

$$\chi''_{loc}(\omega) \sim \frac{1}{|\omega|^s}\mathrm{sgn}(\omega). \tag{1.49}$$

Last but not least Glossop and Ingersent could also plot graphs of the imaginary part of the dynamical local susceptibility at non zero temperature, demonstrating consistency with the $\frac{\omega}{T}$-scaling. To be precise in the case $\omega \ll T_K$ they obtained the following expression

$$T_K \chi''_{loc}(\omega, T) \sim \left(\frac{T}{T_K}\right)^{-s} \Phi\left(\frac{\omega}{T}\right), \tag{1.50}$$

with $\Phi\left(\frac{\omega}{T}\right)$ as an universal function. Zhu *et. al.* [34] received a similar result, however in the case of a multi-channel BFKM within the framework of large N-expansion.

We close this chapter by a little summarisation of possible experimental methods to scrutinise the nature of the quantum phase transition.

1.2.5 Experimental Criterion on the Nature of the QCP

The Fermi surface

By going from the paramagnetic side, governed by the Kondo coupling J_k, to the antiferromagnetic the Kondo screening breaks down close to the QCP. In the paramagnetic domain the local spins become a part of the Fermi surface since they are strongly entangled with the conduction electrons. These f-electrons are now part of the low energy electronic excitation and due to the formation of the Kondo singlet the f-electrons and the conduction electrons are entangled in such a way that the originally charge free local moments 'gain' charge and turn into a charged spin $\frac{1}{2}$ quasi paritcle excitation. So the f-electrons become delocalised and contribute to the Fermi surface. Assuming x is the number of conduction electrons in a unit cell then the Fermi-volume on the paramagnetic side encloses $x + 1$ electrons. Close to the paramagnetic critical point the electronic excitations, consisting of conduction electrons and local moments start to deviate from their Fermi-liquid structure towards a non-Fermi-liquid. These novel excitations become a part of the quantum critical spectrum in the local critical theory. At the magnetic QCP the magnetic fluctuations are dominating, where the spectrum of the paramagnons designate the criticality and the Fermi volume is merely given by the conduction electrons. This means that the f-electrons turn into their localised stage, the entanglement with the conduction electrons breaks up and hence do not contribute to Fermi-surface any more. By tuning the control parameter δ through the critical point the Fermi-surface is going to decrease erratically. But the important new physics are the novel local fluctuation, that change the former non-interacting Gaussian fixed point into an interaction one, and one can hope to built a new critical theory on this basis, explaining the experimental results correctly. Fig.1.3.b refers to this type of quantum phase transition. Moreover, it is possible for the experimentalists to perform an experiment, where the magnitude of the Fermi-surface is measured by Paschen *et.al.* [42] utilising the Haas-van Alphen effect [43, 44]. The local quantum criticality predicts a discontinuous enhancement of the Fermi-surface at the QCP, the Kondo coupling dominated region has a larger Fermi-volume than the antiferromagnetic region. In the spin-density-wave picture no such jump happens, the transition is smooth while the SDW order sets in. Corresponding to Fig.1.3.b, where the onset of the magnetic phase transition already takes place, while the local moments are still quenched. An experimentally very well accessible quantity is the Hall coefficient R_H, the continuity of the Hall coefficient through the critical point is directly related to the Fermi surface so if R_H jumps so the Fermi-surface will do, see also [42].

Grüneisen ratio

As mentioned before, the divergence of thermodynamical quantities come about only in classical phase transitions, though in QPT with zero temperature there is no such divergence, since the third law of thermodynamics has to be obeyed[18]. However, the Grürneisen

[18]The entropy of a system stays finite as the temperature approaches zero.

ratio diverges at the QCP [45]. This ratio is defined in terms of two thermal quantities the specific heat $c_p = \frac{T}{N}\frac{\partial S}{\partial T}|_p$ and the thermal expansion $\alpha = \frac{1}{V}\frac{\partial S}{\partial p}|_{T,N}$,

$$\Gamma = \frac{\alpha}{c_p} = -\frac{N}{TV}\frac{\frac{\partial S}{\partial p}}{\frac{\partial S}{\partial T}}. \tag{1.51}$$

The divergence of the Grüneisen ratio [19] reflects the very fact, that two parameters are necessary to describe the QPT, while c_p measures the response to T, α measures the response to the external tuning parameter. In the classical phase transition parameter like T and p are thermodynamically connected and the typical parameter that controls the transition is proportional to $|T - T_c|$, therefore Γ stays finite at the classical singularity. Zhu *et.al.* [45] showed that the temperature dependence of Γ reads

$$\Gamma \sim \frac{1}{T^{\frac{1}{\nu z}}}, \tag{1.52}$$

the factor νz provides a criterion to classify different types of criticality. One usually refers to the factor $\frac{1}{\nu z}$ as the Grüneisen exponent. Zhu *et.al.* could also disclose that;

a. the divergence of Γ is independent of the kind of QCP

b. the Grüneisen exponent serves a criterion to decide which sort of QCP is present.

For example, for Ce_2Ni_2Ge2, the Grüneisen exponent is $\frac{1}{\nu z} = x = 1$ and is perfectly consitent with the spin-density-wave approach, [20] Küchler *et.al.* [46] pointed out. Otherwise the exponent x differs from one for $CeCu_{6-x}Ag_x$ [47] and the spin-density-wave approach is not appropriate any more, although the local quantum critical theory of Si *et.al.* [19, 16] predicts the right exponent.

[19] If the control parameter is a magnetic field the ratio is defined by $\Gamma = \frac{1}{T}\frac{\partial T}{\partial H}|_S$ and can be measured using the magnetocaloric effect.

[20] If the critical fluctuations are 3-dimensional.

1.3 Flow Equation

Most analytic methods rely on the integrability of the regarded model, therefore such methods like conventional perturbation theory are not applicable and one has to use non-perturbative methods like NRG or flow equation approach. Additionally, the flow equation approach, since it is a non-perturbativ method, can be used for strong coupling calculations, e.g. Sine-Gordon model [48] or a strong weak coupling crossover in the Kondo regime [49]. This is our main motivation to study the Ising type and isotropic Bose Fermi Kondo model with the help of flow equation method.

1.3.1 Overview and the RG-method

The flow equation approach to many body physics was introduced by Wegner [50] and independently by Glazek and Wilson [51, 52] whereas the later ones applied their method mainly to high energy physics. Since this work deals essentially with condensed matter physics, we follow the notation of Wegner. The method itself can be viewed as a kind of generalisation to the usual renormalisation group method due to Wilson [7]. The RG-method works principally as follows, one separates the fast moving from the slow moving modes and integrates out the high energy degrees of freedom that means reducing the dimension of Hilbert space controlled by a certain cut-off parameter Λ_{RG}, which denotes the energy scale one is currently looking at. In order to keep it as simple as possible the considered Hamiltonian depends only on one coupling constant λ that governs the interaction of the system. This coupling constant gets rescaled and the remaining Hamiltonian with the new coupling shall describe the same physics as before within its validity. If this procedure is performed gradually the coupling becomes a function of Λ_{RG} and we talk about a running coupling constant. Assume the considered theory is renomalizable, we can express the behaviour of the running coupling constant in terms of a differential equation

$$\frac{d\lambda(\Lambda_{RG})}{dln\Lambda_{RG}} = \beta(\lambda(\Lambda_{RG})), \tag{1.53}$$

where the β-function is often not known in all detail, though its access via perturbation theory leaded to remarkable results. However, a main disadvantage is that all the information about the high energy sector of the Hamiltonian is gone, which is sort of obvious due to the reduction of Hilbert space dimensions. The reader should see this just as a very basic introduction it was aimed to either refresh some already acquired knowledge or to provide the main idea. For further details we refer to [7], but this is not necessary to follow the rest of this work.

1.3.2 The new Idea

The basic new idea of the flow equation is the unitary transformation of the Hamilton operator, which gradually diagonalises the Hamiltonian, if the transformation is performed several times, see Fig 1.5. For this purpose the unitary transformation will depend on a

parameter $B \in [0, \infty)$ and therefore generate a whole family of transformations $U(B)$ so to say a sequence of successively applied infinitesimal unitary transformation

$$\boxed{H(B) = U(B)H(B=0)U^\dagger(B)}, \tag{1.54}$$

where $U(B) = e^{\eta(B)}$ and $\eta^\dagger(B) = -\eta(B)$ is the antihermitean generator. The main statement is that the solution of the differential equation

$$\boxed{\frac{dH(B)}{dB} = [\eta(B), H(B)]}, \tag{1.55}$$

with $H(B=0)$, with our original Hamiltonian as initial condition, is given by Eqs.(1.54). Generally, we can not demand commutation of the generator with different values of the flow parameter $[\eta(B_1), \eta(B_2)] \neq 0$. Similar to time evolution operator and the time ordering operator we define

$$U(B) = T_B e^{\int_0^B dB' \eta(B')} \tag{1.56}$$

and by taking the derivatve of equation (1.54) we get

$$\begin{aligned} \frac{dH(B)}{dB} &= \frac{dU(B)}{dB} H(B=0)U^\dagger(B) + U(B)H(B=0)\frac{dU^\dagger(B)}{dB} \\ &= \frac{dU(B)}{dB} U^\dagger(B)H(B) + H(B)U(B)\frac{dU^\dagger(B)}{dB} \\ &= \eta(B)H(B) - H(B)\eta(B) = [\eta(B), H(B)], \end{aligned} \tag{1.57}$$

the definition of $U(B)$ and Eqs.(1.54) were used.

Starting from a certain operator product in the interaction term new operator terms will be generated during the flow. In the worst case there will be an infinite number of newly generated terms, thus it can not be expected to solve Eqs.(1.54) generically. It is convenient to rewrite Eqs.(1.54) again, in order to give an approximative solution in terms of perturbation theory. There are cases where the Eqs.(1.54) is exactly solvable (e.g. The Potential Scattering Model [53]), but these are rather limited exceptions. The approximative Hamiltonian, for a fixed value of B, reads

$$\begin{aligned} H(B) &= U(B)H_0U(B)^\dagger = e^{S(B)}H_0e^{S(B)^\dagger} = \sum_n \frac{1}{n!}[S(B), H_0] = \\ &= H_0 + [S(B), H_0] + \frac{1}{2}[S(B), [S(B), H_0]] + \cdots \end{aligned} \tag{1.58}$$

[21] and each term of the expansion can be identified as a higher order term in the sense of newly generated interactions. The Hamiltonian can be expressed through a power law series

$$H_0 = H(B=0) = \sum_{n=0}^{\infty} \varepsilon^n H^{(n)}, \tag{1.59}$$

[21] Since we consider a fixed value of B only, we write $S(B)$ instead of $\eta(B)$ mainly to avoid confusions.

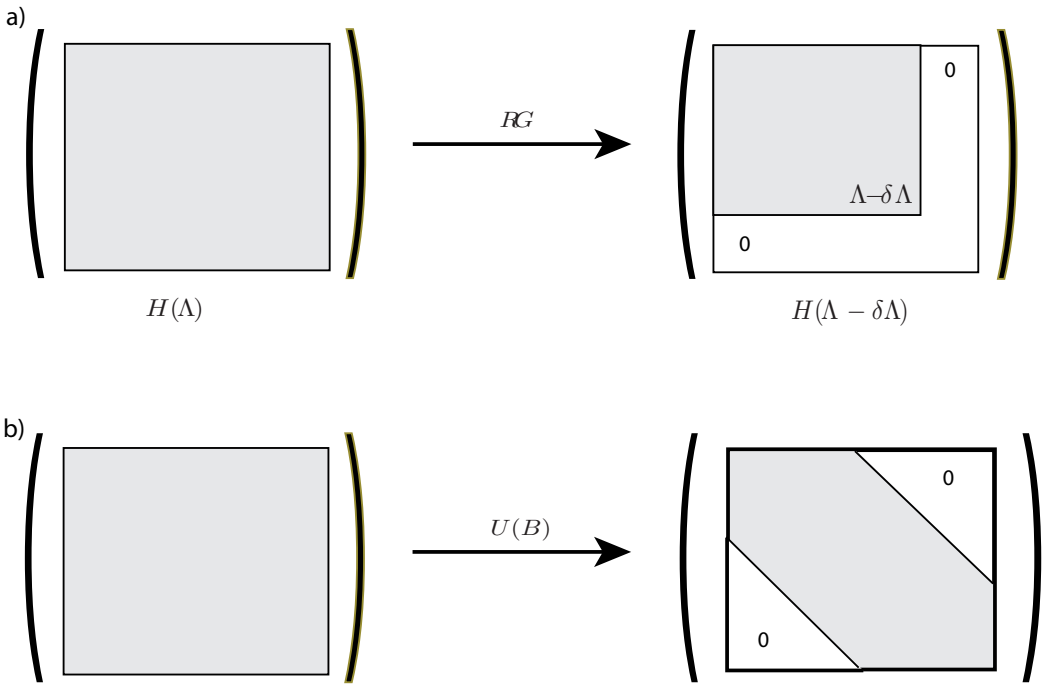

Figure 1.5: Picture a) displays the the conventional RG-approach, integrating out the high energy degrees of freedom. b) shows the flow equation method, making the Hamiltonian more and more diagonal.

where ε is the expansions parameter. $H^{(0)}$ is the free part while all other $H^{(n)}$ are representing the interaction terms. We want to emphasis that the expansion had been applied after the unitary transformation took place, so we are still dealing with a Hamiltonian unitarily equivalent to the original one up to a certain order. As we have already mentioned there is not much hope to solve Eqs.(1.57) exactly for a many body problem, hence we also would like to have $H(B)$ in form of a power series. We assume for the generator to have a power series representation in the parameter ε, keeping in mind that for $\varepsilon = 0$ the Hamiltonian is already diagonal and no more operator products will be generated. A reasonable ansatz will be

$$\eta(B) = \sum_{n=0}^{\infty} \varepsilon^n \eta^n(B), \tag{1.60}$$

plugging this into (1.55) we get a well controlled expansion

$$\frac{dH(B)}{dB} = \varepsilon[\eta^{(1)}, H^{(0)}] + \varepsilon^2([\eta^{(1)}, H^{(1)}] + [\eta^{(2)}, H^{(0)}]) + \mathcal{O}(\varepsilon^3). \tag{1.61}$$

Remember, we took a random but fixed value of B, so our deduction is valid for all

values of B, however, the generalisation is quite subtle, since the $\eta(B)$ does not commute with itself for different B's. For a mathematically rigorous derivation one has to apply the continuous representation Eqs.(1.56)to Eqs. (1.54,). At the end of the day, one gets the same result, which is kind of straightforward to check.

1.3.3 Normal Ordering

One of the main motivations to study flow equation was to obtain non perturbative results, but as we have seen more and more operator products will be generated during the flow. To handle that problem we can e.g. either do an expansion in the coupling and truncate the generated terms by their appearance in power of the coupling or, if the coupling is not small, decide whether the new terms contain any physical significance. A way to do so is the normal ordering procedure.

Wick [54] formally defined the normal ordering of an operator O denoted by $: O :$. The operator O consists of creation and annihilation operators combined written as A_k, where the A_k's can be either fermionic operators or bosonic operators. The following three rules define the normal ordering

$$: 1 := 1 \tag{1.62}$$

$$: \alpha O_1 + \beta O_2 := \alpha : O_1 : + \beta : O_2 : \tag{1.63}$$

$$A_k : O :=: A_k O : + \sum_l \langle A_k A_l \rangle : \frac{\partial O}{\partial A_l} : . \tag{1.64}$$

Note that the here taken expectation value is at zero temperature, nonzero temperature expectation values must be performed with respect to the density matrix. From here Wick first and second theorem can be deduced, but for further details see [53].
In Quantum field theories e.g. Quantum electrodynamics the normal ordering is used to subtract the vacuum energy from the expectation value. But the physical meaning outperform the mere subtraction rule, it actually is given by the fact that the expectation value of an operator O composed of creation and annihilation operators, which is normal ordered, vanishes [53]. There it was also proven that if C_n^1 and C_n^2 are operators each with n creation or annihilation operators in total and $: O_{2m} :$ with $2m$ of such ones, then

$$\langle C_n^1 : O_{2m} : C_n^2 \rangle \neq 0 \tag{1.65}$$

if $n \geq m$. In terms of physics this means, if we look at e.g. $c_p^\dagger c_q^\dagger c_k c_l$ there are also one particle interaction included, whereas $: c_p^\dagger c_q^\dagger c_k c_l :$ just includes two particle interaction, the generalisation is straightforward.

So far we have not talked about any specific representation of the generator, but this is of great importance due to the applicability of normal ordering. The next subsection

will deal with the representation and how this practically looks like will be postponed to the next chapter.

1.3.4 The Meaning of the Generator

The generator lies at the core of this theory while Wegner [50] proposed

$$\boxed{\eta(B) = [H_{diag}(B), H_{int}(B)]} \tag{1.66}$$

for the generator other choices are possible all of them have to share the same property, namely their task is to decouple interaction terms with a non zero energy separation. This means that off diagonal matrix elements will decay and one eventually ends up with a completely diagonalised Hamiltonian, where one easily can read off the eigenenergies of the many body system. If the diagonal matrix elementes are written as $h_{qq} \hat{=} \epsilon_q$, Eqs. (1.66) reads

$$\eta_{pq} = (\epsilon_p - \epsilon_q) h_{pq}, \tag{1.67}$$

where h_{pq} are the matirx elements of the Hamiltonian. In the simple case of a two particle scattering interaction it can be shown [50] that the off diagonal matrix elements decay like

$$h_{pq}(B) \sim h_{pq}(B=0) e^{-B(\epsilon_p - \epsilon_q)^2}. \tag{1.68}$$

It can be seen that the off diagonal terms corresponding to the largest single particle energy separation die off fastest. Eqs.(1.54) and Eqs(1.66) show us the physical dimension of the generator $[\eta] = E^2$ and Eqs. (1.68) gives $[B] = E^{-2}$. At the beginning of the flow (the value of B differs only solely from 0) interaction terms with large energy separation are eliminated, while B increases more and more degenerated energies will be removed. We can at least expect block diagonal form for $H_{int}(B = \infty)$ in the general case, optimally the interaction part of the Hamiltonian itself is diagonal and we get

$$[H_{diag}(B = \infty), H_{int}(B = \infty)] = 0, \tag{1.69}$$

hence $\eta(B = \infty)$=0. As mentioned above within the flow equation method the Hilbert space dimensions do not alter, so the information of the high energy degrees of freedom is still available, though we have to deal with more complicated differential equations of the coupling constants compared to the usual RG-method.

1.4 Obsevables

1.4.1 Expectation value

In many body physics often one is interested in the behaviour of macroscopic properties e.g. the magnetisation due to the expectation value of the Spin operator. In order to calculate expectation values one has to consider what happens with an operator O (always

taken to be hermitian) and the state of the system (mostly one is interested in the ground state) during the flow. The expectation value of an operator at zero temperature is defined by

$$\langle O\rangle_{gs} = \langle \Psi_{gs}|O|\Psi_{gs}\rangle \tag{1.70}$$

and $|\Psi_{gs}\rangle$ solves

$$H|\Psi_{gs}\rangle = E_{gs}|\Psi_{gs}\rangle, \tag{1.71}$$

where E_{gs} denotes the ground state energy and $|\Psi_{gs}\rangle$ the ground state, respectively. Applying the unitary transformation $U(B=\infty)$ a diagonalised Hamiltonian is obtained, whereupon the ground state has been altered

$$H(B=\infty)|\tilde{\Psi}_{gs}\rangle = E_{gs}|\tilde{\Psi}_{gs}\rangle, \tag{1.72}$$

with

$$|\tilde{\Psi}_{gs}\rangle = U(B=\infty)|\Psi_{gs}\rangle. \tag{1.73}$$

The new ground state is plugged into Eqs. (1.70) and one receives

$$\langle O\rangle_{gs} = \langle \tilde{\Psi}_{gs}|U(B=\infty)O(B=0)U^{\dagger}(B=\infty)|\tilde{\Psi}_{gs}\rangle \tag{1.74}$$

by redefining the operator as

$$O(B=\infty) = U(B=\infty)O(B=0)U^{\dagger}(B=\infty) \tag{1.75}$$

one realises the similarity to Eqs.(1.53) and therefore imposes a differential equation for the operator in terms of the flow parameter B

$$\boxed{\frac{dO(B)}{dB} = [\eta(B), O(B)]}. \tag{1.76}$$

The initial condition is $O(B=O)=O$, which is the original operator of interest. The initial operator, even when it is very simply at the beginning, will become more and more complicated durning the flow since new operator terms are generated. Formally one can write the operator as an infinite sum over products of creation and annihilation operators. Say C_n consists of some creation and annihilation operators and $c_n(O)$ are their evolution parameter depending on the original operator

$$O(B=\infty) = \sum_m c_n(O)C_n. \tag{1.77}$$

Once $H(B=\infty)$ is diagonal C_n can be chosen such that it fulfils the following relation

$$[H(B=\infty), C_n] = \Omega_n C_n \tag{1.78}$$

that will be very important to calculate the correlation function.

1.4.2 Correlation Function

Zero Temperature

The correlation function of two operators O_1 and O_2 at different times t_1 and t_2 and zero temperature is given by

$$C_{gs}(t_1, t_2) = \langle O_1(t_1) O_2(t_2) \rangle. \tag{1.79}$$

The operators are in the Heisenberg picture. Firstly, one changes to the Schrödinger picture

$$O(t) = e^{iHt} O(t=0) e^{-iHt} \tag{1.80}$$

and secondly the unitary transformation is used. Moreover Eqs. (1.77) is used to express the operators. After some straightforward manipulation one eventually receives

$$C_{gs}(t_1, t_2) = \sum_{n_1, n_2} c_{n_1}(O_1) c_{n_2}(O_2) e^{i\Omega_{n_1}(t_1 - t_2)} \langle \tilde{\Psi}_{gs} | C_{n_1} C_{n_2} | \tilde{\Psi}_{gs} \rangle, \tag{1.81}$$

where Eqs. (1.78) has also been used. From here one can deduce e.g. the Green's function, response funktion and so on. In the case of a not explicit time dependend Hamoltonian the correlation function just depends on the time difference $t_1 - t_2$. After performing a Fouier transformation one receives the frequency dependend correlation function

$$C_{gs}(\omega) = \int d(t_1 - t_2) e^{i\omega(t_1 - t_2)} C_{gs}(t_1 - t_2). \tag{1.82}$$

Nonzero Temperature

The nonzero temperature correlation function can be ascertained on the some footing as the zero temperature correlation function, though one has to use the density function approach to generalise the notion of expectation value

$$C_\beta(t_1, t_2) = \frac{1}{\tilde{Z}(\beta)} Sp(\rho(\beta) O_1(t_1) O_1(t_2)), \tag{1.83}$$

where β indicates the temperature dependence. So one can get all measurable many body quantities this equation either in terms of time or frequency with nonzero temperature, leading to

$$C_\beta(t_1, t_2) = \frac{1}{Z(\beta)} \sum_{j n_1 n_2} c_{n_1}(O_1) c_{n_2}(O_2) e^{-\beta E_j - i\Omega_{n_1}(t_1 - t_2} \langle j | C_{n_1} C_{n_2} | j \rangle \tag{1.84}$$

Although we will mainly be concerned about the zero temperature Quantum phase transtition that means we are especially interested in the zero temperature correlation function Eqs.(1.82). For a more pedagogical introduction to the flow equation method, see [53].

Chapter 2

The Ising Bose Fermi Kondo Model

The major advantage of the Ising BFKM compared to the isotropic BFKM is its accessibility by numerical methods such as NRG [39] or Quantum Monte Carlo methods [41], moreover, there are also analytical results such as Li *et.al.* [38] to compare with. So we take the Ising BFKM as testing ground for new kind of methods, since the numerical solutions go beyond a perturbative approach. In this chapter we will reproduce already know results, such as the flow of the coupling results, by applying the flow equation on the Ising BFKM. In other words we show the usefulness of the flow equation method for this model, in order to justify a flow equation approach for the isotropic BFKM, where numerical realisations are not present at the moment.

2.1 The Hamiltonian

Generally, the Hamiltonian of the Ising type system can be written as

$$\begin{aligned} H(B) &= \sum_{k,\sigma} \epsilon_k : c^\dagger_{k\sigma} c_{k\sigma} : + \sum_k \omega_k : b^\dagger_k b_k : \\ &\quad + \frac{1}{2} \sum_{pq} J_\parallel(p,q)(B)(: c^\dagger_{p\uparrow} c_{q\uparrow} : - : c^\dagger_{p\downarrow} c_{q\downarrow} :) S^z \\ &\quad + \frac{1}{2} \sum_{pq} J_\perp(p,q)(B)((: c^\dagger_{p\uparrow} c_{q\downarrow} : S^- + : c^\dagger_{q\downarrow} c_{p\uparrow} : S^+) \\ &\quad + S^z \sum_k \lambda_k(B)(b^\dagger_k + b_k) \\ &= H^f_0(B) + H^b_0(B) + H^{\perp,(f)}_{int}(B) + H^{\parallel,(f)}_{int}(B) + H^b_{int}(B), \end{aligned} \tag{2.1}$$

where the first line on the right hand side is related to the non-interacting part, denoted by H_0. The remaining lines combined are the interaction part H_{int}. The normal ordering occurs in two different form, as fermionic ordering and as bosonic ordering, but we think,

at least here, it should not be confusing[1]. Using Eqs.(1.66) we are able to determine the generator by calculating the commutator of the free and the interaction part of the Hamiltonian

$$\begin{aligned}\eta(B) &= [H_0^f(B)+H_0^b(B), H_{int}^{\perp,(f)}(B)+H_{int}^{\parallel,(f)}(B)+H_{int}^b(B)] \\ &= [H_0^f(B), H_{int}^{\perp,(f)}(B)+H_{int}^{\parallel,(f)}(B)]+[H_0^b(B), H_{int}^b(B)], \end{aligned} \tag{2.2}$$

we get

$$\begin{aligned}\eta_0^{\perp,(f)}(B)+\eta_0^{\parallel,(f)}(B) &= [H_0^f(B), H_{int}^{\perp,(f)}(B)+H_{int}^{\parallel,(f)}(B)] \\ &= \frac{1}{2}\sum_{pq}(\epsilon_p-\epsilon_q)J^{\parallel}(p,q)(B)(:c_{p\uparrow}^\dagger c_{q\uparrow}^\dagger:-:c_{p\downarrow}^\dagger c_{q\downarrow}^\dagger:)S^z \\ &\quad +\frac{1}{2}\sum_{pq}(\epsilon_p-\epsilon_q)J^{\perp}(p,q)(B)(:c_{p\uparrow}^\dagger c_{q\downarrow}^\dagger:S^- -:c_{q\downarrow}^\dagger c_{p\uparrow}^\dagger:S^+) \end{aligned} \tag{2.3}$$

and

$$\eta_0^b(B) = [H_0^b(B), H_{int}^b(B)] = S^z\sum_k \omega_k\lambda_k(b_k^\dagger - b_k). \tag{2.4}$$

By calculating

$$[\eta_0^{\parallel}(B)+\eta_0^{\perp}(B)+\eta_0^b(B), H_0^f(B)+H_0^b(B)+H_{int}^{\perp,(f)}(B)+H_{int}^{\parallel,(f)}(B)+H_{int}^b(B)] \tag{2.5}$$

we receive back coupling terms, which have already existed, and new operator product terms, see Appendix(B) for the respective commutators, and so we have to rewrite our ansatz for the Hamiltonian and the generator. But before doing so it is advised to have a closer look to the commutators themselves in order to truncate all terms that contain more then three operators one spin operator and two creation or annihilation operators of whatever sort, but the kept terms must have one creation and one annihilation operator. Normal ordered terms containing a higher number of operators will be omitted, because they produce a higher order in scattering. Likewise we only keep those terms in coupling up to $\mathcal{O}(J^3)$, $\mathcal{O}(J^2\lambda)$ and $\mathcal{O}(J\lambda^2)$. Moreover, we skip terms like Eqs.(A.7), due to the fact, that they only produce a shift of the energy by a constant. Finally, we end up with the

[1] If it is not entirely clear we will make a comment.

following extended version of the Hamiltonian and of the generator respectively

$$\begin{aligned}
\eta(B) &= \frac{1}{2}\sum_{pq} J_{\parallel}(p,q)(B)(\epsilon_p-\epsilon_q)(:c^\dagger_{p\uparrow}c_{q\uparrow}:-:c^\dagger_{p\downarrow}c_{q\downarrow}:)S^z \\
&+\frac{1}{2}\sum_{pq} J_{\perp}(p,q)(B)(\epsilon_p-\epsilon_q)((:c^\dagger_{p\uparrow}c_{q\downarrow}:S^- - :c^\dagger_{q\downarrow}c_{p\uparrow}:S^+) \\
&+\sum_{k}\lambda_k(B)\omega_k(b^\dagger_k-b_k)S^z \\
&+\sum_{kpq} K_{kpq}(B)((\epsilon_p-\epsilon_q)-\omega_k)(b_k:c^\dagger_{p\uparrow}c_{q\downarrow}:S^- - b^\dagger_k:c^\dagger_{q\downarrow}c_{p\uparrow}:S^+) \\
&+\sum_{kpq} L_{kpq}((\epsilon_p-\epsilon_q)+\omega_k)(b^\dagger_k:c^\dagger_{p\uparrow}c_{q\downarrow}:S^- - b_k:c^\dagger_{q\downarrow}c_{p\uparrow}:S^+) \\
&= \eta_0^{\perp,(f)}(B)+\eta_0^{\parallel,(f)}+\eta_0^b(B)+\eta^K_{\rm new}(B)+\eta^L_{\rm new}(B) \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
H(B) &= \sum_{k,\sigma}\epsilon_k:c^\dagger_{k\sigma}c_{k\sigma}:+\sum_k\omega_k:b^\dagger_k b_k: \\
&+\frac{1}{2}\sum_{pq} J_{\parallel}(p,q)(B)(:c^\dagger_{p\uparrow}c_{q\uparrow}:-:c^\dagger_{p\downarrow}c_{q\downarrow}:)S^z \\
&+\frac{1}{2}\sum_{pq} J_{\perp}(p,q)(B)(:c^\dagger_{p\uparrow}c_{q\downarrow}:S^- + :c^\dagger_{q\downarrow}c_{p\uparrow}:S^+) \\
&+\sum_{k}\lambda_k(B)(b^\dagger_k+b_k)S^z \\
&+\sum_{kpq} K_{kpq}(B)(b^\dagger_k:c^\dagger_{q\downarrow}c_{p\uparrow}:S^+ + b_k:c^\dagger_{p\uparrow}c_{q\downarrow}:S^-) \\
&+\sum_{kpq} L_{kpq}(B)(b_k:c^\dagger_{q\downarrow}c_{p\uparrow}:S^+ + b^\dagger_k:c^\dagger_{p\uparrow}c_{q\downarrow}:S^-) \\
&= H_0^f(B)+H_0^b(B)+H_{int}^{\perp,(f)}(B)+H_{int}^{\parallel,(f)}(B)+H_{int}^b(B) \\
&+H^K_{\rm new}(B)+H^L_{\rm new}(B) \quad , \qquad (2.7)
\end{aligned}$$

where the newly introduced couplings are given by

$$K_{kpq}(B)=\frac{1}{2}\lambda_k(B)J_\perp(p,q)(B)((\epsilon_p-\epsilon_q)+\omega_k) \qquad (2.8)$$

and

$$L_{kpq}=\frac{1}{2}\lambda_k(B)J_\perp(p,q)(B)((\epsilon_p-\epsilon_q)-\omega_k). \qquad (2.9)$$

The new coupling constants have been already produced in order of $\mathcal{O}(J\lambda)$, which simplifies our task of calculating the flow equations for the couplings, by disregarding higher

coupling terms like $[\eta^K_{\rm new}(B)+\eta^L_{\rm new}(B), H^K_{\rm new}(B)+H^L_{\rm new}(B)]$. The question is, why we even ought to take $K_{kpq}(B)$ and $L_{kpq}(B)$ into account?
The main reason is, the commingling of both the fermionic degrees of freedom and the bosonic degrees of freedom with the local spin operator, leading to an effective coupling of the fermionic part with the bosonic part, have to be taken into account to grasp the essential physics of the system.

We have already calculated the commutators of $[\eta_{\rm old}, H_{\rm old}]$, those commutators, which are left such as $[\eta_{\rm new}, H_{\rm old}]$ and $[\eta_{\rm old}, H_{\rm new}]$, are presented in the Appendix(B.2).

2.2 Flow equations

Eventually, the differential equations for the couplings are obtained by considering the back coupling of the commutators to the new Hamiltonian in leading order of the flow equation analysis. Certain precautions are necessary, since ordinary scattering, energy shift and higher order terms are going to occure and it is easy to get lost in the sometimes quite messy expression of particular commutators. In the following we will present the coupling equations. We do not even try to solve them analytically in the below exposed form and further approximations have to be made

$$\begin{aligned}\frac{dJ^{\parallel}(p,q)(B)}{dB} &= -(\epsilon_p-\epsilon_q)^2 J^{\parallel}(p,q)(B)\\ &\quad +\frac{1}{2}\sum_n (1-2n_f(n))(2\epsilon_n-\epsilon_p-\epsilon_q)J^{\perp}(p,n)(B)J^{\perp}(q,n)(B) \quad (2.10)\end{aligned}$$

and

$$\begin{aligned}\frac{dJ^{\perp}(p,q)(B)}{dB} &= -(\epsilon_p-\epsilon_q)^2 J^{\perp}(p,q)(B)\\ &\quad +\frac{1}{4}\sum_n (1-2n_f(n))(2\epsilon_n-\epsilon_p-\epsilon_q)J^{\parallel}(p,n)J^{\perp}(q,n)\\ &\quad +\frac{1}{4}\sum_n (1-2n_f(n))(2\epsilon_n-\epsilon_p-\epsilon_q)J_{\perp}(p,n)(B)J_{\parallel}(q,n)(B)\\ &\quad +2\sum_k \omega_k\lambda_k(B)(L_{kpq}(B)-K_{kpq}(B))(1+2n_b(k))\\ &\quad +\sum_k (\epsilon_p-\epsilon_q)\lambda_k(B)(L_{kpq}(B)+K_{kpq}(B))(1+2n_b(k)) \quad (2.11)\end{aligned}$$

and

$$\begin{aligned}\frac{d\lambda_k(B)}{dB} &= -\omega_k^2\lambda_k(B)+\frac{1}{2}\sum_{pq}\omega_k J^{\perp}(p,q)(B)(K_{kpq}(B)-L_{kpq}(B))\\ &\quad \times(n_f(p)+n_f(q)-2n_f(p)n_f(q)) \quad (2.12)\end{aligned}$$

and

$$\begin{aligned}\frac{dK_{kpq}(B)}{dB} &= -K_{kpq}((\epsilon_p-\epsilon_q)-\omega_k)^2+\frac{1}{2}\lambda_k(B)J^{\perp}(p,q)((\epsilon_p-\epsilon_q)+\omega_k)\\ &+\frac{1}{4}\sum_n(2\epsilon_n-\epsilon_p-\epsilon_q-\omega_k)J^{\parallel}(p,n)(B)K_{knq}(B)(1-2n_f(n))\\ &+\frac{1}{4}\sum_n(2\epsilon_n-\epsilon_p-\epsilon_q+\omega_k)J^{\parallel}(q,n)(B)K_{kpn}(B)(1-2n_f(n)) \quad (2.13)\end{aligned}$$

and

$$\begin{aligned}\frac{dL_{kpq}(B)}{dB} &= -L_{kpq}((\epsilon_p-\epsilon_q)+\omega_k)^2+\frac{1}{2}\lambda_k(B)J^{\perp}(p,q)(B)((\epsilon_p-\epsilon_q)-\omega_k)\\ &+\frac{1}{4}\sum_n(2\epsilon_n-\epsilon_p-\epsilon_q-\omega_k)J^{\parallel}(n,q)(B)L_{kpn}(1-2n_f(n))\\ &+\frac{1}{4}\sum_n(2\epsilon_n-\epsilon_p-\epsilon_q+\omega_k)J^{\parallel}(p,n)(B)L_{knq}(B)(1-2n_f(n)). \quad (2.14)\end{aligned}$$

2.3 The infrared Approximation

As it was mentioned in the previous section we need to find an appropriate approximation to the flow equations. A good choice is the so called infrared approximation, where we separate the exponential factor that comprises the momentum dependence from the bare infrared part of the coupling constant. Except for the newly generated terms the momentum dependence of the residual coupling is needed for the ongoing calculation of the Kondo and bosonic coupling. By doing so we obtain the following ansatz

$$J^{\perp}(p,q)(B) = J^{\perp}_{IR}(B)e^{-B(\epsilon_p-\epsilon_q)^2} \tag{2.15}$$

$$J^{\parallel}(p,q)(B) = J^{\parallel}_{IR}(B)e^{-B(\epsilon_p-\epsilon_q)^2} \tag{2.16}$$

$$\lambda_k(B) = \lambda_{IR}(B)e^{-B\omega_k^2} \tag{2.17}$$

$$K_{kpq}(B) = \tilde{K}_{kpq}(B)e^{-B((\epsilon_p-\epsilon_q)-\omega_k)^2} \tag{2.18}$$

$$L_{kpq}(B) = \tilde{K}_{kpq}(B)e^{-B((\epsilon_p-\epsilon_q)+\omega_k)^2}. \tag{2.19}$$

The strategy is to determine the coupling constants $L_{pqk}(B)$ and $K_{pqk}(B)$ in terms of infrared Kondo couplings $J^{\parallel}_{IR}(B)$ and $J^{\perp}_{IR}(B)$, and the infrared bosonic bath coupling $\lambda_{IR}(B)$ that means we first instert Eqs.(2.15), Eqs.(2.17) and Eqs.(2.19) into Eqs.(2.14), furthermore, we just want to know the behaviour of the couplings up to the order of $\mathcal{O}(J^2)$ or $\mathcal{O}(J\lambda)$, hence we omit the last two lines in Eqs.(2.14), because $L_{pqk}(B)$ is already generated in second order. The actual flow equation of $L_{pqk}(B)$ reduces to

$$\frac{d\tilde{L}_{pqk}(B)}{dB} = \frac{1}{2}\lambda_{IR}(B)J^{\perp}_{IR}(B)((\epsilon_p-\epsilon_q)-\omega_k)e^{2B\omega_k(\epsilon_p-\epsilon_q)}. \tag{2.20}$$

Integration over B, from zero up to B' leads to

$$\tilde{L}_{kpq}(B') = \frac{1}{2}((\epsilon_p-\epsilon_q)-\omega_k)\int_0^{B'} dB \quad \lambda_{IR}(B)J^{\perp}_{IR}(B)e^{2B\omega_k(\epsilon_p-\epsilon_q)}, \tag{2.21}$$

where $L_{pqk}(B=0)=0$, since our original Hamiltonian did not depend on it at all. By using the mean value theorem for integration we get

$$\tilde{L}_{kpq}(B') = \frac{1}{2}((\epsilon_p-\epsilon_q)-\omega_k)\frac{1}{B}\int_0^{B'} dB \quad \lambda_{IR}(B)J^{\perp}_{IR}(B)\int_0^{B'} dB \quad e^{2B\omega_k(\epsilon_p-\epsilon_q)}. \tag{2.22}$$

Remember, the infrared coupling varies slowly with B in comparison to the exponential factor, and so we consider the product $\lambda_{IR}(B)\cdot J^{\perp}_{IR}(B)$ as almost independent of B, but

notice, this is only a valid approximation, if we take the exponential factor explicitly into account. In other words, the exponentially fast decay of the e-function suppresses the onset of the variation of the infrared couplings. After integration, the $L_{pqk}(B)$-coupling is then given by

$$\tilde{L}_{kpq}(B) = \frac{1}{2}\frac{\lambda_{IR}(B)J^{\perp}_{IR}(B)((\epsilon_p - \epsilon_q) - \omega_k)}{2\omega_k(\epsilon_p - \epsilon_q)}\left(e^{2B\omega_k(\epsilon_p-\epsilon_q)} - 1\right), \tag{2.23}$$

inserting into Eqs.(2.19) gives

$$L_{kpq}(B) = \frac{1}{2}\frac{\lambda_{IR}(B)J^{\perp}_{IR}(B)((\epsilon_p - \epsilon_q) - \omega_k)}{2\omega_k(\epsilon_p - \epsilon_q)}\left(e^{-B((\epsilon_p-\epsilon_q)^2+\omega_k^2)} - e^{-B((\epsilon_p-\epsilon_q)+\omega_k)^2}\right). \tag{2.24}$$

Further simplifications can be made by merely considering the physics close to the Fermi surface, where the Fermi-energy is defined as $\epsilon_F = 0$, which treads the physics at temperature $T = 0$ and is of our main concern. On the one hand we take the limit where the momentum dependent energies go to zero, e.g. ϵ_p and $\epsilon_p \to 0$ and on the other hand the Fermi-Dirac function reduces at $T = 0$ to

$$n_f(\epsilon) = \begin{cases} 1, & \text{if } \epsilon < 0, \\ 0, & \text{if } \epsilon > 0, \end{cases} \tag{2.25}$$

and the Bose-Einstein statistic to

$$n_b(\omega) = 0 \quad \text{for all } \omega > 0. \tag{2.26}$$

By taking the limits ϵ_p and $\epsilon_p \to 0$ in Eqs.(2.24) we finally have the following expression for the $L_{pqk}(B)$-coupling

$$Ł_{k00} = -\frac{1}{2}\lambda_{IR}(B)J^{\perp}_{IR}(B)\omega_k B e^{-B\omega_k^2}. \tag{2.27}$$

On the very same footing we get an expression for the $K_{pqk}(B)$-coupling

$$K_{kpq} = +\frac{1}{2}\lambda_{IR}(B)J^{\perp}_{IR}(B)\omega_k B e^{-B\omega_k^2}. \tag{2.28}$$

Now, with these approximations we can calculate the flow of the couplings $J^{\perp}_{IR}(B)$ and $J^{\parallel}_{IR}(B)$, so Eqs.(2.10) simplifies as follows

$$\begin{aligned} \frac{J^{\parallel}_{IR}(B)}{dB} &= (J^{\perp}_{IR}(B))^2 \sum_n \epsilon_n e^{-2B\epsilon_n^2}(1 - 2n_f(n)) \\ &= \rho_0 (J^{\perp}_{IR}(B))^2 \int_{-D}^{D} d\epsilon \quad \epsilon e^{-2B\epsilon^2}(1 - 2n_f(\epsilon)) \\ &= \frac{\rho_0}{2}\frac{(J^{\perp}_{IR}(B))^2}{B}(1 - e^{-2BD^2}). \end{aligned} \tag{2.29}$$

As we can read off, it is perfectly fine to neglect the flow of the coupling, while $B \leq D^{-2}$. For greater B's the flow of the coupling sets in and the flow equation becomes

$$\frac{J_{IR}^{\parallel}(B)}{dB} = \frac{\rho_0}{2}\frac{(J_{IR}^{\perp}(B))^2}{B}. \tag{2.30}$$

Notice that we have gained the same coupling eqaution for $J_{IR}^{\parallel}(B)$, we have already been familiar with from the Kondo model [3]. We implicitly assumed a constant density of states ρ_0 for the fermions, which have been introduced by the change of the sum into an integral

$$\sum \longrightarrow \int \rho(\epsilon) \quad d\epsilon. \tag{2.31}$$

But, before we procced on in establishing the flow equation for $J_{IR}^{\perp}(B)$ we have to talk about the bosonic part, which explicitly enters here and will alter the flow of $J_{IR}^{\perp}(B)$ in comparison to the Kondo model. The bosonic density of states, often denoted as spectral density, is given by

$$S(\omega) = \sum_k \lambda_k^2 \delta(\omega - \omega_k) = 2\alpha\omega^s \Theta(\omega_c - \omega)^2, \tag{2.32}$$

where ω_c is a cut-off parameter, and α determines decoherence effects, see [55].

We go back to Eqs.(2.11), and by using the assumption stated before, we receive

$$\begin{aligned}\frac{dJ_{IR}^{\perp}(B)}{dB} &= \sum_m (1-2n_f(n))\epsilon_n J_{IR}^{\perp}(B) J_{IR}^{\parallel}(B) e^{-2B\epsilon_n} \\ &\quad -2\sum_k \omega_k^2 J_{IR}^{\perp}(B)\lambda_{IR}^2(B) e^{-2B\omega_k}.\end{aligned} \tag{2.33}$$

Now, we use the spectral density to rewrite the sum over k as an integral and the same reasoning as for the $J_{IR}^{\parallel}(B)$-coupling holds here for the fermionic part, therefore the flow equation becomes

$$\begin{aligned}\frac{dJ_{IR}^{\perp}(B)}{dB} &= \frac{\rho_0}{2}\frac{J_{IR}^{\perp}(B)J_{IR}^{\parallel}(B)}{B}(1-e^{-2BD^2}) \\ &\quad -4\alpha\tilde{J}_{IR}^{\perp}(B)B\int_0^{\omega_c} d\omega \quad \omega^{s+2}e^{-2B\omega^2} \\ &= \frac{\rho_0}{2}\frac{J_{IR}^{\perp}(B)J_{IR}^{\parallel}(B)}{B} - \frac{4\alpha J_{IR}^{\perp}(B)}{(2B)^{\frac{1}{2}(s+3)}}B\int_0^{\sqrt{2B}\omega_c} dx \quad x^{s+2}e^{-x^2} \\ &= \frac{\rho_0}{2}\frac{J_{IR}^{\perp}(B)J_{IR}^{\parallel}(B)}{B} - \frac{4\alpha J_{IR}^{\perp}(B)}{(2B)^{\frac{1}{2}(s+5)}}B\underbrace{\int_0^{2B\omega_c^2} dy \quad y^{(\frac{1}{2}(s+3))-1}e^{-y}}_{=\Gamma(\frac{1}{2}(s+3)),\quad B \text{ large enough}},\end{aligned} \tag{2.34}$$

[2]The meaning of the parameter s has been explained in Chapter 1.

the second line is obtained by substituting $\omega = \frac{x}{\sqrt{2B}}$ and the last line by $x = \sqrt{y}$.

The bosonic coupling equation is simply just

$$\frac{d\lambda_k(B)}{dB} = -\omega_k^2 \lambda_k(B) \tag{2.35}$$

in leading order. We insert our ansatz and see that this equation is only fulfilled as long as $\lambda_{IR}(B)$ is a constant, and thus its value is determined by the intitial strength of the bosonic coupling.

The remaining unsolved flow equations are

$$\boxed{\frac{dJ_{IR}^{\perp}(B)}{dB} = \frac{\rho_0}{2}\frac{J_{IR}^{\perp}(B)J_{IR}^{\parallel}(B)}{B} - \frac{\alpha J_{IR}^{\perp}(B)}{2^{\frac{1}{2}(s+1)}}\frac{1}{B^{\frac{1}{2}(s+1)}}\Gamma(\frac{1}{2}(s+3))} \tag{2.36}$$

and

$$\boxed{\frac{dJ_{IR}^{\parallel}(B)}{dB} = \frac{\rho_0}{2}\frac{(J_{IR}^{\perp}(B))^2}{B}}. \tag{2.37}$$

For our purposes its enough to consider the $s = 1$ (ohmic) case, which reduces the complexity of the flow equations to an easily solvable set of differential equations

$$\frac{dJ_{IR}^{\perp}(B)}{dB} = \frac{J_{IR}^{\perp}(B)}{B}(\frac{\rho_0}{2}J_{IR}^{\parallel}(B) - \frac{\alpha}{2}) \tag{2.38}$$

and

$$\frac{dJ_{IR}^{\parallel}(B)}{dB} = \frac{\rho_0}{2}\frac{(J_{IR}^{\perp}(B))^2}{B}, \tag{2.39}$$

by redefining $g^{\parallel}(B) = \frac{\rho_0}{2}J_{IR}^{\parallel}(B) - \frac{\alpha}{2}$ and $g^{\perp}(B) = \frac{\rho_0}{2}J_{IR}^{\perp}(B)$ we receive the well known differential equations for the anisotropic Kondo model, shifted only by a constant,

$$\boxed{\frac{dg^{\parallel}(B)}{dB} = \frac{(g^{\perp}(B))^2}{B}} \tag{2.40}$$

and

$$\boxed{\frac{dg^{\perp}(B)}{dB} = \frac{g^{\perp}(B)g^{\parallel}(B)}{B}}. \tag{2.41}$$

The shift of the coupling in the parallel direction is give by $\frac{\alpha}{2}$ and is exactly the result obtained by Li *et.al.* [38].

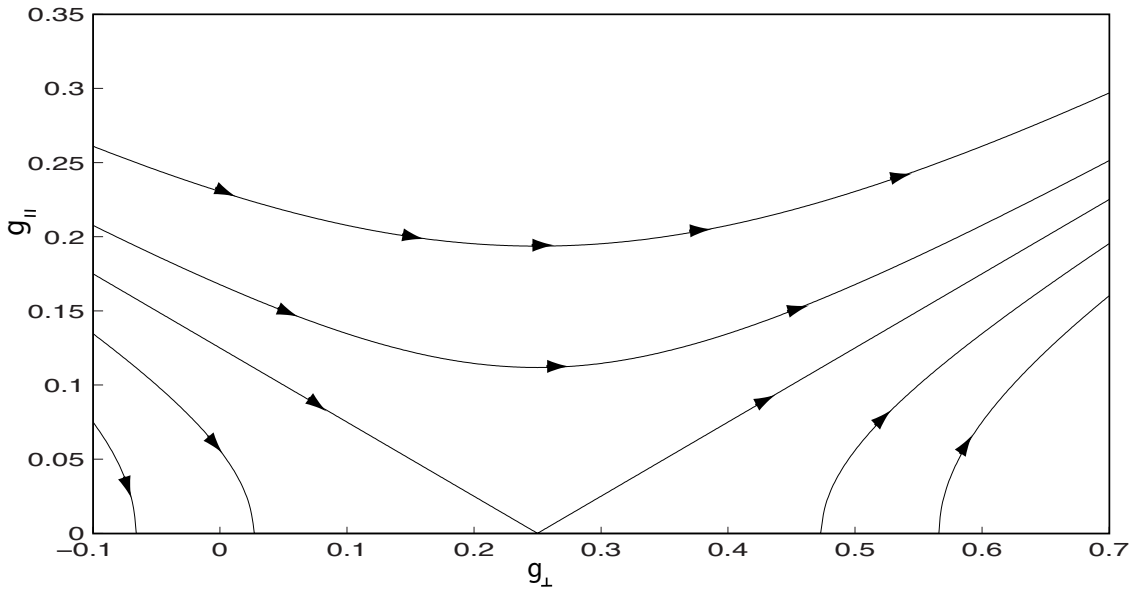

Figure 2.1: The flow of the new defined dimensionsless coupling constants $g^{\parallel}(B) = \frac{\rho_0}{2} J^{\parallel}_{IR}(B) - \frac{\alpha}{2}$ and $g^{\perp}(B) = \frac{\rho_0}{2} J^{\perp}_{IR}(B)$. On the contrary to the flow of the original Kondo-couplings Fig.1.1 the finite coupling region is extended by the dissipation parameter α (In this plot the value of α is taken to be 0.25). So if we increase the value of α the region, with non divergent coupling constants, is enlarged. Actually, the set of the value of possible initial conditions that end up having a finite value after the flow, is enlarged.

2.3.1 Solution of the Flow Equation with $T \neq 0$

Now, we can ask the question, how does a non zero temperature change the flow equations, and will they still have similarities to the ordinary anisotropic Kondo model? We go back to the to the flow equations before we set $T = 0$ and use the well known Bose-Enstein and Fermi-Dirac statistic.

We rewrite the couplings as $J^{\perp}_{IR}\rho_0 = \tilde{g}_{\perp}$ and $J^{\parallel}_{IR}\rho_0 = \tilde{g}_{\parallel}$ and hence the differential equations become

$$\frac{d\tilde{g}_{\parallel}}{dB} = \tilde{g}_{\perp}^2 \int_{-\infty}^{\infty} d\epsilon \quad \epsilon e^{-2B\epsilon^2}(1 - 2n_f(\epsilon)) \tag{2.42}$$

and

$$\begin{aligned} \frac{d\tilde{g}_{\perp}}{dB} &= \tilde{g}_{\perp}\tilde{g}_{\parallel} \int_{-\infty}^{\infty} d\epsilon \quad \epsilon e^{-2B\epsilon^2}(1 - 2n_f(\epsilon)) \\ &\quad -4\alpha\tilde{g}_{\perp} B \int_0^{\infty} d\omega \quad \omega^{s+2} e^{-2B\omega^2}(1 + 2n_b(\omega)). \end{aligned} \tag{2.43}$$

We also rewrite the Bose-Einstein and Fermi-Dirac statistic in a different representation

$$(1 - 2n_f(\epsilon)) = 1 - \frac{2}{e^{\frac{\epsilon}{T}} + 1} = \tanh(\frac{\epsilon}{2T}) \tag{2.44}$$

and

$$(1+2n_b(\omega)) = 1+\frac{2}{e^{\frac{\omega}{T}}-1} = \coth(\frac{\omega}{2T}), \tag{2.45}$$

which has the advantage of a straightforward linearisation of the new expressions around $\frac{\epsilon}{2T}=0$. But before we make use of this, it has to be mentioned that there are basically two different limiting cases we are interested, in the first one, we consider the region where $BT^2 \ll 1$. As B is the inverse of the squared cut off energy, we refer to this limit as the high energy limit, in other words the energy is much higher then the temperature and therefore the scaling equations for the couplings Eqs.(2.36) and Eqs.(2.37) we have obtained by setting $T=0$ can be adopt for this case as well.

In the other case the temperature is greater than the energy $BT^2 \gg 1$ and therefore the respective statistics have to be taken into account. First we expand $\tanh(\frac{\epsilon}{2T}) = \frac{\epsilon}{2T}$ up to first order and plug it into Eqs.(2.42) and we get

$$\begin{aligned}
\frac{d\tilde{g}_{\parallel}}{dB} &= \tilde{g}_{\perp}^2 \int_{-\infty}^{\infty} d\epsilon \quad \epsilon e^{-2B\epsilon^2}(1-2n_f(\epsilon)) \\
&= \tilde{g}_{\perp}^2 \int_{-\infty}^{\infty} d\epsilon \quad \epsilon tanh(\frac{\epsilon}{2T}) \\
&= \tilde{g}_{\perp}^2 \int_{-\infty}^{\infty} d\epsilon \frac{\epsilon^2}{2T} \\
&= \frac{\sqrt{2\pi}}{16}\frac{1}{TB^{\frac{3}{2}}}
\end{aligned} \tag{2.46}$$

For the second differential equation we consider the ohmic case where s=1, only. We expand $\coth(\frac{\omega}{2T})$ for small arguments hence for small frequencies or large temperatures

$$coth(\frac{\omega}{2T}) = \underbrace{\frac{2T}{\omega}}_{A} + \underbrace{\frac{1}{3}\frac{\omega}{2T}}_{B}. \tag{2.47}$$

So for term A we obtain

$$-8\alpha BT\int_0^{\infty} d\omega\omega^2 e^{-2B\omega^2} \underbrace{=}_{\omega=\frac{x}{\sqrt{2B}}} -\frac{4}{\sqrt{2}}\frac{\alpha T}{B^{\frac{1}{2}}}\int_0^{\infty} dx x^2 e^{-x^2} = \sqrt{\frac{\pi}{2}}\frac{\alpha T}{B^{\frac{1}{2}}} \tag{2.48}$$

and for the B-term

$$-\frac{3}{2}\frac{\alpha B}{T}\int_0^{\infty} d\omega\omega^4 e^{-2B\omega^2} \underbrace{=}_{\omega=\frac{x}{\sqrt{2B}}} -\frac{1}{6\sqrt{2}}\frac{\alpha}{TB^{\frac{3}{2}}}\int_0^{\infty} dx x^4 e^{-x^2} = -\frac{\sqrt{\pi}}{16\sqrt{2}}\frac{\alpha}{TB^{\frac{3}{2}}}. \tag{2.49}$$

If we put all this together we receive for the flow equation of $\tilde{g}_{\perp}$

$$\frac{d\tilde{g}_{\perp}}{dB} = \frac{\sqrt{2\pi}}{16}\frac{1}{TB^{\frac{3}{2}}}(\tilde{g}_{\perp}\tilde{g}_{\parallel} - 8\alpha\underbrace{T^2B}_{\gg 1} - \underbrace{\frac{\alpha}{2}}_{=const}), \tag{2.50}$$

since we are in the $BT^2 \gg 1$ region and the third term on the right hand side is constant, we can omit the third term in favor of the second term on the right hand side. So in the limit of $BT^2 \gg 1$ we get for the complete flow equations

$$\boxed{\frac{d\tilde{g}_\perp}{dB} = \tilde{g}_\perp \tilde{g}_\parallel \frac{\sqrt{2\pi}}{16} \frac{1}{TB^{\frac{3}{2}}} - \frac{\sqrt{2\pi}}{2} \frac{\alpha T}{B^{\frac{1}{2}}}} \tag{2.51}$$

and

$$\boxed{\frac{d\tilde{g}_\parallel}{dB} = \tilde{g}_\perp^2 \frac{\sqrt{2\pi}}{16} \frac{1}{TB^{\frac{3}{2}}}} \tag{2.52}$$

Note that the temperature dependence of the dissipative term is, irrespective of the prefactors, the same as the second loop contribution in the Kondo model, see [53].
Even more can be said as a final remark of this chapter, the one loop calculations of the Ising BFKM, to be precise the contributions coming form the dissipative bath, have lots of similarities to the second loop contribution of the ordinary anisotropic Kondo model.

Chapter 3

The isotropic Bose Fermi Kondo Model

In the isotropic case all of the fermionic couplings are equal and likewise the bosonic couplings. The investigation of the isotropic BFKM in terms of numerics, such as NRG, could not be done so far, since it is not possible to model three bosonic baths at once, yet the flow equation method is in principal able to solve the model. In this section we are going to present not an exact solution but we give a sufficiently well perturbative solution. The perturbatively obtained correlation function can be used to analyse the EDMFT self consistency equation. Additionally, we consider the idea of the existence of a conformal field theory in the immediate vicinity of the quantum phase transition proposed by Kirchner and Si [56].

3.1 Hamiltonian

The Hamiltonian of the BFKM is given by

$$\begin{aligned} H &= \sum_{p\alpha} \epsilon_p : c^\dagger_{p\alpha} c_{p\alpha} : + \sum_{pq} J(p,q) : \mathbf{S} \cdot \mathbf{s}_{pq} : \\ &+ \sum_k \omega_k : \mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}_k : + \sum_k \lambda_k \mathbf{S} \cdot (\mathbf{\Phi}^\dagger_k + \mathbf{\Phi}_k), \end{aligned} \tag{3.1}$$

where $\mathbf{s}_{pq} = \sum_{\alpha\beta=1}^{2} c^\dagger_{p\alpha} \frac{1}{2} \sigma_{\alpha\beta} c_{q\beta}$ and $\mathbf{\Phi}^\dagger_k$, $\mathbf{\Phi}_k$ are bosonic vector operators.

The generator is gained by Eqs.(1.66) and reads

$$\begin{aligned} \eta(B) &= \sum_{pq} (\epsilon_p - \epsilon_q) J(p,q)(B) : \mathbf{S} \cdot \mathbf{s}_{pq} : \\ &+ \sum_k \lambda_k(B) \omega_k \mathbf{S} \cdot (\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k). \end{aligned} \tag{3.2}$$

While inserting this into Eqs.(1.55) and performing the commutator new terms are generated durning the flow, hence we have to rewrite the original Hamiltonian, which as well changes the generator, as it follows

$$\begin{aligned} H(B) &= \sum_{p\alpha} \epsilon_p : c^\dagger_{p\alpha} c_{p\alpha} : + \sum_{pq} J(p,q)(B) : \mathbf{S}\cdot\mathbf{s}_{pq} : \\ &+ \sum_k \omega_k : \mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}_k : + \sum_k \lambda_k(B) \mathbf{S}\cdot(\mathbf{\Phi}^\dagger_k + \mathbf{\Phi}_k) \\ &+ i\sum_{pqk} K_{pqk}(B) \mathbf{\Phi}^\dagger_k \cdot : \mathbf{s}_{pq} \times \mathbf{S} : \\ &+ i\sum_{pqk} L_{pqk}(B) \mathbf{\Phi}^{\cdot}_k : \mathbf{s}_{pq} \times \mathbf{S} : \\ &+ i\sum_{kl} \mu_{kl}(B) \mathbf{S}\cdot : \mathbf{\Phi}^\dagger_k \times \mathbf{\Phi}_l : \\ &+ i\sum_{kl} \Psi_{kl}(B) \mathbf{S}\cdot(\mathbf{\Phi}^\dagger_k \times \mathbf{\Phi}^\dagger_l - \mathbf{\Phi}_k \times \mathbf{\Phi}_l) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \eta(B) &= \sum_{pq} (\epsilon_p - \epsilon_q) J(p,q)(B) : \mathbf{S}\cdot\mathbf{s}_{pq} : + \sum_k \lambda_k(B)\omega_k \mathbf{S}\cdot(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k) \\ &+ i\sum_{pqk} K_{pqk}(B)((\epsilon_p - \epsilon_q) + \omega_k) \mathbf{\Phi}^\dagger_k \cdot : \mathbf{s}_{pq} \times \mathbf{S} : \\ &+ i\sum_{pqk} L_{pqk}(B)((\epsilon_p - \epsilon_q) - \omega_k) \mathbf{\Phi}^{\cdot}_k : \mathbf{s}_{pq} \times \mathbf{S} : \\ &+ i\sum_{kl} \mu_{kl}(B)(\omega_k - \omega_l) \mathbf{S}\cdot : \mathbf{\Phi}^\dagger_k \times \mathbf{\Phi}_l : \\ &+ i\sum_{kl} \Psi_{kl}(B)(\omega_k + \omega_l) \mathbf{S}\cdot(\mathbf{\Phi}^\dagger_k \times \mathbf{\Phi}^\dagger_l + \mathbf{\Phi}_k \times \mathbf{\Phi}_l), \end{aligned} \tag{3.4}$$

where we introduced new couplings

$$K_{pqk}(B) = -J(p,q)(B)\lambda_k(B)((\epsilon_p - \epsilon_q) - \omega_k) \tag{3.5}$$

$$L_{pqk}(B) = -J(p,q)(B)\lambda_k(B)((\epsilon_p - \epsilon_q) + \omega_k) \tag{3.6}$$

$$\mu_{kl}(B) = (\omega_k + \omega_l)\lambda_l(B)\lambda_k(B) \tag{3.7}$$

$$\Psi_{kl}(B) = \omega_k \lambda_k(B)\lambda_l.(B) \tag{3.8}$$

The coupling constants obey the following relations, for any B , since the Hamiltonian has to be hermitian

$$K_{pqk} = -L_{qpk} \qquad \text{and} \qquad \mu_{kl} = \mu_{lk}. \tag{3.9}$$

3.1.1 The Commutators

The overall commutation relation reads:

$$[\eta_0^f(B) + \eta_0^b(B) + \eta_{new}(B), H_0^f(B) + H_0^b(B) + H_{int}^f(B) + H_{int}^b(B) + H_{new}(B)] \quad (3.10)$$

where

$$\eta_{new}(B) = \eta_{new}^K(B) + \eta_{new}^L(B) + \eta_{new}^\mu(B) + \eta_{new}^\Psi(B) \quad (3.11)$$

and

$$H_{new}(B) = H_{new}^K(B) + H_{new}^L(B) + H_{new}^\mu(B) + H_{new}^\Psi(B). \quad (3.12)$$

Notice that we neglect terms like

$$[\eta_{new}^K(B)+\eta_{new}^L(B)+\eta_{new}^\mu(B)+\eta_{new}^\Psi(B), H_{new}^K(B)+H_{new}^L(B)+H_{new}^\mu(B)+H_{new}^\Psi(B)] \quad (3.13)$$

from the outset, because they will produce couplings in higher order than we are interested. To be precise we neglect couplings of order $\mathcal{O}(\lambda^4)$,$\mathcal{O}(J\lambda^3)$ and $\mathcal{O}(J^2\lambda^2)$. The calculations of the commutator are rather cumbersome and therefore we delayed the quite lengthy expression into the Appendix(B), since the calculations are not at all necessary for the understanding of the following discussion, nevertheless, they have to be done. In the next section we will present the flow equations of the couplings in all its detail, at least up to the order of our interest.

3.2 Flow Equation

3.2.1 The Flow Equations

Once again Appendix(B) displays the commutator relations of the generator and Hamiltonian, however, we only consider those terms, which couple back to the Hamiltonian given in Eqs(3.3) and so we end up with a quite complicated set of coupled differential equations

$$\begin{aligned}
\frac{dJ(p,q)(B)}{dB} &= -(\epsilon_p - \epsilon_q)^2 J(p,q)(B) \\
&\quad - \sum_k K_{pqk}(B)((\epsilon_p - \epsilon_q) + 2\omega_k)\lambda_k(B)(1 + 2n(k)) \\
&\quad - \sum_k L_{pqk}(B)((\epsilon_p - \epsilon_q) - 2\omega_k)\lambda_k(B)(1 + 2n(k)) \\
&\quad + \sum_m (\epsilon_p + \epsilon_q - 2\epsilon_m) J(p,m)(B) J(m,q)(B)(n(m) - \frac{1}{2})
\end{aligned} \quad (3.14)$$

and

$$\begin{aligned}\frac{d\lambda_k(B)}{dB} &= -\omega_k^2\lambda_k(B) \\ &\quad +\frac{1}{2}\sum_{pq} K_{pqk}(B)\omega_k J(p,q)(B)(n(p)+n(q)-2n(p)n(q)) \\ &\quad +\sum_l(\omega_k-2\omega_l)\lambda_l(B)\mu_{kl}(B)(1+2_b(l)) \\ &\quad +\sum_{kl}(\omega_k+2\omega_l)\lambda_l(B)(\Psi_{kl}(B)-\Psi_{lk}(B))(1+2n_b(l)) \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\frac{dK_{pqk}(B)}{dB} &= -((\epsilon_p-\epsilon_q)+\omega_k)^2K_{pqk}(B) \\ &\quad -J(p,q)(B)\lambda_k(B)((\epsilon_p-\epsilon_q)-\omega_k) \\ &\quad \frac{1}{4}\sum_m K_{pmk}(B)J(m,q)(B)((\epsilon_p-\epsilon_q)+\omega_k-2\epsilon_m)(1-2n(m)) \\ &\quad \frac{1}{4}\sum_m K_{mqk}(B)J(p,m)(B)((\epsilon_p-\epsilon_q)-\omega_k-2\epsilon_m)(1-2n(m)) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\frac{dL_{pqk(B)}}{dB} &= -((\epsilon_p-\epsilon_q)-\omega_k)^2L_{pqk}(B) \\ &\quad -J(p,q)(B)\lambda_k((\epsilon_p-\epsilon_q)+\omega_k) \\ &\quad \frac{1}{4}\sum_m L_{pmk}(B)J(m,q)(B)((\epsilon_p-\epsilon_q)-\omega_k-2\epsilon_m)(1-2n(m)) \\ &\quad \frac{1}{4}\sum_m L_{mqk}(B)J(p,m)(B)((\epsilon_p-\epsilon_q)+\omega_k-2\epsilon_m)(1-2n(m)) \end{aligned} \tag{3.17}$$

and

$$\frac{d\mu_{kl}(B)}{dB} = -(\omega_k-\omega_l)^2\mu_{kl}(B)+(\omega_k+\omega_l)\lambda_k(B)\lambda_l(B) \tag{3.18}$$

and

$$\frac{d\Psi_{kl}(B)}{dB} = -(\omega_k+\omega_l)^2\Psi_{kl}(B)+\lambda_l(B)\lambda_k(B)\omega_k. \tag{3.19}$$

3.2.2 Solution of the Flow Equation

Of course, we can not expect to solve the above flow equations analytically, however, if we search for a solution in the low energy sector by applying an infrared approximation of the couplings we can extract the relevant information of their behaviour in the $B \to \infty$ limit.

Deriving the fixed point equation

But before we are getting started it is important to know, how do we detect from the flow equations, whether a quantum phase transition is present or a stable phase. A stable fixed point where all flow lines go in, is identified by a certain phase, because once you are in that particular fixed point there is no chance to leave it. Whereas, if we consider an unstable fixed point, some flow lines go into the point and others go out of it, and we identify this with a transition point see Figure (1.4). So we just follow the flow lines of the coupling constants by taking $B \to \infty$ and eventually end up in either a stable or an unstable fixed point. Just to get a felling how, the couplings behave, we first consider two limiting cases;
1.Kondo phase
The Kondo regime is obtained by setting the bosonic coupling λ equals to zero and the flow equations reduce to

$$\begin{aligned}\frac{dJ(p,q)(B)}{dB} &= -(\epsilon_p - \epsilon_q)^2 J(p,q)(B) \\ &\quad + \sum_m (\epsilon_p + \epsilon_q - 2\epsilon_m) J(p,m)(B) J(m,q)(B)(n(m) - \frac{1}{2}) \end{aligned} \tag{3.20}$$

see S. Kehrein[53], therefore the Kondo phase is identified.

2.SU(2) bosonic phase
The purely bosonic SU(2)-invariant domain is governed by the λ-coupling, which can be achieved by setting J=0 and gives the following equations

$$\begin{aligned}\frac{d\lambda_k}{dB} &= -\omega_k^2 \lambda_k(B) \\ &\quad + \sum_l (\omega_k - 2\omega_l)\lambda_l \mu_{kl}(B)(1 + 2n_b(l)) \\ &\quad + \sum_{kl} (\omega_k + 2\omega_l)\lambda_l(B)(\Psi_{kl}(B) - \Psi_{lk}(B))(1 + 2n_b(l)) \end{aligned} \tag{3.21}$$

and

$$\frac{d\mu_{kl}(B)}{dB} = -(\omega_k - \omega_l)^2 \mu_{kl}(B) + (\omega_k + \omega_l)\lambda_k(B)\lambda_l(B) \tag{3.22}$$

and

$$\frac{d\Psi_{kl}(B)}{dB} = -(\omega_k + \omega_l)^2 \Psi_{kl}(B) + \lambda_l(B)\lambda_k(B)\omega_k. \tag{3.23}$$

It is convenient to define

$$\lambda_k = \sqrt{2\omega_k^s \alpha(B)} e^{-B\omega_k^2}, \tag{3.24}$$

which is justified by the representation of the spectral function of bosons, given by

$$S(w) = \sum_k \lambda_k^2 \delta(\omega_k - \omega) = 2\alpha\omega^s \Theta(\omega_c - \omega), \tag{3.25}$$

where ω_c is the cut off parameter and α describes the coupling of spin to the bosonic bath. We redefine the parameter s as $1-s=\varepsilon$. The s parameter is used to describe the characteristic behaviour of the spectral function.(e.g. ohmic, sub-ohmic , super-ohmic). The ε parameter will be our small expansion parameter.

By making the ansatz

$$\mu_{kl}(B)=\tilde{\mu}_{kl}(B)e^{-B(\omega_k-\omega_l)^2} \tag{3.26}$$

and plugging this into Eqs.(3.18) we receive

$$\mu_{kl}(B)=\frac{(\omega_l+\omega_k)}{2\omega_k\omega_l}\sqrt{2\omega_k^{1-\varepsilon}}\sqrt{2\omega_l^{1-\varepsilon}}\alpha(B)(1-e^{-2B\omega_k\omega_l})e^{-B(\omega_k-\omega_l)^2}. \tag{3.27}$$

Additionally, we make the ansatz

$$\Psi_{kl}(B)=\tilde{\Psi}_{kl}(B)e^{-B(\omega_k+\omega_l)^2} \tag{3.28}$$

and inserting into Eqs.(3.19)

$$\Psi_{kl}(B)=\frac{\sqrt{2\omega_l^{1-\varepsilon}}\sqrt{2\omega_k^{1-\varepsilon}}}{2\omega_l}\alpha(B)(e^{2B\omega_k\omega_l)}-1)e^{-B(\omega_k+\omega_l)^2}. \tag{3.29}$$

Finally, we rewrite Eqs.(3.15) in terms of α, as it has been defined in Eqs.(3.24). Now, we are in the position to solve the flow equation for the actual coupling so Eqs.(3.27) and Eqs.(3.29) are combined with the rewritten Eqs.(3.15) we get

$$\begin{aligned}\sqrt{2\omega_k^{1-\varepsilon}}\frac{d\sqrt{\alpha(B)}}{dB} &= -\sum_l(\omega_k-2\omega_l)2\omega_l^{1-\varepsilon}\sqrt{2\omega_k^{1-\varepsilon}}\alpha(B)^{\frac{3}{2}}\frac{(\omega_l+\omega_k)}{2\omega_k\omega_l}\\ &\quad\times(e^{-2B\omega_k\omega_l}-1)e^{-B(\omega_k-\omega_l)^2}e^{B\omega_k^2}e^{-B\omega_l^2}\\ &\quad+\sum_l(\omega_k+2\omega_l)2\omega_l^{1-\varepsilon}\alpha(B)^{\frac{3}{2}}\frac{\sqrt{2\omega_k^{1-\varepsilon}}}{2}(\frac{1}{\omega_l}-\frac{1}{\omega_k})\\ &\quad\times(e^{2B\omega_k\omega_l)}-1)e^{-B(\omega_k+\omega_l)^2}e^{B\omega_k^2}e^{-B\omega_l^2}.\end{aligned} \tag{3.30}$$

Since we just consider the zero temperature case, ω_k is set equal to zero and

$$\begin{aligned}\frac{d\sqrt{\alpha(B)}}{dB} &= 2\sum_l\omega_l^{1-\varepsilon}\alpha(B)^{\frac{3}{2}}(-2B\omega_l)e^{-2B\omega_l^2}\omega_l\\ &\quad+2\sum_l\omega_l^{1-\varepsilon}\alpha(B)^{\frac{3}{2}}(-2B\omega_l)e^{-2B\omega_l^2}\omega_l.\end{aligned} \tag{3.31}$$

Obviously, the first term and the second one on the right hand side are the same and we can simplify the equation

$$\begin{aligned}\frac{d\sqrt{\alpha(B)}}{dB} &= -8B\sum_l \omega_l^{3-\varepsilon}\alpha^{\frac{3}{2}}e^{-2B\omega_l^2} \\ &= -8B\alpha^{\frac{3}{2}}\int d\omega \quad \omega^{3-\varepsilon}e^{-2B\omega^2} \\ &= -2\frac{\alpha^{\frac{3}{2}}}{(2B)^{\frac{1}{2}(2-\varepsilon)}}\Gamma(\frac{1}{2}(4-\varepsilon)) \end{aligned} \tag{3.32}$$

$$\Longrightarrow \frac{d\alpha(B)}{dB} = -\frac{4\alpha(B)^2}{(2B)^{\frac{1}{2}(2-\varepsilon)}}\Gamma(\frac{1}{2}(4-\varepsilon)) \tag{3.33}$$

E.g. for the ohmic case we get

$$\frac{d\alpha}{dB} = -2\frac{\alpha^2}{B}, \tag{3.34}$$

which is the first term of the β-function therefore this result indicates a stable fixed point. In the $s \neq 1$ case, which means that $\varepsilon \neq 0$, the above equation is solved by the following ansatz $\alpha(B) \propto B^x$(power law, x has to be determined), which also indicates a stable fixed point. These two limiting cases only occur, if in the first case λ either goes to zero or becomes vanishing small, and in the second case $J \to 0$, which, in both cases, depends on the initial value of the respective coupling constants.

3. SU(2) Bose Fermi unstable fixed point

Somewhere between there should be the unstable SU(2) Bose-Fermi fixed point. We solve the flow equation for the K_{pqk} and L_{pqk} coupling in first order expansion. In order to do so we use the following ansatz eliminating more and more degenerated energies durning the flow

$$K_{pqk}(B) = \tilde{K}_{pqk}(B)e^{-B((\epsilon_p-\epsilon_q)+\omega_k)^2} \tag{3.35}$$

$$L_{pqk}(B) = \tilde{L}_{pqk}(B)e^{-B((\epsilon_p-\epsilon_q)-\omega_k)^2} \tag{3.36}$$

$$\lambda_k(B) = \sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}e^{-B\omega_k^2} \tag{3.37}$$

$$J(p,q)(B) = J_{IR}(B)e^{-(\epsilon_p-\epsilon_q)^2}. \tag{3.38}$$

Plugging these into Eqs.(3.16) and neglect couplings that go with $\mathcal{O}(J^2)$

$$\frac{\tilde{K}_{pqk}(B)}{dB} = -J_{IR}(B)\sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}((\epsilon_p-\epsilon_q)-\omega_k)e^{2B(\epsilon_p-\epsilon_q)\omega_k}. \tag{3.39}$$

After a straightforward integration we take the ϵ_p,$\epsilon_q \longrightarrow 0$ limit, since we only regard the physics close to the fermi surface

$$K_{00k}(B) = BJ_{IR}(B)\sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}\omega_k e^{-B\omega_k^2}, \tag{3.40}$$

in analogy L_{pqk}

$$L_{00k}(B) = -BJ_{IR}(B)\sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}\omega_k e^{-B\omega_k^2} \tag{3.41}$$

therefore we get a solvable differential equation for the coupling constant $J_{IR}(B)$ by considering the physics close to the fermi surface. Recall that a quantum phase transition takes place at $T = 0$, this causes a drastic simplification of the particle number function of both, bosonic as well as fermionic function

$$n_f(\epsilon) = \begin{cases} 1, & \text{if } \epsilon < 0, \\ 0, & \text{if } \epsilon > 0, \end{cases} \tag{3.42}$$

and

$$n_b(\omega) = 0 \quad \text{for all } \omega > 0. \tag{3.43}$$

Eqs.(3.16) can be written as follows

$$\begin{aligned} \frac{dJ_{IR}}{dB} &= \sum_m \epsilon_m J_{IR}(B)^2 e^{-2B\epsilon_m^2}(1 - 2n_f(m)) - 8\alpha(B)J_{IR}(B)B\sum_k \omega_k^{3-\varepsilon} e^{-2B\omega_k^2} \\ &= J_{IR}(B)^2\rho_0 \int d\epsilon \quad \epsilon e^{-2B\epsilon^2}(1 - 2n_f(\epsilon)) - 8\alpha(B)J_{IR}(B)B\int \omega^{3-\varepsilon} e^{-2B\omega^2} \\ &= \frac{J_{IR}^2(B)\rho_0}{2B} - 2\frac{J_{IR}(B)\alpha(B)\Gamma(\frac{1}{2}(4-\varepsilon))}{(2B)^{\frac{1}{2}(2-\varepsilon)}}. \end{aligned} \tag{3.44}$$

Redefining $J_{IR}(B)\rho_0 = \xi(B)$ gives

$$\frac{d\xi(B)}{dB} = \frac{1}{2B}(\xi(B)^2 - 2\frac{\xi(B)\alpha(B)\Gamma(\frac{1}{2}(4-\varepsilon))}{(2B)^{-\frac{\varepsilon}{2}}}). \tag{3.45}$$

On the bosonic side we take the limit $\omega_k \to 0$ in order to investigate the low energy physics. In Eqs.(3.15), the term $\omega_k K_{pqk}(B)$ appears, which is zero in the considered limit. As a result we can say, that a non-zero value of $J_{IR}(B)$ are not going to alter the flow equation for the bosonic coupling $\alpha(B)$

$$\frac{d\alpha(B)}{dB} = -\frac{4\alpha(B)^2}{(2B)^{\frac{1}{2}(2-\varepsilon)}}\Gamma(\frac{1}{2}(4-\varepsilon)). \tag{3.46}$$

A good choice for the ansatz is $\alpha(B) = g^* B^{-x}$

$$-xg^* B^{-x-1} = -\frac{4g^{*2}}{2(2)^{-\frac{\varepsilon}{2}}}B^{-2x-1+\frac{\varepsilon}{2}}\Gamma(\frac{1}{2}(4-\varepsilon)), \tag{3.47}$$

this equation has to hold for all B, hence

$$x = \frac{\varepsilon}{2} \text{ and therefore } g^* = \frac{\varepsilon}{4}\frac{1}{\Gamma(\frac{1}{2}(4-\varepsilon))2^{\frac{\varepsilon}{2}}}. \tag{3.48}$$

We assumed no B dependence for g^*, so g^* is already the fixed point value. For a more general ansatz we let g depend on B, so $\alpha(B) = g(B)B^{-\frac{\varepsilon}{2}}$,

$$\begin{aligned} \frac{dg(B)}{dB}B^{-\frac{\varepsilon}{2}} - \frac{\varepsilon}{2}B^{\frac{\varepsilon}{2}-1}g(B) &= -\frac{4g(B)^2B^{-\varepsilon}}{(2B)^{\frac{1}{2}(2-\varepsilon)}}\Gamma(\frac{1}{2}(4-\varepsilon)) \\ \iff \qquad \frac{dg(B)}{dB} &= \frac{\varepsilon}{2B}g(B) - \frac{4g(B)^2(B)^{-\frac{\varepsilon}{2}}}{2B^{\frac{1}{2}(2-\varepsilon)}}\Gamma(\frac{1}{2}(4-\varepsilon)) \end{aligned} \tag{3.49}$$

leading to

$$\frac{dg(B)}{dB} = \frac{1}{2B}\left(\varepsilon g(B) - 4g(B)^2 2^{\frac{\varepsilon}{2}}\Gamma(\frac{1}{2}(4-\varepsilon))\right). \tag{3.50}$$

Due to the redefinition of the coupling constant $\alpha(B)$ the new coupling constant $g(B)$ is dimensionless. We plug the ansatz into Eqs.(3.45) and obtain

$$\frac{d\xi(B)}{dB} = \frac{1}{2B}(\xi(B)^2 - 2\frac{\xi(B)g(B)\Gamma(\frac{1}{2}(4-\varepsilon))}{2^{-\frac{\varepsilon}{2}}}). \tag{3.51}$$

It can easily be seen that the above founded fixed point solves Eqs.(3.50). We insert g^* into Eqs.(3.45) and obtain,

$$\frac{d\xi(B)}{dB} = \frac{\xi(B)}{2B}(\xi(B) - \frac{\varepsilon}{2}) \tag{3.52}$$

it is easily seen that $\xi = \frac{\varepsilon}{2}$ is a solution and at the same time the desired fixed point value.

In order to get the full solution we have to define the initial value of ξ,

$$\xi(B = D^{-2}) = \xi_0 \tag{3.53}$$

The solution of the fermionic coupling in terms of B at the bosonic fixed point value is given by

$$\Rightarrow \xi = \frac{\varepsilon}{2 + (\frac{\varepsilon}{\xi_0} - 2)(BD^{-2})^{\frac{\varepsilon}{4}}} \tag{3.54}$$

From here we can see the nature of the fixed, in the present case an unstable fixed point is identified. If

$\xi_0 < \frac{\varepsilon}{2} \Rightarrow \xi(B) \to 0$ as $B \to \infty$
we find yourselves in the purely bosonic phase, since the Kondo coupling eventually approaches zero, however, if we look at the case
$\xi_0 > \frac{\varepsilon}{2} \Rightarrow \xi(B) \to \infty$ as $B \to \infty$,
so the Kondo coupling goes to infinity and we are in the strong coupling regime, means, the paramagnetic Kondo phase is occupied. But if,
$\xi_0 = \frac{\varepsilon}{2} \Rightarrow \xi(B) = \frac{\varepsilon}{2}$ for all B,
the fixed point value is found. So a slight deviation of the initial condition form $\xi_0 = \frac{\varepsilon}{2}$ causes as significant different outcome.

In the following we would like to write the flow equation, now depending on B, in terms of the cut off parameter Λ[1]. The resulting equations can then be compared to previously obtained equations by Zaránd and Demler[33]

$$\underbrace{-\frac{dg}{dln\Lambda}}_{\frac{dg}{dln\Lambda^{-1}}} = \left(\varepsilon g - 4g^2 2^{\frac{\varepsilon}{2}}\Gamma(\frac{1}{2}(4-\varepsilon))\right) \tag{3.55}$$

$$\Rightarrow \underbrace{-\frac{d\xi}{dln\Lambda}}_{\frac{d\xi}{dln\Lambda^{-1}}} = (\xi^2 - 2\frac{\xi g\Gamma(\frac{1}{2}(4-\varepsilon))}{2^{-\frac{\varepsilon}{2}}}). \tag{3.56}$$

If we expand g^* up to second order in ε we find

$$g^* = \frac{\varepsilon}{4} + \frac{1}{8}(ln2 - \gamma + 1)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{3.57}$$

where γ is the Euler constant, and approximate $2^{\frac{\varepsilon}{2}}\Gamma(\frac{1}{2}(4-\varepsilon) = 1$, to ensure that we are merely dealing with first order terms in ε. For further discussions of the Bose-Fermi Kondo problem we stick to the first order expansion of the fixed point value of the bosonic coupling as well as the fermionic coupling, namely

$$\boxed{g^* = \frac{\varepsilon}{4} \qquad \text{and} \qquad \xi^* = \frac{\varepsilon}{2}}. \tag{3.58}$$

Remember, the parameter ε was an external input parameter. Actually, its exact value is determine by the solution of the self-consistency condition, but we assumed it to be small in order ensure the validity of the perturbative approach. Combining these approximation with Eqs.(3.55) and Eqs.(3.56) the flow equations finally reduces to

$$\boxed{\frac{d\xi}{dln\Lambda^{-1}} = \xi(\xi - 2g)}, \tag{3.59}$$

$$\boxed{\frac{dg}{dln\Lambda^{-1}} = (\varepsilon g - 4g^4)}. \tag{3.60}$$

[1] Just as a reminder $\Lambda = B^{-\frac{1}{2}}$.

3.3 The Spin S Operator

As it was mentioned in section 1.2 it is stringently neccesary to find an expression for the local correlation function, mainly to have an educated guess for the self consistency equation. But before talking about the correlation function, we have to figure out the general behaviour of the Spin-operator in the new basis[2] and from there we are able to calculate the correlation function or the susceptibility.

3.3.1 Commutators and the Flow Equation

By solving Eqs.(1.76) we will receive the Spin-operator in the new basis, which is achieved in more or less the same manner as it has been done by the Hamilton operator. We can use the same generator as in Eqs.(3.2)

$$\eta(B) = \sum_{pq}(\epsilon_p - \epsilon_q)J(p,q)(B) : \mathbf{S}\cdot\mathbf{s}_{pq} : + \sum_k \omega_k \lambda_k(B)\mathbf{S}\cdot(\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k). \tag{3.61}$$

During the flow of $[\eta, S^a]$, where $a = x, y, x$ new terms are generated so it is not unreasonable to use a more sophisticated ansatz than simply the bare spin operator itself

$$S(B)^a = h(B)S^a + i\sum_{pq}\mu_{pq}(B)(: \mathbf{S}\times\mathbf{s}_{pq} :)^a + i\sum_k \Psi_k(B)(\mathbf{S}\times(\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k))^a. \tag{3.62}$$

The commutator $[\eta_f + \eta_b, S_0^a + S_f^a + S_b^a]$ can be divided into certain pieces, which are given by

$$\begin{aligned}[\eta_f(B), S^a(B)_b] &= -\sum_{pqk}(\epsilon_p - \epsilon_q)J(p,q)(B) \\ &\quad\times(: \mathbf{s}_{p,q} : (\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)\cdot\mathbf{S} - \mathbf{S}^a(: \mathbf{s}_{p,q} : \cdot(\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)))\end{aligned} \tag{3.63}$$

and

$$\begin{aligned}[\eta_b(B), S^a(B)_f] &= -\sum_{pqk}\omega_k\lambda_k(B)\mu_{pq}(B) \\ &\quad\times((\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)^a(: \mathbf{s}_{p,q} : \cdot\mathbf{S}) - \mathbf{S}^a(: \mathbf{s}_{p,q} : \cdot(\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)))\end{aligned} \tag{3.64}$$

[2]While we have gradually diagonalised the Hamilton operator, we simultaneously changed the basis representation of the eigenstate.

and

$$\begin{aligned}
[\eta_b(B), S^a(B)_b] &= -\sum_{kl} \omega_k \lambda_k(B) \Psi_l(B) (\mathbf{\Phi}_k^{\dagger(a)} (\mathbf{\Phi}_l^\dagger \cdot \mathbf{S}) - \mathbf{S}^a (\mathbf{\Phi}_k^\dagger \cdot \mathbf{\Phi}_l^\dagger)) \\
&\quad - \sum_{kl} \omega_k \lambda_k(B) \Psi_l(B) (\mathbf{\Phi}_k^a (\mathbf{\Phi}_l \cdot \mathbf{S}) - \mathbf{S}^a (\mathbf{\Phi}_k \cdot \mathbf{\Phi}_l)) \\
&\quad + \sum_{kl} \omega_k \lambda_k(B) \Psi_l(B) (: \mathbf{\Phi}_k^{\dagger(a)} (\mathbf{\Phi}_l \cdot \mathbf{S} :) - (: \mathbf{S}^a (\mathbf{\Phi}_k^\dagger \cdot \mathbf{\Phi}_l :)) \\
&\quad + \sum_{kl} \omega_k \lambda_k(B) \Psi_l(B) (: \mathbf{\Phi}_l^\dagger \cdot \mathbf{S} (\mathbf{\Phi}_k^a :) - (: \mathbf{S}^a (\mathbf{\Phi}_l^\dagger \cdot \mathbf{\Phi}_k :)) \\
&\quad - 2 \sum_k \omega_k \lambda_k \Psi_k (1 + 2 n_b(k))
\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}
[\eta_f(B), S^a(B)_f] &= i \sum_{pqmn} (: \mathbf{s}_{mn}^a (\mathbf{S} \cdot \mathbf{s}_{pq}) : - \mathbf{S}^a : \mathbf{s}_{pq} \cdot \mathbf{s}_{mn} :) \\
&\quad + \frac{i}{4} \sum_{pqmn} (\delta_{qm} : \mathbf{s}_{pn}^a : + \delta_{pn} : \mathbf{s}_{qm}^a :) \\
&\quad - \frac{i}{2} \mathbf{S}^a \sum_{pqmn} \Bigg(\frac{1}{2} \delta_{qm} (1 - 2 n_f(q)) \sum_\sigma : c_{p\sigma}^\dagger c_{n\sigma} : \\
&\qquad + \frac{1}{2} \delta_{pn} (1 - 2 n_f(p)) \sum_\sigma : c_{m\sigma}^\dagger c_{q\sigma} : \\
&\qquad + 2 \delta_{qm} \delta_{pn} (1 - n_f(q) n_f(p)) \Bigg) \\
&\quad - \frac{1}{4} \sum_{pqmn} \delta_{qm} (1 - n_f(q)) : (\mathbf{S} \cdot \mathbf{s}_{pn})^a : \\
&\quad - \frac{1}{4} \sum_{pqmn} \delta_{pn} (1 - n_f(p)) : (\mathbf{S} \cdot \mathbf{s}_{qm})^a : .
\end{aligned} \tag{3.66}$$

Again, we merely consider the terms that have already been taken into account in the original ansatz of our spin operator. The newly appeared terms, however, are initially generated in higher order of the coupling, and therefore will be neglected during the following calculation. So we can easily read of the differential equation for the couplings

$$\begin{aligned}
\frac{dh(B)}{dB} &= \sum_{pq} (\epsilon_p - \epsilon_q) J(p,q)(B) \mu_{qp}(B) n(p)(1 - n(q)) \\
&\quad - 2 \sum_k \omega_k \lambda_k(B) \Psi_k(B) (1 + 2 n_b(k))
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}\frac{d\mu_{pq}(B)}{dB} &= h(B)(\epsilon_p-\epsilon_q)J(p,q)(B)\\ &+\frac{1}{4}\sum_m\left((\epsilon_p-\epsilon_m)J(p,m)(B)\mu_{mq}(B)\right.\\ &\left.+(\epsilon_q-\epsilon_m)J(m,q)(B)\mu_{pm}(B)\right)(1-2n_f(m))\end{aligned} \tag{3.68}$$

and

$$\frac{d\Psi_k(B)}{dB} = h\omega_k(B)\lambda_k(B). \tag{3.69}$$

Notice, the structure of the differential equation of $\mu_{pq}(B)$ is exactly the same as for the ordinary Kondo model without a bosonic bath [53], although, the bosonic bath influences Eqs.(3.68) by the modified Eqs.(3.67), where the bosonic part affects the spin decay. Before we proceed on and try to solve the above equation we perform a little consistency check by proving that the sum rule is fulfilled for any value of B. This is done in subsequent section and the hasty reader may skip it.

3.3.2 Sum Rules

During the unitary flow certain sum rules should stay fulfilled, e.g. the anti-commutation relations of the fermionic creation and annihilation operators and relation for the Spin operators $\langle(S^a(B))^2\rangle = \frac{3}{4}$, obviously this is satisfied when $B = 0$. The expectation value takes the normal ordering of the operators into account. The structure of the flow equation is such that we can consider the sum rule of the fermionic part and the bosonic part separately. In Ref.[53] it is shown that the fermionic part obeys this sum rule exact, at least, up to $\mathcal{O}(J^2)$. We check the bosonic part by setting $J = 0$, because in our problem the bosonic part and the fermionic part are independent of each other. First we calculate the expectation value of the spin operator and then we sum over all directions

$$\begin{aligned}\Rightarrow\sum_{a=1}^{3}\langle S^a(B)S^a(B)\rangle &= h^2(B)\frac{3}{4}-\sum_{kla}\Psi_k(B)\Psi_l(B)\langle\epsilon_{abc}\epsilon_{ade}S^b(\Phi_k^{\dagger c}-\Phi_k^c)S^d(\Phi_l^{\dagger e}-\Phi_l^e)\rangle\\ &= h^2(B)\frac{3}{4}+\frac{3}{2}\sum_k\Psi_k(B)^2(1+2n_b(k)) \stackrel{!}{=} \text{const.}\end{aligned} \tag{3.70}$$

We take the derivative of this expression with respect to B and get

$$\frac{d\sum_{a=1}^{3}\langle S^a(B)S^a(B)\rangle}{dB} = \frac{3}{2}h(B)h'(B)+3\sum_k\Psi_k(B)\Psi_k(B)'(1+2n_b(k)), \tag{3.71}$$

where the prime denotes the derivation with respect to B and with the use of Eqs.(3.67) we receive

$$\begin{aligned}\frac{d\sum_{a=1}^{3}\langle S^a(B)S^a(B)\rangle}{dB} &= -3h(B)\sum_k \omega_k\lambda_k\Psi_k(1+2n_b(k)) \\ &\quad +3h(B)\sum_k \omega_k\lambda_k\Psi_k(1+2n_b(k)) = 0, \end{aligned} \tag{3.72}$$

and the sum rule is valid.

3.3.3 Discussion of the Spin operator near the unstable fixed point

Spin operator within the h=1 ansatz (T=0)

Obviously, it is not that easy to solve the differential equations Eqs.(3.67), Eqs.(3.68) and Eqs.(3.69) analytically, at least in the present form, hence we have to make a few reasonable assumptions. Firstly, notice that $\mu_{pq}(B)$ is already generated in order $\mathcal{O}(J)$, so we neglect the second term in Eqs.(3.68), and secondly, we look at that part of the flow, where the deviation of $h(B)$ from 1 can be considered as negligible small, in other words $h(B)$ varies slowly, and in Eqs.(3.68) and Eqs.(3.69) we set

$$h(B) = 1. \tag{3.73}$$

Surely, the validity of this approximation is rather limited, but we can hope for a clue how a better solution might look like. In addition, the ansatz for the coupling is again

$$\lambda_k(B) = \sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}e^{-B\omega_k^2} \tag{3.74}$$

and

$$J(p,q)(B) = J_{IR}(B)e^{-B(\epsilon_p-\epsilon_q)^2}. \tag{3.75}$$

Plugging into Eqs.(3.68), respectively Eqs.(3.69) we get

$$\mu_{pq}(B) = \frac{J_{IR}(B)}{(\epsilon_p-\epsilon_q)}(1-e^{-B(\epsilon_p-\epsilon_q)^2}) \tag{3.76}$$

and

$$\Psi_k(B) = \frac{\sqrt{2\omega_k^{1-\varepsilon}\alpha(B)}}{\omega_k}(1-e^{-B\omega_k^2}). \tag{3.77}$$

Now, we insert these two equations into Eqs.(3.67)

$$\begin{aligned}\frac{dh(B)}{dB} &= -\sum_{pq} J_{IR}^2(B)(1-e^{-B(\epsilon_p-\epsilon_q)^2})e^{-B(\epsilon_p-\epsilon_q)^2}n_f(p)(1-n_f(q)) \\ &\quad -4\sum_k \omega_k^{1-\varepsilon}\alpha(B)(1-e^{-B\omega_k^2})e^{-\omega_k^2 B}(1+2n_b(k)) \\ &= -J_{IR}^2(B)\rho_0^2\int_{-\infty}^{0}d\epsilon'\int_0^{\infty}d\epsilon(1-e^{-B(\epsilon'-\epsilon)^2})e^{-B(\epsilon'-\epsilon)^2} \\ &\quad -4\alpha(B)\int d\omega \quad \omega^{1-\varepsilon}(1-e^{-B\omega^2})e^{-\omega^2 B} \\ &= -\frac{\xi(B)^2}{4B}-\frac{2\alpha(B)\Gamma(\frac{1}{2}(2-\varepsilon))}{B^{\frac{1}{2}(2-\varepsilon)}}\left(1-\frac{1}{2^{\frac{1}{2}(2-\varepsilon)}}\right), \end{aligned} \tag{3.78}$$

and the differential equation, ultimately, has got the form

$$\frac{dh(B)}{dB} = -\frac{1}{B}\left(\frac{\xi^2(B)}{4}+\frac{2\alpha(B)\Gamma(\frac{1}{2}(2-\varepsilon))}{B^{-\frac{\varepsilon}{2}}}\left(1-\frac{1}{2^{\frac{1}{2}(2-\varepsilon)}}\right)\right). \tag{3.79}$$

We restrict ourselves to the behaviour of the Spin operator at the unstable fixed point, namely $\xi^* = \frac{\varepsilon}{2}$ and $g^* = \frac{\varepsilon}{2^{2+\frac{\varepsilon}{2}}\Gamma(\frac{1}{2}(4-\varepsilon))}$, have to be inserted which leads to the following expression

$$\frac{dh(B)}{dB} = -\frac{1}{B}\left(\frac{\varepsilon^2}{64}+\frac{\varepsilon}{2}\frac{\Gamma(\frac{1}{2}(2-\varepsilon))}{\Gamma(\frac{1}{2}(4-\varepsilon))}(\frac{1}{2^{\frac{\varepsilon}{2}}}-\frac{1}{2})\right). \tag{3.80}$$

The solution is given by

$$\boxed{h(B) = 1 - m(\varepsilon) ln(BD^{-2})} \tag{3.81}$$

where

$$m(\varepsilon) = \frac{\varepsilon^2}{64}+\frac{\varepsilon}{2}\frac{\Gamma(\frac{1}{2}(2-\varepsilon))}{\Gamma(\frac{1}{2}(4-\varepsilon))}(\frac{1}{2^{\frac{\varepsilon}{2}}}-\frac{1}{2}), \tag{3.82}$$

with the initial condition is $h(B = D^{-2}) = 1$. It is easy to see that $m(\varepsilon)$ goes to zero as ε goes to zero, so we can choose ε such that $m(\varepsilon)$ is arbitrarily small, which means that the deviation rate of $h(B)$ from 1 is controlled by the value of ε. Additionally, we can deduce that the fermionic coupling influences the evolution of the Spin operator in $\mathcal{O}(\varepsilon^2)$ and the bosonic part solely in first order of ε. The logarithmic behaviour of $h(B)$ points towards a power law e.g. $h \sim B^x$ in a more sophisticated ansatz, since a expansion in x will produce a $h \sim 1 - xln(B)$ up to higher terms of x. So just to make a rough guess, we expand $m(\varepsilon)$ to first order in ε, so $m(\varepsilon) = \frac{\varepsilon}{4} + \mathcal{O}(\varepsilon^2)$, which means that the power $x = \frac{\varepsilon}{4}$. Of course, this does not count as a prove, and we have to make a better ansatz for $h(B)$.

Spin operator within the more sophisticated ansatz of h

We would like to know the behaviour of $h(B)$ over the entire B range, which is achieved by making no assumption of any form of $h(B)$ in the first place going back to the original

differential equations Eqs.(3.68) and Eqs.(3.69) which are at least formaly solved by

$$\mu_{pq}(B) = (\epsilon_p - \epsilon_q) \int_0^B dB' \quad h(B') J_{IR}(B') e^{-B'(\epsilon_p - \epsilon_q)} \tag{3.83}$$

and

$$\Psi_k(B) = \sqrt{2\omega_k^{1-\varepsilon}}\omega_k \int_0^B dB' \quad h(B')\sqrt{\alpha(B')} e^{-B'\omega_k^2}. \tag{3.84}$$

We limit ourselves just to the fixed point that means we replace the coupling constants by their critical value

$$\begin{aligned}
\frac{dh(B)}{dB} &= -\sum_{pq} (\epsilon_p - \epsilon_q)^2 J^2(B) e^{-B(\epsilon_p-\epsilon_q)^2} n_f(q)(1-n_f(p)) \int_0^B dB' \quad h(B') e^{-B'(\epsilon_p-\epsilon_q)^2} \\
&\quad -4\sum_k \omega_k^{3-\varepsilon}\sqrt{\alpha(B)}(1+2n_b(k)) \int_0^B dB' \quad h(B')\sqrt{\alpha(B')} e^{-B'\omega_k^2} \qquad (3.85) \\
&= -\frac{\varepsilon^2}{16}\int_{-\infty}^0 d\epsilon' \int_0^\infty d\epsilon \quad (\epsilon'-\epsilon)^2 e^{-B(\epsilon'-\epsilon)^2} \int_0^B dB' \quad h(B') e^{-B'(\epsilon'-\epsilon)^2} \\
&\quad -\varepsilon \int d\omega \quad \omega^{3-\varepsilon} B^{-\frac{\varepsilon}{4}} e^{-B\omega^2} \int_0^B dB' \quad h(B') B'^{-\frac{\varepsilon}{4}} e^{-B'\omega^2}. \qquad (3.86)
\end{aligned}$$

Note that the fermionic part is a second order contribution and can be neglected in first order calculation, whereas the bosonic part is of first order as one can see very easily. Since we are working in first order expansion we can set ε equals to zero if it appears in an exponent, which gives the following

$$\begin{aligned}
\frac{dh(B)}{dB} &= -\varepsilon \int_0^{\omega_c} .d\omega \quad \omega^3 \int_0^B dB' \quad h(B') e^{-(B+B')\omega^2} \\
&\underbrace{=}_{\omega = \frac{x}{\sqrt{B+B'}}} -\varepsilon \int_0^B dB' \quad \frac{h(B')}{(B+B')^2} \int_0^{\sqrt{B+B'}\omega_c} dx \quad x^3 e^{-x^2} \\
&= -\frac{\varepsilon}{2}\int_0^B dB' \quad \frac{h(B')}{(B+B')^2}. \qquad (3.87)
\end{aligned}$$

Above the logarithmic dependence of $h(B)$ was shown which can be viewed as an indication of a power law behaviour of $h(B)$. It is reasonable to make the following ansatz $h(B) \sim B^{-x}$ and plug this into the above equation

$\Rightarrow$

$$\begin{aligned} -xB^{-x-1} &= -\frac{\varepsilon}{2}\int_0^B dB' \quad \frac{B'^{-x}}{(B+B')^2} \\ &= -\frac{\varepsilon}{2}B^{-2}\int_0^B dB' \quad \frac{B'^{-x}}{(1+(\frac{B'}{B}))^2} \\ &\underbrace{=}_{B'=Bz} -\frac{\varepsilon}{2}B^{-1-x}\underbrace{\int_0^1 dz \frac{z^{-x}}{(1+z^2)}}_{=\frac{1}{2} \quad z^{-x}=1} \end{aligned} \tag{3.88}$$

$\Rightarrow x = \frac{\varepsilon}{4}$ thus

$$\boxed{h(B) \sim B^{-\frac{\varepsilon}{4}}}, \tag{3.89}$$

and the power law behaviour is proved up to the considered order. So our previously guessed power law behaviour was correct.

3.3.4 Correlation function

The general expression

The S-operator, given in Eqs.(3.62), can be written in the structure $S(t) = F(t) + B(t)$, as in the limit B goes to infinity $h(B = \infty) = 0$. Remember, the frequency dependent symmetric correlation function is given by

$$C(\omega) = \frac{1}{2}\int \quad dt \quad e^{i\omega t}\langle\{S(0), S(t)\}\rangle. \tag{3.90}$$

Taking a closer look at the anti-commutator products of the spin operator we see that we either get a purely bosonic and fermionic part or so called mixed terms, conisting of a product of the bosonic part and the fermionic part

$$\begin{aligned} \langle\{S(0), S(t)\}\rangle &= \langle\{F(0)+B(0), F(t)+B(t)\}\rangle \\ &= \langle F(0)F(t)+F(t)F(0)\rangle + \langle B(0)B(t)+B(t)B(0)\rangle \\ &\quad +\langle B(0)F(t)+B(t)F(0)+F(0)B(t)+F(t)B(0)\rangle \\ &= \langle F(0)F(t)+F(t)F(0)\rangle + \langle B(0)B(t)+B(t)B(0)\rangle \end{aligned} \tag{3.91}$$

The bosionic part of the spin operator contains only a single annihilation opertator or creation operator and the expectation value of the mixed term gives no contribution, only the square terms of the bosonic and fermionic operators will give a non zero expectation value. So once again we can split the correlation function into a purely bosonic and fermionic part

$$C(\omega) = C_f(\omega) + C_b(\omega). \tag{3.92}$$

The $C_f(\omega)$ has already been calculated by S. Kehrein [53] and reads

$$\begin{aligned} C_f(\omega) &= \frac{\pi}{4}\sum_p \mu^2_{\epsilon_p+\omega,\epsilon_p}(B=\infty) \\ &\quad \times(n_f(\epsilon_p)(1-n_f(\epsilon_p+\omega))+n_f(\epsilon_p+\omega)(1-n_f(\epsilon_p))). \end{aligned} \tag{3.93}$$

We use Eqs.(1.84) in order to calculate the bosonic bit of the correlation function

$$\begin{aligned} C_b(\omega) &= -\frac{\pi}{\tilde{Z}(\beta)}\sum_j e^{-E_j\beta}\sum_{mk}\Psi_m(B=\infty)\Psi_k(B=\infty)<j|\epsilon_{abc}\epsilon_{ade}S^bS^d(-\phi^{c\dagger}_m\phi^e_k)|j> \\ &\quad \times(\delta(\omega-(-\omega_k))+\delta(\omega+(-\omega_k))) \\ &\quad -\frac{\pi}{\tilde{Z}(\beta)}\sum_j e^{-E_j\beta}\sum_{mk}\Psi_m(B=\infty)\Psi_k(B=\infty)<j|\epsilon_{abc}\epsilon_{ade}S^bS^d(-\phi^{c}_m\phi^{e\dagger}_k)|j> \\ &\quad \times(\delta(\omega-(\omega_k))+\delta(\omega+(\omega_k))) \\ &= \frac{\pi}{2}\sum_k \Psi_k(B=\infty)(1+2n_b(\omega_k))(\delta(\omega-(\omega_k))+\delta(\omega+(\omega_k))) \\ &= \frac{\pi}{2}\left(\Psi^2_\omega(B=\infty)(1+2n_b(\omega))+\Psi^2_{-\omega}(B=\infty)(1+2n_b(-\omega))\right). \end{aligned} \tag{3.94}$$

We usually use the Einstein convention, however, in this case we do not sum over the index a, but over all others that appear twice. A summation over a would lead to an overall factor of three, due to the isotropic nature of the model.

On the one hand $C_f(\omega)\sim\mathcal{O}(J^2)$ and on the other hand $C_b(\omega)\sim\mathcal{O}(\lambda^2)$. At the fixed point $J\sim\varepsilon$ and $\lambda\sim(\varepsilon)^{\frac{1}{2}}$ up to first order, thus we can omit the fermionic part of the correlation function as it is of higher order in the expansions parameter ε and the remaining correlation function is

$$\boxed{C(\omega)=\frac{\pi}{2}\Psi^2_{|\omega|}(B=\infty)(1+2n_b(|\omega|))}. \tag{3.95}$$

Spin correlation function and dynamical susceptibility in first order calculation

In order to calculate the spin correlation function we have to reinsert Eqs.(3.89) into Eqs.(3.84)

$$\begin{aligned} \Psi_k(B) &\simeq \sqrt{2\omega_k^{1-\varepsilon}}\omega_k\int_0^B dB' \quad B'^{-\frac{\varepsilon}{4}}\sqrt{\alpha(B')}e^{-B'\omega_k^2} \\ &\simeq \sqrt{\frac{\varepsilon}{4}}\frac{\sqrt{2\omega_k^{1-\varepsilon}}}{\omega_k^{-\varepsilon}\omega_k}\int_0^{B\omega_k^2}dx \quad x^{-\frac{3}{4}\varepsilon}e^{-x} \\ &\simeq \sqrt{\frac{\varepsilon}{4}}\frac{\sqrt{2\omega_k^{1-\varepsilon}}}{\omega_k^{-\varepsilon}\omega_k}\Gamma(1-\frac{\varepsilon}{2}). \end{aligned} \tag{3.96}$$

In the integral expression we took the limit B to infintiy so that the last line is the needed expression for $\Psi_k(B=\infty)$. By inserting this into Eqs.(3.95) and setting $n_b(\omega)=0$ due to $T=0$ we finally receive the following ω-dependence of the correlation function up to first order in the ε-expansion

$$\boxed{C(\omega) \sim \Psi(|\omega|)^2 \sim \frac{sgn(\omega)}{|\omega|^{1-\varepsilon}}}. \tag{3.97}$$

This is exactly the same result as it was obtained by Zárand and Demler [33], but, new is that now we have a full expression for the correlation function, so if we solved Eqs.(3.67), Eqs.(3.68)and Eqs.(3.69) we could extract all the information of the correlation function, not just the approximate ω-dependence. In the $T \neq 0$ we have to keep track of the momentum dependence of couplings and therefore the infrared approximation is not appropriate any more, so analytically it seems futile to solve the full set of differential equations, however, numerically it can be done, even for $T \neq 0$, and this is very important. On the one hand, there is the $\frac{\omega}{T}$-scaling and of course we would like to know, if this can be deduced within our framework, on the other hand, there is a kind of conjecture on a possible connection between the BFKM at the fixpoint and a certain conformal field theory.

Before we discusse those two topics in detail, we would like to compare once again the above result with the one we would get if we took the simple ansatz for $h(B)$, namely $h(B)=1$. After some fairly straightforward calculations we have

$$C(\omega) \sim \frac{1}{\omega}. \tag{3.98}$$

So no ε dependence of the exponent is present in first order calculation and we do not have the desired behaviour of the susceptibility which would be important for the kind of phase transition we are looking for and by taking the limit $\varepsilon \to 0$ in Eqs.(3.97) we obtain Eqs.(3.98).

3.4 Conformal Field Theory Conjecture

A further step to get a better understanding of the nature of the quantum phase transition was taken by Kirchner and Si [56], by suggesting a kind of connection between the BFK-system[3] at the quantum critical point and a conformal field theory. On the conformal field theory side it is a well known fact that with the help of mapping the half plane on a half cylinder, the expression for the finite temperature correlator reads

$$\boxed{\langle \mathbf{\Phi}(\tau,T), \mathbf{\Phi}(0,T)\rangle = C\left(\frac{\pi/\beta}{\sin(\pi\tau/\beta)}\right)^{2\Delta}}, \tag{3.99}$$

where τ is the imaginary time, $\beta=\frac{1}{T}$, Δ the scaling dimension of $\mathbf{\Phi}$ a conformal primary field and C a constant, see [57, 58]. And if the dynamical spin susceptibility of the BFKM

[3] In their paper they considered both types, the Ising type as well as the isotropic model.

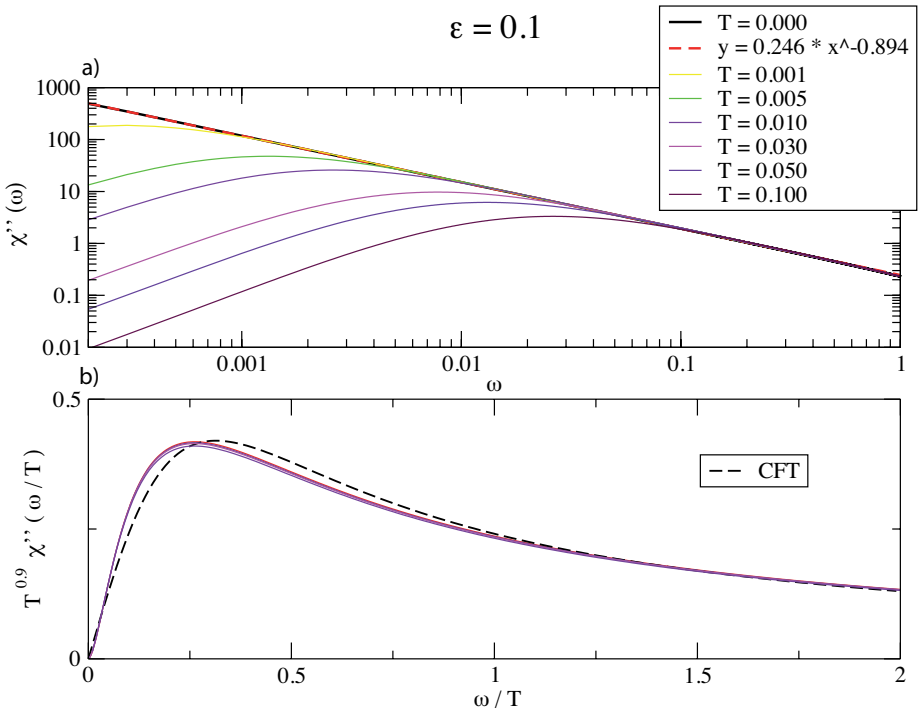

Figure 3.1: The expansion parameter ε is 0.1, a) shows a double logarithmic plot of the imaginary part of the dynamical spin susceptibility $\chi''(\omega)$ at different temperatures. If we take $T = 0$ the analytically obtained power law is recovered, but, if the temperature is non-zero, $\chi''(\omega)$ begins to deviate from the power low behaviour as $\frac{\omega}{T} < 1$. In other words, in the case of non-zero temperature, the temperature only matters, as long as the energy is sufficiently small, otherwise in the high energy region the temperature has got no effect. In b) the rescaled dynamical spin susceptibility is plotted for different temperatures and compared to the conformal field theory result.

at the quantum critical point has the same form, by matching the parameters, a connection would be very likely. The trueness of this conjecture causes different new insights, such as, although the BFK-Hamiltonian possesses no conformal invariance, due to the sub-ohimc nature of the bosonic spectrum, the BFK-Hamiltonian at the critical point, somehow gains symmetry, thus making an underlying boundary conformal field theory possible. So far it is not clear, where these additional symmetries should come from, or in other words, how the system enhances its symmetries.

As aforementioned we have to solve the differential equations Eqs.(3.67), Eqs.(3.68)and Eqs.(3.69) and plug the solution into the expression for the correlation function 3.94. This has been done by using numerical methods for solving differential equations, by Peter Fritsch. In order to compare the result with Eqs.(3.99), we have to convert the symmetric

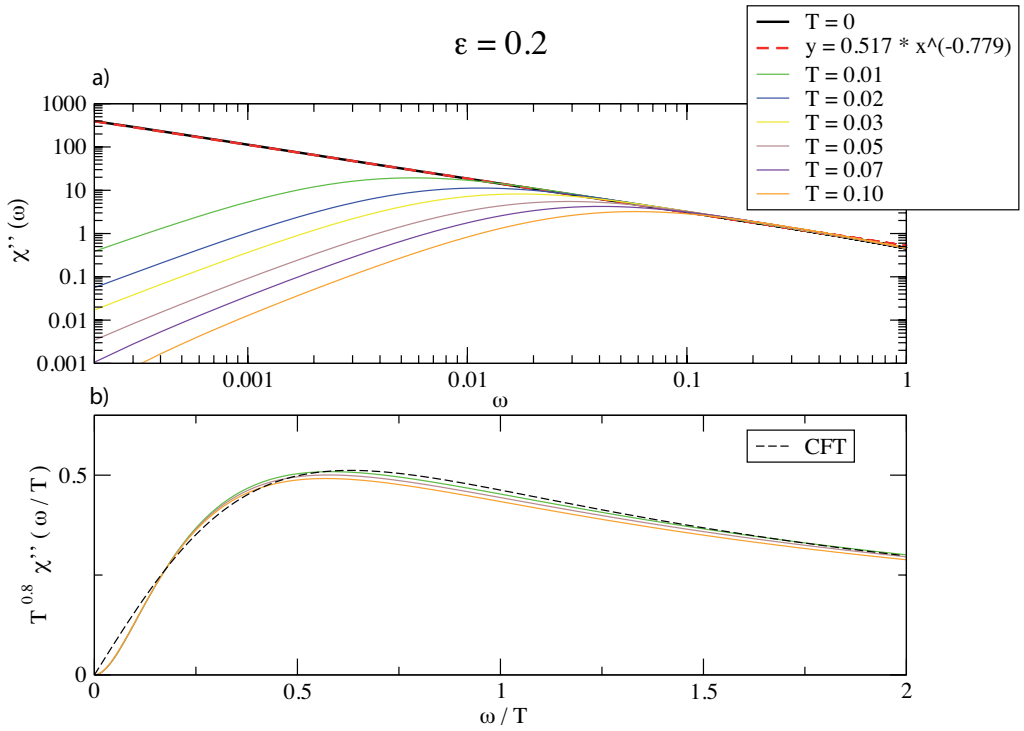

Figure 3.2: The expansion parameter ε is 0.2, a) and b) display, at least, qualitatively the same behaviour as in Fig.3.1

correlation function into the susceptibility, by using the fluctuation dissipation theorem

$$C_{\beta}^{(sym)}(\omega) = \coth\left(\frac{\beta\omega}{2}\right) \mathrm{Im}R_{\beta}(\omega), \tag{3.100}$$

and $R_{\beta}(\omega)$ is the general response function, which, if we consider spin operators, is the dynamical susceptibility. In Fig.3.1 and Fig.3.2 we see that in the high energy domain $\frac{\omega}{T} > 1$, firstly the bare imaginary part of the dynamical spin susceptibility $\chi''(\omega)$ features power law behaviour and secondly, the matching of the rescaled susceptibility with the conformal field theory result is quite good. Considering the low energy domain $\frac{\omega}{T} < 1$, on the one hand $\chi''(\omega)$ begins to deviate from the power low, and on the other hand the matching of the rescaled susceptibility and the conformal field theory result loses its quality. We want to note that the maximum of the conformal field theory result can only be altered by its absolute value not by its position. In Fig. 3.4 we have zoomed into the small frequency region, where the discrepance between the conformal field theory and the BFKM at the QCP is most noticeable. It has to be said that the region $\frac{\omega}{T} < 1$ is very difficult to access, not just for the flow equation methods but also for the ordinary RG-method or NRG, besides this fact, the agreement of curves is quite remarkable. There is an almost perfect

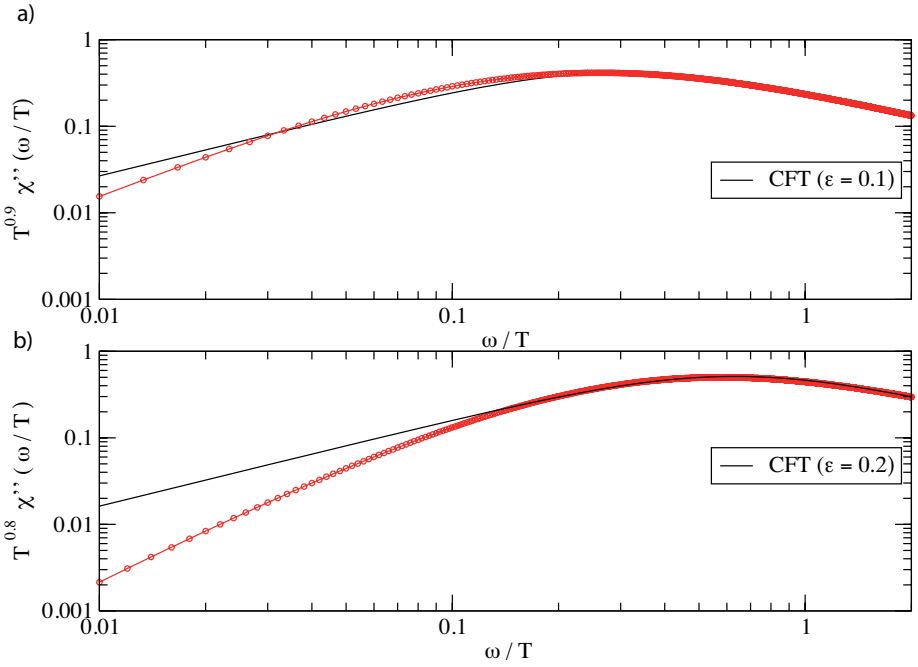

Figure 3.3: A douple logarithmic plot of the BFKM rescaled dynamical spin susceptibility in the low frequency region. a) $\varepsilon = 0.1$ and b) $\varepsilon = 0.2$, both plots display the some behaviour. If $\frac{\omega}{T} < 1$ the BFKM-susceptibility start to deviate from the conformal field theory solution.

match in the high frequency, when the frequency is greater than the temperature, domain.

As a final remark we can say, there are energy regions where the agreement of the imaginary part of the dynamical spin susceptibility and the conformal field theory result is quite astonishing, but on the other side there are energy regions where we see a not negligible deviation of the results. Our statement is, that the conformal field theory is a good approximation to the BFKM at the quantum phase transition point, however, whether there is an exact correspondence can not be decided yet.

Chapter 4

Outlook

The outlook deals with, firstly the time evolution of the BFKM, to be precise, the time evolution of the correlation function, and secondly with a principle determination of the $\mathcal{T}$-matrix.

4.1 Time Evolution of the Bose Model

In chapter 3 we saw that in first order of the ε-expansion only the bosonic part contribute to the leading behaviour of the correlation function and when we want to know how the istoropic BFKM evolves in time up to first order, it is very well justified to look at the so called Bose-Kondo model only, with the fermionic part is totally left aside.

4.1.1 Introduction

Real time evolution in many-body physics is an outstanding problem, since an evolution of an operator in the Heisenberg picture

$$A(t) = \sum_j \frac{(it)^j}{j!} [H, A]_n{}^{1\prime 2} \tag{4.1}$$

in powers of time, leads to serious difficulties. One is that the utility of the above expression is of restricted practicality, because an infinite number of commutators have to be calculated. Even in a perturbative way this is still hardly useful, as far as the long-time behaviour is concerned. In the long-time limit all higher terms of t eventually contribute, means, if the time value is larger than one over the (small) coupling constant, the higher terms cannot be neglected any more. But in the flow equation method the Hamiltonian finally becomes diagonal in the $B = \infty$ representation and typically the regarded operator also becomes fairly simple in that representation, which enables us to solve the real time

[1] $[H, A]_n$ denotes the n-fold commutator

[2] The operator A in the commutator is time independent, given in the Schrödinger picture

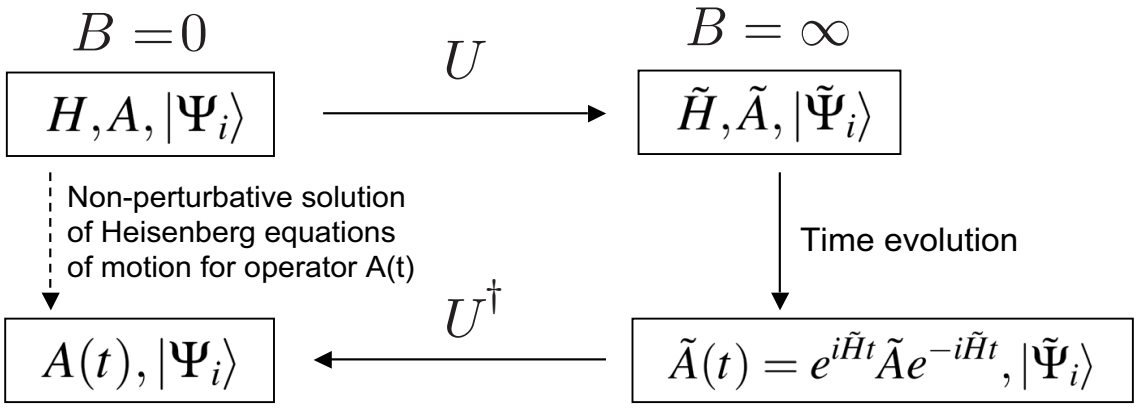

Figure 4.1: The time evolution of a physical system represented in terms of forward $B \to \infty$ at time $t = 0$ transformation and backward $B \to 0$ at time t transformation. Taken from [59, 60]

evolution problem. No secular terms, generated by the grow in time, are going to spoil the perturbative calculation.

Basically, the scheme works as follows, first one applies the unitary transformation depending on B on the system to make the Hamiltonian diagonal($B = \infty$). In this representation the Heisenberg equation of motion can be solved exactly, after that a backward transformation is performed, going from $B = \infty$ to $B = 0$. Finally, the considered operator is expressed in the original basis merely depending on time and not on the flow parameter B. So we end up with an effective way to solve the Heisenberg equation of motion, even non-perturbativly, at least in principle, and therefore non-equillibrium many body systems can be studied. In Figure 4.1, it is visualised, how the time evolution procedure is done in principal.
This idea has mainly been created by Andreas Hackl and Stefan Kehrein [59, 60]. In their work they investigate the spin evolution of the Spin Boson model.

4.2 Time Evolution

The BFKM Hamiltonian becomes diagonal

$$H(B=\infty) = \sum_{p\alpha} \epsilon_p : c^\dagger_{p\alpha} c_{p\alpha} : + \sum_k \omega_k : \mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}_k : \tag{4.2}$$

once the $B \to \infty$ limit is reached.
Now, we are in the position to make an reasonable attempt to obtain a solution of the Heisenberg equation of motion in this new basis. The Heisenberg equation of motion is

$$i\hbar \frac{dA(t)}{dt} = [A(t), H], \tag{4.3}$$

if A has got no explicit time dependence and is at least formally solved by

$$A(t) = e^{iHt}A(0)e^{-iHt}. \tag{4.4}$$

We use the operator relation

$$e^A B = Be^{A+D} \quad \text{where D satisfies} \quad [A,B] = DB \quad \text{where} \quad [A,D] = [B,D] = 0 \tag{4.5}$$

to solve Eqs(4.4).

We are mainly interested in the time dependent evolution of the spin operator S^z, together with the initial condition that S_z is fully aligned in $+\frac{1}{2}$ direction at $t = 0$[3]. In the previous section we made the following ansatz

$$S(B)^a = h(B)S^a + i\sum_k \Psi_k(\mathbf{S} \times (\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k))^a$$

in the purely bosonic case, since only the vector bosons governs the power law behaviour of the spin susceptibility. Further $H(B = \infty)$ reduces to

$$H(B = \infty) = \sum_k \omega_k : \mathbf{\Phi}_k^\dagger \cdot \mathbf{\Phi}_k :$$

and as a reminder the decisive part of the generator is

$$\eta(B) = \sum_k \omega_k \lambda_k \mathbf{S} \cdot (\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k), \tag{4.6}$$

so only two commutators has to be calculated

$$\sum_k \omega_k \varepsilon_{abc}[\Phi_k^{\dagger d}\Phi_k^d, S^b\Phi_l^{\dagger c}] = \sum_k \underbrace{(\omega_k)}_{=D}(\mathbf{S} \times \mathbf{\Phi}_k^\dagger)^a \tag{4.7}$$

and analogous

$$-\sum_k \omega_k \varepsilon_{abc}[\Phi_k^{\dagger d}\Phi_k^d, S^b\Phi_l^c] = -\sum_k \underbrace{-(\omega_k)}_{=D}(\mathbf{S} \times \mathbf{\Phi}_k)^a, \tag{4.8}$$

which obviously satisfy the above requirements. So the time dependent spin operator reads

$$S^a(B = \infty, t) = h(B = \infty)S^a + i\sum_k \Psi_k \mathbf{S} \times (\mathbf{\Phi}_k^\dagger e^{i\omega_k t} - \mathbf{\Phi}_k e^{-i\omega_k t}), \tag{4.9}$$

by redefining the old parameter as

$$\tilde{\Psi}_k := \Psi_k e^{i\omega_k t} \quad \text{and} \quad \tilde{\Psi}_k{}^* := \Psi_k e^{-i\omega_k t} \tag{4.10}$$

[3] This may be achieved by applying an external magnetic field for negative t and suddenly switch it off at $t = 0$.

we make a 'new' ansatz for the spin-operator

$$S^a(B,t) = h(B)S^a + i\sum_k \tilde{\Psi}_k(B,t)\mathbf{S}\times\mathbf{\Phi}_k^\dagger - i\sum_k \tilde{\Psi}_k{}^*(B,t)\mathbf{S}\times\mathbf{\Phi}_k. \tag{4.11}$$

And then we apply the backwards transformation on it, by integrating B from infinity down to zero and the time dependent solution of the Heisenberg equation of motion acts as the new initial condition.

The commutators are given by

$$\begin{aligned}[\eta(B), S^a_{(h)}] &= i\sum_k h(B)\omega_k\lambda_k\mathbf{S}\cdot\mathbf{\Phi}_k^\dagger \\ &\quad -i\sum_k h(B)\omega_k\lambda_k\mathbf{S}\cdot\mathbf{\Phi}_k \end{aligned}\tag{4.12}$$

and

$$\begin{aligned}[\eta(B), S^a_{(\tilde{\Psi})}] &= -\sum_{kl}\omega_k\lambda_k\tilde{\Psi}_l(\mathbf{\Phi}_k^{\dagger(a)}(\mathbf{\Phi}_l^\dagger\cdot\mathbf{S}) - \mathbf{S}^a(\mathbf{\Phi}_k^\dagger\cdot\mathbf{\Phi}_l^\dagger)) \\ &\quad +\sum_{kl}\omega_k\lambda_k\tilde{\Psi}_l(:\mathbf{\Phi}_k^{\dagger(a)}(\mathbf{\Phi}_l\cdot\mathbf{S}:) - (:\mathbf{S}^a(\mathbf{\Phi}_k^\dagger\cdot\mathbf{\Phi}_l:)) \\ &\quad -\sum_k\omega_k\lambda_k\tilde{\Psi}_k(1+2n_b(k)) \end{aligned}\tag{4.13}$$

and

$$\begin{aligned}[\eta(B), S^a_{(\tilde{\Psi}^*)}] &= -\sum_{kl}\omega_k\lambda_k\tilde{\Psi}_l^*(\mathbf{\Phi}_k^a(\mathbf{\Phi}_l\cdot\mathbf{S}) - \mathbf{S}^a(\mathbf{\Phi}_k\cdot\mathbf{\Phi}_l)) \\ &\quad +\sum_{kl}\omega_k\lambda_k\tilde{\Psi}_l^*(:\mathbf{\Phi}_l^\dagger\cdot\mathbf{S}(\mathbf{\Phi}_k^a:) - (:\mathbf{S}^a(\mathbf{\Phi}_l^\dagger\cdot\mathbf{\Phi}_k:)) \\ &\quad -\sum_k\omega_k\lambda_k\tilde{\Psi}_k^*(1+2n_b(k)). \end{aligned}\tag{4.14}$$

Now, we can easily read off the flow equations

$$\boxed{\frac{dh(B)}{dB} = -\sum_k\omega_k\lambda_k(\tilde{\Psi}_k+\tilde{\Psi}_k^*)(1+2n(k))} \tag{4.15}$$

and

$$\boxed{\frac{d\tilde{\Psi}_k(B)}{dB} = \omega_k\lambda_k h(B)} \tag{4.16}$$

and

$$\boxed{\frac{d\tilde{\Psi}_k^*(B)}{dB} = \omega_k \lambda_k h(B)}. \tag{4.17}$$

After inserting λ_k into Eqs.(4.16) in favour of α and integrating down to $B = 0$ we obtain

$$\begin{aligned}
\int_{\tilde{\Psi}_k(B,t)}^{\tilde{\Psi}_k(0,t)} d\tilde{\Psi}_k &= \omega_k \sqrt{2\omega_k^{1-\varepsilon}} \int_B^0 dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} \\
\Rightarrow \tilde{\Psi}_k(0,t) &= \tilde{\Psi}_k(B) e^{i\omega_k t} \\
&\quad + \omega_k \sqrt{2\omega_k^{1-\varepsilon}} \int_B^0 dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} \\
&= \omega_k \sqrt{2\omega_k^{1-\varepsilon}} \int_0^B dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} (e^{i\omega_k t} - 1),
\end{aligned} \tag{4.18}$$

on the same footing this can be done for $\tilde{\Psi}_k^*$,

$$\tilde{\Psi}_k^*(0,t) = \omega_k \sqrt{2\omega_k^{1-\varepsilon}} \int_0^B dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} (e^{-i\omega_k t} - 1). \tag{4.19}$$

We plug these into Eqs.(4.15) and receive a flow equation merely for the time dependent h. The evolution of $h(0,t)$ governs the time-behaviour of the spin-operator after the switch off, of the external constrain e.g. magnetic field

$$\begin{aligned}
\frac{dh(B,t)}{dB} &= 4 \sum_k \omega_k^{3-\varepsilon} e^{-B\omega_k^2} \sqrt{\alpha(B)} \int_0^B dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} \\
&\quad - 2 \sum_k \omega_k^{3-\varepsilon} e^{-B\omega_k^2} \sqrt{\alpha(B)} \int_0^B dB' \sqrt{\alpha(B')} h(B') e^{-B'\omega_k^2} \\
&\quad \times \underbrace{(e^{i\omega_k t} + e^{-i\omega_k t})}_{2\cos(\omega_k t)} \\
&= \varepsilon \sum_k \omega_k^{3-\varepsilon} e^{-B\omega_k^2} B^{-\frac{\varepsilon}{4}} \int_0^B dB' B'^{-\frac{\varepsilon}{4}} h(B') e^{-B'\omega_k^2} \\
&\quad \times (1 - \cos(\omega_k t)),
\end{aligned} \tag{4.20}$$

where we used the fixed point expression for $\alpha(B)$ and neglected again the gamma function by just considering first order calculations in ε.

Substituting $\omega = \frac{x}{\sqrt{(B+B')}}$ and expanding $x^{3-\varepsilon} = x^3 + \mathcal{O}(\varepsilon)$, it follows

$$\begin{aligned}\frac{dh(B,t)}{dB} &= \varepsilon \int_0^B dB' \frac{h(B')}{(B+B')^2} \int_0^{\sqrt{B+B'}\omega_c} dx x^3 e^{-x^2} (1 - \cos(x \frac{t}{\sqrt{B+B'}})) \\ &= \varepsilon \int_0^B dB' \frac{h(B')}{(B+B')^{3/2}} \frac{t}{16} e^{-\frac{1}{4}\frac{t^2}{B+B'}} \\ &\quad \times \left(-6i\sqrt{\pi}\mathrm{erf}((\frac{1}{2} i \frac{t}{\sqrt{B+B'}}) + 2\frac{t}{\sqrt{B+B'}} e^{\frac{1}{4}\frac{t^2}{B+B'}} \right. \\ &\quad \left. + i\frac{t^2}{B+B'}\sqrt{\pi}\mathrm{erf}(\frac{1}{2} i \frac{t}{\sqrt{B+B'}}) \right). \end{aligned} \tag{4.21}$$

Substituting $\frac{t}{\sqrt{B+B'}} = x$ yields to (note we change the integration limit and picking up a minus sign)

$$\begin{aligned}\frac{dh(B,t)}{dB} &= \frac{\varepsilon}{8} \int_{\frac{t}{\sqrt{2B}}}^{\frac{t}{\sqrt{B}}} dx h((\frac{t^2}{x^2} + B) e^{-\frac{1}{4}x^2} \\ &\quad \times \left(-6i\sqrt{\pi}\mathrm{erf}((\frac{1}{2} ix) + 2xe^{\frac{1}{4}x^2} + ix^2\sqrt{\pi}\mathrm{erf}(\frac{1}{2} ix) \right) \end{aligned} \tag{4.22}$$

Interim we are just concerned in the behaviour of small time evolution, (additionally we could say that we are coming from $B \sim \infty$) so we integrate over a very small region of small x-values

$$\left(-6i\sqrt{\pi}\mathrm{erf}((\frac{1}{2} ix) + 2xe^{\frac{1}{4}x^2} + ix^2\sqrt{\pi}\mathrm{erf}(\frac{1}{2} ix) \right) e^{-\frac{1}{4}x^2} = 8x - 2x^3 + \mathcal{O}(x^5), \tag{4.23}$$

this leads to the following simplified differential equation for $h(B,t)$, namely

$$\frac{dh(B,t)}{dB} = \frac{\varepsilon}{8} \int_{\frac{t}{\sqrt{2B}}}^{\frac{t}{\sqrt{B}}} dx h((\frac{t^2}{x^2} + B)(8x - 2x^3). \tag{4.24}$$

At this stage there is not much hope to find an analytic expression for $h(B,t)$ that solves this differential equation, so what we can do, is to go back to the time dependent flow quations and try to solve them numerically[4].

[4]At the time this thesis has been written, the work on the numerical solution was still in progress. The work was done in collaboration with Anderas Hackl in Köln.

4.3 The $\mathcal{T}$-matrix

The correlation function, we obtained in the previous chapter, is not the only quantity of interest in the local system. The second very important one is the $\mathcal{T}$-matrix. In order to get a solution of the self-consistency equation we need to know an expression for the $\mathcal{T}$-matrix of the impurity system. In fact, the self-consistency equations are given in terms of the susceptibility and the Green's function, which is directly related to the $\mathcal{T}$-matrix.

The $\mathcal{T}$-matrix for the scattering of the conduction electrons on the impurity is defined by

$$\mathcal{G}_{pq.\sigma}(\omega) = \delta_{pq}\mathcal{G}^0_{pq,\sigma}(\omega) + \mathcal{G}^0_{pq,\sigma}(\omega)\mathcal{T}_\sigma(\omega)\mathcal{G}^0_{pq,\sigma}(\omega), \tag{4.25}$$

$\mathcal{G}_{pq.\sigma}(\omega) = \langle\langle c_{p,\sigma}; c^\dagger_{q,\sigma}\rangle\rangle$ is the retarded Green's function and $\mathcal{G}^0_{pq,\sigma}(\omega)$ the unperturbated Green's function, for a more detailed description see [3]. On the contrary to the bare Kondo model, the BFKM has one significantly different setting, not only the conduction electrons influence the spin of the impurity but also the bosonic degrees of freedom. So we expect a mixture of bosonic and fermionic operators in the $\mathcal{T}$-matrix, due to the fact that both, the bosonic and fermionic degrees of freedom interact with the spin operator. But, the $\mathcal{T}$-matrix is still local in our model similar to the Kondo model. By solving the Heisenberg equation of motion of the Green's function Costi and Rosch *et.al.* [61] and [62], respectively, could show that $\mathcal{T}_\sigma(\omega)$ has the form

$$\boxed{\mathcal{T}_\sigma(\omega) = J_K^2\langle\langle O_\sigma; O^\dagger_\sigma\rangle\rangle}, \tag{4.26}$$

where $O_\sigma = \sum_{p\alpha} c^\dagger_{p\alpha}\frac{\sigma_{\alpha\sigma}}{2}\cdot\mathbf{S}$ and J_K is the Kondo coupling. The retarded Green's function is purely imaginary and therefore we take only the imaginary part of the $\mathcal{T}$-matrix, furthermore, we perform a Fourier transformation as we are interested in its ω-dependece

$$Im(\hat{T}_\sigma(\omega)) = Im\left(-i\int_\infty^{-\infty} dt \quad \Theta(t)\langle\{O_\sigma(t), O^\dagger_\sigma(0)\}\rangle e^{i\omega t}\right). \tag{4.27}$$

Now, we have the basic ingredients, to calculate the $\mathcal{T}$-matrix, by making use of the flow equation method, which means we must transform the operator O_σ into the diagonal basis[5]. Again, we need to solve Eqs.(1.76), by simply taking

$$\eta(B) = \sum_{pq}(\epsilon_p - \epsilon_q)J(p,q)(B) : \mathbf{S}\cdot\mathbf{s}_{p,q} : + \sum_k \lambda_k(B)\omega_k\mathbf{S}\cdot(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k) \tag{4.28}$$

as your generator and our initial operator is

$$O_\sigma = \sum_{p\alpha} T_p(B)c^\dagger_{p\alpha}\frac{\sigma_{\alpha\sigma}}{2}\cdot\mathbf{S}, \tag{4.29}$$

[5]See chapter 1 for further information.

where we set $J_K = T_p(B = 0)$. In the same manner, as in the previous chapters, we can see that during the flow new terms are going to be generated, thus we make a better ansatz for the operator

$$O_\sigma = \sum_{p\alpha} T_p(B) c^\dagger_{p\alpha} \frac{\sigma_{\alpha\sigma}}{2} \cdot \mathbf{S} + i \sum_{kp} \sum_{\alpha} P_{kp}(B)(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k) \cdot (c^\dagger_{p\alpha} \frac{\sigma_{\alpha\sigma}}{2} \times \mathbf{S}), \tag{4.30}$$

with $P_{kp}(B) = \lambda_k(B)\omega_k T_p(B)$. From the expression for the newly generated coupling $P_{kp}(B)$ we can read off the connection of the bosonic degrees and fermionic degrees of freedom as aforementioned. The actual calculation of the $\mathcal{T}$-matrix is postponed to the Appendix(C) as it was not complete at the time this thesis has been written.

Conclusion

After introducing the Kondo lattice and the BFKM we briefly described the mapping of the Kondo lattice onto the BFKM, via EDMFT. Then we applied the flow equation method on both the Ising BFKM and the isotropic BFKM.
The Ising BFKM was merely used to check the efficiency of the flow equation method, since non-perturbative results form NRG are available to compare with. The essential part of the thesis dealt with applying the flow equation method on the isotropic BFKM. We could reproduce already known results such as, the correct flow of the coupling constants in the infrared limit and its non trivial solution displaying a perturbatively accessible unstable fixed point, which actually is the transition point. Anymore, the approximate ω-dependence of the spin correlation function at the quantum critical point matches also with the known one. More important is, however, that in this thesis the non zero temperature dynamical spin susceptibility was given for the very first time. Remarkably, the temperature dependent spin susceptibility showed the desired $\frac{\omega}{T}$-scaling.
We used that result to compare it with a recently proposed conjecture between the isotropic BFKM at the quantum critical point and a certain conformal field theory, by looking at the dynamical spin susceptibility and respectively the correlation function of the primary fields. The astonishing agreements provide certainly no prove for the conjecture, but at least we can say that for a small expansion parameter the conformal field theory is a very good approximation. We thus justified the conformal field theory assumption used in the literature.
In the outlook chapter we analytically obtained a set of differential equations for the time evolution of the impurity spin, after switching on the couplings of the spin impurity to the bathes. Further analysing of the differential equations in terms of numerical studies need to be done. Additionally, we provided a way how to calculate the $\mathcal{T}$-matrix, as the last ingredient for solving the self-consistency equation of the EDMFT. All the calculations have been made using a power law like bosonic density of states $S(\omega) \sim \omega^{1-\varepsilon}$, as it has been done in the literature concerning this issue. The flow equation method is principally capable of dealing with an arbitrary kind of bosonic density of states, however this remains to be done in future work.
So with the help of the flow equation method we could give a full solution, for the dynamical spin susceptibility, and a recipe for the $\mathcal{T}$-matrix. With the help of this recipe the first expression for the $\mathcal{T}$-matrix could be received in order to understand the nature of the emerging local quantum phase transition in certain Kondo lattices. Based on this work new

insights into the quantum phase transition can be gained, by solving the self-consitency equations.

Appendix A

Ising Bose Fermi Kondo Model

A.1 The first newly Generated Terms

The required commutators read

$$
\begin{aligned}
[\eta_0^{\perp}(B), H_{\rm int}^{\parallel,(f)}(B)] &= \frac{1}{4}\sum_{pqmn}(\epsilon_p-\epsilon_q)J^{\perp}(p,q)(B)J^{\parallel}(m,n)(B) \\
&\times\Big((:c^{\dagger}_{p\uparrow}c_{q\downarrow}c^{\dagger}_{m\uparrow}c_{n\uparrow}:S^{-}+:c^{\dagger}_{q\downarrow}c_{p\uparrow}c^{\dagger}_{m\uparrow}c_{n\uparrow}:S^{+}) \\
&\quad -(:c^{\dagger}_{p\uparrow}c_{q\downarrow}c^{\dagger}_{m\downarrow}c_{n\downarrow}:S^{-}+:c^{\dagger}_{q\downarrow}c_{p\downarrow}c^{\dagger}_{m\uparrow}c_{n\uparrow}S^{+}:)\Big) \\
&+\frac{1}{8}\sum_{pqm}(1-2n_f(m))\Big((\epsilon_m-\epsilon_q)J^{\perp}(m,q)(B)J^{\parallel}(p,m)(B) \\
&\quad -(\epsilon_p-\epsilon_m)J^{\perp}(p,m)(B)J^{\parallel}(q,m)(B)\Big) \\
&\quad \times(:c^{\dagger}_{p\uparrow}c_{q\downarrow}:S^{-}+:c^{\dagger}_{q\downarrow}c_{p\uparrow}:S^{+}) \qquad \text{(A.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^{\parallel}(B), H_{\rm int}^{\perp,(f)}(B)] &= \frac{1}{4}\sum_{pqmn}(\epsilon_p-\epsilon_q)J^{\perp}(m,n)(B)J^{\parallel}(p,q)(B) \\
&\times\Big((-:c^{\dagger}_{p\uparrow}c_{q\uparrow}c^{\dagger}_{m\uparrow}c_{n\downarrow}:S^{-}+:c^{\dagger}_{p\uparrow}c_{q\uparrow}c^{\dagger}_{m\downarrow}c_{n\uparrow}:S^{+}) \\
&\quad -(-:c^{\dagger}_{p\downarrow}c_{q\downarrow}c^{\dagger}_{m\uparrow}c_{n\downarrow}:S^{-}+:c^{\dagger}_{p\downarrow}c_{q\downarrow}c^{\dagger}_{m\downarrow}c_{n\uparrow}S^{+}:)\Big) \\
&+\frac{1}{8}\sum_{pqm}(1-2n_f(m))\Big((\epsilon_m-\epsilon_p)J^{\perp}(m,q)(B)J^{\parallel}(p,m)(B) \\
&\quad -(\epsilon_q-\epsilon_m)J^{\perp}(p,m)(B)J^{\parallel}(q,m)(B)\Big) \\
&\quad \times(:c^{\dagger}_{p\uparrow}c_{q\downarrow}:S^{-}+:c^{\dagger}_{q\downarrow}c_{p\uparrow}:S^{+}) \qquad \text{(A.2)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^{\perp}(B), H_{\rm int}^{\perp,(f)}(B)] \;=\; & -\frac{1}{2}\sum_{pqmn}(\epsilon_p-\epsilon_q+\epsilon_m-\epsilon_n)J^{\perp}(p,q)(B)J^{\perp}(m,n)(B) \\
& \times(:c_{n\downarrow}^{\dagger}c_{m\uparrow}c_{p\uparrow}^{\dagger}c_{q\downarrow}:S^z) \\
& -\frac{1}{2}\sum_{pq}(n_f(p)+n_f(q)-2n_f(p)n_f(q))(\epsilon_p-\epsilon_q)(J^{\perp}(p,q)(B))^2S^z \\
& -\frac{1}{4}\sum_{pqm}(1-2n_f(m))(2\epsilon_m-\epsilon_p-\epsilon_q) \\
& \times J^{\perp}(m,q)(B)J^{\perp}(m,p):c_{p\downarrow}^{\dagger}c_{q\downarrow}:S^z \\
& -\frac{1}{4}\sum_{pqm}(1-2n_f(m))(2\epsilon_m-\epsilon_p-\epsilon_q) \\
& \times J^{\perp}(p,m)(B)J^{\perp}(q,m):c_{p\uparrow}^{\dagger}c_{q\uparrow}:S^z \\
& -\frac{1}{8}\sum_{pqm}(2\epsilon_m-\epsilon_p-\epsilon_q)J^{\perp}(m,q)(B)J^{\perp}(m,p):c_{p\downarrow}^{\dagger}c_{q\downarrow}: \\
& -\frac{1}{8}\sum_{pqm}(2\epsilon_m-\epsilon_p-\epsilon_q)J^{\perp}(q,m)(B)J^{\perp}(p,m):c_{p\uparrow}^{\dagger}c_{q\uparrow}: \qquad \text{(A.3)}
\end{aligned}
$$

and

$$
[\eta_0^b(B), H_0^b(B)] = -S^z\sum_k \omega_k^2\lambda_k(b_k^{\dagger}+b_k) \qquad \text{(A.4)}
$$

and

$$
[\eta_0^b(B), H_{\rm int}^{\perp,(f)}+H_{\rm int}^{\parallel,(f)}] = \frac{1}{2}\sum_{kpq}\omega_k\lambda_k J_{\perp}(p,q)(b_k^{\dagger}-b_k)(:c_{q\downarrow}^{\dagger}c_{p\uparrow}:S^+-:c_{p\uparrow}^{\dagger}c_{q\downarrow}:S^-) \qquad \text{(A.5)}
$$

and

$$
[\eta_0^{\parallel}+\eta_0^{\perp}, H_{\rm int}^b] = \frac{1}{2}\sum_{kpq}(\epsilon_p-\epsilon_q)J_{\perp}(p,q)\lambda_k(b_k^{\dagger}+b_k)(:c_{p\uparrow}^{\dagger}c_{q\downarrow}:S^-+:c_{q\downarrow}^{\dagger}c_{p\uparrow}:S^-) \qquad \text{(A.6)}
$$

and

$$
[\eta_0^b(B), H_{\rm int}^b(B)] = -\frac{1}{2}\sum_k \omega_k\lambda_k^2. \qquad \text{(A.7)}
$$

$$
\begin{aligned}
[\eta_0^{\perp,(f)}(B)+\eta_0^{\parallel,(f)}(B),H_0(B)] &= \frac{1}{2}\sum_{pq}(\epsilon_p-\epsilon_q)^2 J^{\parallel}(p,q)(B)(:c_{p\uparrow}^{\dagger}c_{q\uparrow}^{\dagger}:-:c_{p\downarrow}^{\dagger}c_{q\downarrow}^{\dagger}:)S^z \\
&\quad +\frac{1}{2}\sum_{pq}(\epsilon_p-\epsilon_q)^2 J^{\perp}(p,q)(B) \\
&\quad \times(:c_{p\uparrow}^{\dagger}c_{q\downarrow}^{\dagger}:S^- -:c_{q\downarrow}^{\dagger}c_{p\uparrow}^{\dagger}:S^+), \qquad \text{(A.8)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^{\parallel}(B),H_{\text{int}}^{\parallel,(f)}(B)] &= \frac{1}{8}\sum_{pq}(n_f(p)-n_f(q))(\epsilon_p-\epsilon_q)J^{\parallel}(p.q)(B)^2 \\
&\quad +\frac{1}{16}\sum pqm(\epsilon_p+\epsilon_q-2\epsilon_m)J^{\parallel}(p,m)(B)J^{\parallel}(m,q)(B) \\
&\quad \times(:c_{p\uparrow}^{\dagger}c_{q\uparrow}+:c_{p\downarrow}^{\dagger}c_{q\downarrow}) \qquad \text{(A.9)}
\end{aligned}
$$

A.2 Correction Terms

Here we present the remaining commutator relations, which are needed in order to get the dynamicas of the couplings in terms of the flow parmeter B up to leading order in the couplings

$$
\begin{aligned}
[\eta_0^{\parallel}(B),H_{new}^{K}(B)] &= \frac{1}{2}\sum_{mnpqk}(\epsilon_m-\epsilon_n)J^{\parallel}(m,n)(B)K_{kpq}(B) \\
&\quad \times\Big(:c_{m\uparrow}^{\dagger}c_{n\uparrow}c_{q\downarrow}^{\dagger}c_{p\uparrow}:-:c_{m\downarrow}^{\dagger}c_{n\downarrow}c_{q\downarrow}^{\dagger}c_{p\uparrow}:)b_k^{\dagger}S^+ \\
&\qquad +(:c_{m\downarrow}^{\dagger}c_{n\downarrow}c_{p\uparrow}^{\dagger}c_{q\downarrow}:-:c_{m\uparrow}^{\dagger}c_{n\uparrow}c_{p\uparrow}^{\dagger}c_{q\downarrow}:)b_kS^-\Big) \\
&\quad +\frac{1}{4}\sum_{npqk}\big((\epsilon_q-\epsilon_n)J^{\parallel}(n,q)(B)K_{kpq}(B)-(\epsilon_p-\epsilon_q)J^{\parallel}(q,p)(B)K_{kqn}(B)\big) \\
&\quad \times(1-2n(q))(b_k^{\dagger}:c_{n\downarrow}^{\dagger}c_{p\uparrow}:S^+ + b_k:c_{p\uparrow}^{\dagger}c_{n\downarrow}:S^-) \qquad \text{(A.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^{\perp}(B), H_{new}^K(B)] &= \frac{1}{2}\sum_{mnpqk}(\epsilon_m-\epsilon_n)J^{\perp}(m,n)(B)K_{kpq}(B) \\
&\times(:c_{m\uparrow}^{\dagger}c_{n\downarrow}c_{q\downarrow}^{\dagger}c_{p\uparrow}:b_k^{\dagger}+:c_{n\downarrow}^{\dagger}c_{m\uparrow}c_{p\uparrow}^{\dagger}c_{p\downarrow}:b_k)S^z \\
&+\frac{1}{2}\sum_{mpqk}(\epsilon_q-\epsilon_m)J^{\perp}(m,q)(B)K_{kpq}(B)(1-2n(q)) \\
&\times\left(b_k^{\dagger}(:c_{m\uparrow}^{\dagger}c_{p\uparrow}:-:c_{p\downarrow}^{\dagger}c_{m\downarrow}:)+b_k(:c_{p\uparrow}^{\dagger}c_{m\uparrow}:-:c_{m\downarrow}^{\dagger}c_{p\downarrow}:)\right)S_z \\
&+\frac{1}{4}\sum_{mnpqk}(\epsilon_m-\epsilon_q)J^{\perp}(m,q)(B)K_{kpq}(B) \\
&\times\left(b_k^{\dagger}(:c_{m\uparrow}^{\dagger}c_{p\uparrow}:+:c_{p\downarrow}^{\dagger}c_{m\downarrow}:)+b_k(:c_{p\uparrow}^{\dagger}c_{m\uparrow}:+:c_{m\downarrow}^{\dagger}c_{p\downarrow}:)\right) \\
&+\frac{1}{2}\sum_{pqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,q)(B)K_{kpq}(B) \\
&\quad\times(n(p)+n(q)-2n(p)n(q))(b_k^{\dagger}+b_k)S^z \\
&+\frac{1}{4}\sum_{pqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,q)(B)K_{kpq}(B)(n(p)-n(q))(b_k^{\dagger}+b_k)
\end{aligned}
\tag{A.11}
$$

and

$$
\begin{aligned}
[\eta_0^{b}(B), H_{\rm new}^K(B)] &= \sum_{pqkl}\omega_k\lambda_k(B)K_{lpq}(B)(:c_{q\downarrow}^{\dagger}c_{p\uparrow}:b_k^{\dagger}b_l^{\dagger}S^{+}+:c_{p\uparrow}^{\dagger}c_{q\downarrow}:b_kb_lS^{-}) \\
&-\sum_{pqkl}\omega_k\lambda_k(B)K_{lpq}(B)(:c_{q\downarrow}^{\dagger}c_{p\uparrow}::b_l^{\dagger}b_k:S^{+}+:c_{p\uparrow}^{\dagger}c_{q\downarrow}::b_k^{\dagger}b_l:S^{-}) \\
&-\frac{1}{2}\sum_{pqk}\omega_k\lambda_k(B)K_{kpq}(B)(1+2n_b(k)) \\
&\quad\times(:c_{q\downarrow}^{\dagger}c_{p\uparrow}:S^{+}+:c_{p\uparrow}^{\dagger}c_{q\downarrow}:S^{-})
\end{aligned}
\tag{A.12}
$$

and

$$
\begin{aligned}
[\eta_0^{\parallel}(B), H_{\rm new}^L(B)] &= \frac{1}{2}\sum_{mnpqk}(\epsilon_m-\epsilon_n)J^{\parallel}(m,n)(B)L_{kpq}(B) \\
&\quad\times\Big((:c_{m\uparrow}^{\dagger}c_{n\uparrow}c_{q\downarrow}^{\dagger}c_{p\uparrow}:-:c_{m\downarrow}^{\dagger}c_{n\downarrow}c_{q\downarrow}^{\dagger}c_{p\uparrow}:)b_kS^{+} \\
&+(:c_{m\downarrow}^{\dagger}c_{n\downarrow}c_{p\uparrow}^{\dagger}c_{q\downarrow}:-:c_{m\uparrow}^{\dagger}c_{n\uparrow}c_{p\uparrow}^{\dagger}c_{q\downarrow}:)b_k^{\dagger}S^{-}\Big) \\
&+\frac{1}{4}\sum_{npqk}\left((\epsilon_q-\epsilon_n)J^{\parallel}(n,q)(B)L_{kpq}(B)-(\epsilon_p-\epsilon_q)J^{\parallel}(q,p)(B)L_{kqn}(B)\right) \\
&\quad\times(1-2n(q))(b_k:c_{n\downarrow}^{\dagger}c_{p\uparrow}:S^{+}+b_k^{\dagger}:c_{p\uparrow}^{\dagger}c_{n\downarrow}:S^{-})
\end{aligned}
\tag{A.13}
$$

and

$$
\begin{aligned}
[\eta_0^{\perp}(B), H_{\text{new}}^{L}(B)] \;=\; & \sum_{mnpqk} (\epsilon_n - \epsilon_m) J^{\perp}(m,n)(B) L_{kpq}(B) \\
& \times \Big(: c_{m\uparrow}^{\dagger} c_{n\downarrow} c_{q\downarrow}^{\dagger} c_{p\uparrow} : b_k + : c_{n\downarrow}^{\dagger} c_{m\uparrow} c_{p\uparrow}^{\dagger} c_{p\downarrow} : b_k^{\dagger} \Big) S^z \\
& + \frac{1}{2} \sum_{mnpqk} (\epsilon_q - \epsilon_m) J^{\perp}(m,q)(B) L_{kpq}(B)(1 - 2n(q)) \\
& \times \Big(b_k (: c_{m\uparrow}^{\dagger} c_{p\uparrow} : - : c_{p\downarrow}^{\dagger} c_{m\downarrow} :) + b_k^{\dagger} (: c_{p\uparrow}^{\dagger} c_{m\uparrow} : - : c_{m\downarrow}^{\dagger} c_{p\downarrow} :) \Big) S_z \\
& + \frac{1}{4} \sum_{mnpqk} (\epsilon_m - \epsilon_q) J^{\perp}(m,q)(B) L_{kpq}(B) \\
& \times (b_k (: c_{m\uparrow}^{\dagger} c_{p\uparrow} : + : c_{p\downarrow}^{\dagger} c_{m\downarrow} :) + b_k^{\dagger} (: c_{p\uparrow}^{\dagger} c_{m\uparrow} : + : c_{m\downarrow}^{\dagger} c_{p\downarrow} :)) \\
& + \frac{1}{2} \sum_{pqk} (\epsilon_p - \epsilon_q) J^{\perp}(p,q)(B) L_{kpq}(B)(n(p) + n(q) - 2n(p)n(q))(b_k^{\dagger} + b_k) S^z \\
& + \frac{1}{4} \sum_{pqk} (\epsilon_p - \epsilon_q) J^{\perp}(p,q)(B) L_{kpq}(B)(n(p) - n(q))(b_k^{\dagger} + b_k) \qquad \text{(A.14)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^{b}(B), H_{\text{new}}^{L}(B)] \;=\; & - \sum_{pqkl} \omega_k \lambda_k(B) L_{lpq}(B) (: c_{q\downarrow}^{\dagger} c_{p\uparrow} : b_k b_l S^+ + : c_{p\uparrow}^{\dagger} c_{q\downarrow} : b_k^{\dagger} b_l^{\dagger} S^-) \\
& + \sum_{pqkl} \omega_k \lambda_k(B) L_{lpq}(B) (: c_{q\downarrow}^{\dagger} c_{p\uparrow} :: b_k^{\dagger} b_l : S^+ + : c_{p\uparrow}^{\dagger} c_{q\downarrow} :: b_l^{\dagger} b_k : S^-) \\
& + \frac{1}{2} \sum_{pqk} \omega_k \lambda_k(B) L_{kpq}(B)(1 + 2n_b(k)) \\
& \times (: c_{q\downarrow}^{\dagger} c_{p\uparrow} : S^+ + : c_{p\uparrow}^{\dagger} c_{q\downarrow} : S^-) \qquad \text{(A.15)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_{\text{new}}^{K}(B), H_{\text{int}}^{\parallel,(f)}(B)] \;=\; & \frac{1}{2} \sum_{mnpqk} K_{kpq}(B)((\epsilon_p - \epsilon_q) - \omega_k) J^{\parallel}(m,n) \\
& \times \Big(: c_{n\uparrow}^{\dagger} c_{m\uparrow} c_{p\uparrow}^{\dagger} c_{q\downarrow} : b_k S^- + : c_{m\uparrow}^{\dagger} c_{n\uparrow} c_{q\downarrow}^{\dagger} c_{p\uparrow} : b_k^{\dagger} S^+ \\
& \quad - (: c_{m\downarrow}^{\dagger} c_{n\downarrow} c_{p\uparrow}^{\dagger} c_{q\downarrow} : b_k S^- + : c_{n\downarrow}^{\dagger} c_{n\downarrow} c_{q\downarrow}^{\dagger} c_{p\uparrow} : b_k^{\dagger} S^+) \Big) \\
& + \frac{1}{4} \sum_{mpqk} (K_{kpq}(B)((\epsilon_p - \epsilon_q) - \omega_k) J^{\parallel}(m,p)(B) \\
& \quad - K_{kmp}(B)((\epsilon_m - \epsilon_p) - \omega_k) J^{\parallel}(p,q))(B) \\
& \times (1 - 2n(p))(: c_{m\uparrow}^{\dagger} c_{q\downarrow} : b_k S^- + : c_{q\downarrow}^{\dagger} c_{m\uparrow} : b_k^{\dagger} S^+) \qquad \text{(A.16)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^K_{\rm new}(B), H^{\perp,(f)}_{\rm int}(B)] \;=\; & -\sum_{mnpqk}(\epsilon_n-\epsilon_m)J^{\perp}(m,n)(B)K_{kpq}(B) \\
& \times(:c^\dagger_{m\uparrow}c_{n\downarrow}c^\dagger_{q\downarrow}c_{p\uparrow}:b^\dagger_k+:c^\dagger_{n\downarrow}c_{m\uparrow}c^\dagger_{p\uparrow}c_{p\downarrow}:b_k)S^z \\
& -\frac{1}{2}\sum_{npqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,n)(B)K_{kpq}(B)(1-2n(p)) \\
& \times\Big((b^\dagger_k:c^\dagger_{n\uparrow}c_{q\uparrow}:-:c^\dagger_{n\downarrow}c_{q\downarrow}:b_k)S^z+(b_k:c^\dagger_{q\uparrow}c_{n\uparrow}:-:c^\dagger_{n\downarrow}c_{p\downarrow}:b^\dagger_k)S^z\Big) \\
& +\frac{1}{2}\sum_{npqk}\omega_k J^{\perp}(p,n)(B)K_{kpq}(B)(1-2n(p)) \\
& \times\Big((b^\dagger_k:c^\dagger_{n\uparrow}c_{q\uparrow}:+:c^\dagger_{n\downarrow}c_{q\downarrow}:b_k)S^z+(b_k:c^\dagger_{q\uparrow}c_{n\uparrow}:+:c^\dagger_{n\downarrow}c_{p\downarrow}:b^\dagger_k)S^z\Big) \\
& -\frac{1}{4}\sum_{npqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,n)(B)K_{kpq}(B) \\
& \times\Big((b^\dagger_k:c^\dagger_{n\uparrow}c_{q\uparrow}:+:c^\dagger_{n\downarrow}c_{q\downarrow}:b_k)+(b_k:c^\dagger_{q\uparrow}c_{n\uparrow}:+:c^\dagger_{n\downarrow}c_{p\downarrow}:b^\dagger_k)\Big) \\
& -\frac{1}{4}\sum_{npqk}\omega_k J^{\perp}(p,n)(B)K_{kpq}(B) \\
& \times\Big(b_k(:c^\dagger_{n\downarrow}c_{q\downarrow}:-:c^\dagger_{q\uparrow}c_{n\uparrow}:)+(:c^\dagger_{q\downarrow}c_{n\downarrow}:-:c^\dagger_{n\uparrow}c_{p\uparrow}:)b^\dagger_k\Big) \\
& -\frac{1}{2}\sum_{pqk}((\epsilon_p-\epsilon_q)-\omega_k)J_{\perp}(p,q)(B)K_{kpq}(B) \\
& \times(n(p)+n(q)-2n(p)n(q))(b^\dagger_k+b_k)S^z \\
& +\frac{1}{4}\sum_{pqk}((\epsilon_p-\epsilon_q)-\omega_k)J^{\perp}(p,q)K_{kpq}(B)(n(p)-n(q))(b^\dagger_k+b_k) \qquad \text{(A.17)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^K_{\rm new}(B), H^B_0(B)] \;=\; & \sum_{pqkl}((\epsilon_p-\epsilon_q)-\omega_k)\lambda_l(B)K_{kpq}(B) \\
& \times(:c^\dagger_{q\downarrow}c_{p\uparrow}:b^\dagger_k b^\dagger_l S^+ +:c^\dagger_{p\uparrow}c_{q\downarrow}:b_k b_l S^-) \\
& +\sum_{pqkl}((\epsilon_p-\epsilon_q)-\omega_k)\lambda_l(B)K_{kpq}(B)(:c^\dagger_{q\downarrow}c_{p\uparrow}::b^\dagger_k b_l:S^+ +:c^\dagger_{p\uparrow}c_{q\downarrow}::b^\dagger_l b_k:S^-) \\
& +\frac{1}{2}\sum_{pqk}((\epsilon_p-\epsilon_q)-\omega_k)\lambda_k(B)K_{kpq}(B) \\
& \times(:c^\dagger_{q\downarrow}c_{p\uparrow}:S^+ +:c^\dagger_{p\uparrow}c_{q\downarrow}:S^-)(1+2n_b(k)) \qquad \text{(A.18)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{\rm new}(B), H^b_0(B) + H^f_0(B)] &= -\sum_{pqk} L_{kpq}(B)((\epsilon_p - \epsilon_q) + \omega_k)^2 \\
&\quad \times (b^\dagger_k : c^\dagger_{p\uparrow} c_{q\downarrow} : S^- + b_k : c^\dagger_{q\uparrow} c_{p\downarrow} : S^+) \qquad \text{(A.19)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{\rm new}(B), H^b_0(B) + H^f_0(B)] &= -\sum_{pqk} K_{kpq}(B)((\epsilon_p - \epsilon_q) - \omega_k)^2 \\
&\quad \times (b^\dagger_k : c^\dagger_{p\uparrow} c_{q\downarrow} : S^- + b^\dagger_k : c^\dagger_{q\uparrow} c_{p\downarrow} : S^+) \qquad \text{(A.20)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{\rm new}(B), H^\parallel_0(B)] &= \frac{1}{2} \sum_{mnpqk} L_{kpq}(B)((\epsilon_p - \epsilon_q) + \omega_k) J^\parallel(m,n)(B) \\
&\quad \times \Big(: c^\dagger_{n\uparrow} c_{m\uparrow} c^\dagger_{p\uparrow} c_{q\downarrow} : b^\dagger_k S^- + : c^\dagger_{m\uparrow} c_{n\uparrow} c^\dagger_{q\downarrow} c_{p\uparrow} : b_k S^+ \\
&\quad - (: c^\dagger_{m\downarrow} c_{n\downarrow} c^\dagger_{p\uparrow} c_{q\downarrow} : b^\dagger_k S^- + : c^\dagger_{n\downarrow} c_{n\downarrow} c^\dagger_{q\downarrow} c_{p\uparrow} : b_k S^+) \Big) \\
&\quad + \frac{1}{4} \sum_{mpqk} (L_{kpq}(B)((\epsilon_p - \epsilon_q) + \omega_k) J^\parallel(m,p)(B) \\
&\qquad - L_{kmp}(B)((\epsilon_m - \epsilon_p) + \omega_k) J^\parallel(p,q))(B) \\
&\quad \times (1 - 2n(p))(: c^\dagger_{m\uparrow} c_{q\downarrow} : b^\dagger_k S^- + : c^\dagger_{q\downarrow} c_{m\uparrow} : b_k S^+) \qquad \text{(A.21)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{\rm new}(B), H^b_0(B)] &= \sum_{pqkl} ((\epsilon_p - \epsilon_q) + \omega_k) \lambda_l(B) L_{kpq}(B) \\
&\quad \times (: c^\dagger_{q\downarrow} c_{p\uparrow} : b_k b_l S^+ + : c^\dagger_{p\uparrow} c_{q\downarrow} : b^\dagger_k b^\dagger_l S^-) \\
&\quad + \sum_{pqkl} ((\epsilon_p - \epsilon_q) + \omega_k) \lambda_l(B) L_{kpq}(B) \\
&\quad \times (: c^\dagger_{q\downarrow} c_{p\uparrow} :: b^\dagger_l b_k : S^+ + : c^\dagger_{p\uparrow} c_{q\downarrow} :: b^\dagger_k b_l : S^-) \\
&\quad + \frac{1}{2} \sum_{pqk} ((\epsilon_p - \epsilon_q) + \omega_k) \lambda_k(B) L_{kpq}(B) \\
&\quad \times (: c^\dagger_{q\downarrow} c_{p\uparrow} : S^+ + : c^\dagger_{p\uparrow} c_{q\downarrow} : S^-)(1 + 2n_b(k)). \qquad \text{(A.22)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^{L}_{\mathrm{new}}(B), H_0^{\perp,(f)}(B)] \;=\; & -\sum_{mnpqk}(\epsilon_n-\epsilon_m)J^{\perp}(m,n)(B)L_{kpq}(B) \\
& \times(:c^{\dagger}_{m\uparrow}c_{n\downarrow}c^{\dagger}_{q\downarrow}c_{p\uparrow}:b_k+:c^{\dagger}_{n\downarrow}c_{m\uparrow}c^{\dagger}_{p\uparrow}c_{p\downarrow}:b^{\dagger}_k)S^z \\
& -\frac{1}{2}\sum_{npqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,n)(B)L_{kpq}(B)(1-2n_f(p)) \\
& \times\Big((b_k:c^{\dagger}_{n\uparrow}c_{q\uparrow}:-:c^{\dagger}_{n\downarrow}c_{q\downarrow}:b^{\dagger}_k)S^z+(b^{\dagger}_k:c^{\dagger}_{q\uparrow}c_{n\uparrow}:-:c^{\dagger}_{n\downarrow}c_{p\downarrow}:b_k)S^z\Big) \\
& -\frac{1}{2}\sum_{npqk}\omega_k J^{\perp}(p,n)(B)L_{kpq}(B)(1-2n(p)) \\
& \times\Big((b_k:c^{\dagger}_{n\uparrow}c_{q\uparrow}:+:c^{\dagger}_{n\downarrow}c_{q\downarrow}:b^{\dagger}_k)S^z+(b^{\dagger}_k:c^{\dagger}_{q\uparrow}c_{n\uparrow}:+:c^{\dagger}_{n\downarrow}c_{p\downarrow}:b_k)S^z\Big) \\
& -\frac{1}{4}\sum_{npqk}(\epsilon_p-\epsilon_q)J^{\perp}(p,n)(B)L_{kpq}(B) \\
& \times\Big((b_k:c^{\dagger}_{n\uparrow}c_{q\uparrow}:+:c^{\dagger}_{n\downarrow}c_{q\downarrow}:b^{\dagger}_k)+(b^{\dagger}_k:c^{\dagger}_{q\uparrow}c_{n\uparrow}:+:c^{\dagger}_{n\downarrow}c_{p\downarrow}:b_k)\Big) \\
& +\frac{1}{4}\sum_{npqk}\omega_k J^{\perp}(p,n)(B)L_{kpq}(B) \\
& \times\Big(b^{\dagger}_k(:c^{\dagger}_{n\downarrow}c_{q\downarrow}:-:c^{\dagger}_{q\uparrow}c_{n\uparrow}:)+(:c^{\dagger}_{q\downarrow}c_{n\downarrow}:-:c^{\dagger}_{n\uparrow}c_{p\uparrow}:)b_k\Big) \\
& -\frac{1}{2}\sum_{pqk}((\epsilon_p-\epsilon_q)+\omega_k)J^{\perp}(p,q)(B)L_{kpq}(B) \\
& \times(n(p)+n(q)-2n(p)n(q))(b^{\dagger}_k+b_k)S^z \\
& +\frac{1}{4}\sum_{pqk}((\epsilon_p-\epsilon_q)+\omega_k)J^{\perp}(p,q)(B)L_{kpq}(B) \\
& \times(n(p)-n(q))(b^{\dagger}_k+b_k) \qquad\qquad (A.23)
\end{aligned}
$$

A.3 The non-ohmic Cases

For the sake of completeness we consider the case s≠1 here in the Appendix. The reasons, why this is put here and not in the actual work, are, on the one hand we discussed this model solely as a kind of check for applicability of our method, and therefore we want to keep the considered model as simple as possible. On the other hand, it will turn out that the non-ohmic case exhibits a fixed point, which is, at least within our method, not perturbatively accessible.

We define:$x = \frac{1}{2}(s+1)$ and $\kappa = \frac{\alpha\Gamma(\frac{1}{2}(s+3))}{2^{\frac{1}{2}(s+1)}}$ and make the ansatz

$$J_{IR}^{\perp}(B) = g(B)e^{-\frac{\kappa B^{1-x}}{1-x}}, \tag{A.24}$$

taking the derivative thereof with respect to B leads to

$$\frac{dJ_{IR}^{\perp}(B)}{dB} = \frac{dg}{dB}e^{-\frac{\kappa B^{1-x}}{1-x}} - g(B)\kappa e^{-\frac{\kappa B^{1-x}}{1-x}}B^{-x}. \tag{A.25}$$

Inserting into Eqs.(2.36) gives

$$\frac{dg}{dB} = \frac{\rho_0}{2}\frac{g(B)J_{IR}^{\parallel}(B)}{B} \tag{A.26}$$

$$\frac{dJ_{IR}^{\parallel}(B)}{dB} = \frac{\rho_0}{2}\frac{g^2}{B}e^{-\frac{2\kappa B^{1-x}}{1-x}}. \tag{A.27}$$

Appendix B

The isotropic Bose Fermi Kondo Model

B.1 The Commutators

We present all the commutators that are needed in order to derive the flow eqautions of the running couplings

$$[\eta^K_{new}(B), H^f_0(B) + H^b_0(B)] = -i \sum_{pqk} K_{pqk}(B)((\epsilon_p - \epsilon_q) + \omega_k)^2 \mathbf{\Phi}^\dagger_k \cdot (: \mathbf{s}_{pq} \times \mathbf{S} :) \tag{B.1}$$

and

$$[\eta^L_{new}(B), H^f_0(B) + H^b_0(B)] = -i \sum_{pqk} L_{pqk}(B)((\epsilon_p - \epsilon_q) - \omega_k)^2 \mathbf{\Phi}_k \cdot (: \mathbf{s}_{pq} \times \mathbf{S} :) \tag{B.2}$$

and

$$\begin{aligned}
[\eta^K_{new}(B), H^b_{int}(B)] \;=\; & -\sum_{pqkl} K_{pqk}(B)((\epsilon_p - \epsilon_q) + \omega_k)\lambda_l(B) \\
& \times \Big(: \mathbf{s}_{pq} \cdot \mathbf{S} : (\mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}^\dagger_l + : \mathbf{\Phi}^\dagger_k \cdot \mathbf{\Phi}_l :) \\
& -(\mathbf{S} \cdot \mathbf{\Phi}^\dagger_k : \mathbf{s}_{pq} : \cdot \mathbf{\Phi}^\dagger_l + : \mathbf{S} \cdot \mathbf{\Phi}^\dagger_k : \mathbf{s}_{pq} :_f \cdot \mathbf{\Phi}_l :_b +)\Big) \\
& -\sum_{pqk} \lambda_k(B) K_{pqk}(B)((\epsilon_p - \epsilon_q) + \omega_k) \\
& \times (1 + 2n_b(k))(: \mathbf{s}_{pq} \cdot \mathbf{S} :)
\end{aligned} \tag{B.3}$$

and

$$
\begin{aligned}
[\eta^K_{new}(B), H^f_{int}(B)] \;=\; & -\sum_{pqmnk} K_{mnk}(B)((\epsilon_m-\epsilon_n)+\omega_k)J(p,q)(B) \\
& \times((:\mathbf{s}_{mn}\cdot\mathbf{S})(\mathbf{s}_{pq}\cdot\mathbf{\Phi}^\dagger_k:) - :(\mathbf{\Phi}^\dagger_k\cdot\mathbf{S})(\mathbf{s}_{mn}\cdot\mathbf{s}_{pq}:)) \\
& +\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} K_{pmk}(B)((\epsilon_p-\epsilon_m)+\omega_k)J(m,q)(B) \\
& \times(1-2n_f(m))(:c^\dagger_{p\alpha}c_{q\alpha}:\mathbf{\Phi}^\dagger_k\cdot\mathbf{S}) \\
& +\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} K_{mqk}(B)((\epsilon_m-\epsilon_q)+\omega_k)J(p,m)(B) \\
& \times(1-2n_f(m))(:c^\dagger_{p\alpha}c_{q\alpha}:\mathbf{\Phi}^\dagger_k\cdot\mathbf{S}) \\
& +\frac{i}{4}\sum_{pqmk} K_{pmk}(B)((\epsilon_p-\epsilon_m)+\omega_k)J(m,q)(B) \\
& \times(1-2n_f(m))(\mathbf{\Phi}^\dagger_k\cdot:\mathbf{S}\times\mathbf{s}_{pq}:) \\
& -\frac{i}{4}\sum_{pqmk} K_{mqk}(B)((\epsilon_m-\epsilon_q)+\omega_k)J(p,m)(B) \\
& \times(1-2n_f(m))(\mathbf{\Phi}^\dagger_k\cdot:\mathbf{S}\times\mathbf{s}_{pq}:) \\
& +\frac{1}{2}\sum_{pqk} K_{qpk}(B)((\epsilon_q-\epsilon_p)+\omega_k)J(p,q)(B) \\
& \times(n_f(p)+n_f(q)-2n_f(p)n_f(q))(\mathbf{\Phi}^\dagger_k\cdot\mathbf{S}) \\
& +\frac{i}{4}\sum_{pqmk} K_{pmk}(B)((\epsilon_p-\epsilon_m)+\omega_k)J(m,q)(B)(\mathbf{\Phi}^\dagger_k\cdot:\mathbf{s}_{pq}:) \\
& +\frac{i}{4}\sum_{pqmk} K_{mqk}(B)((\epsilon_m-\epsilon_q)+\omega_k)J(p,m)(B)(\mathbf{\Phi}^\dagger_k\cdot:\mathbf{s}_{pq}:) \qquad \text{(B.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{new}(B), H^b_{int}(B)] \;=\; & -\sum_{pqkl} L_{pqk}(B)((\epsilon_p-\epsilon_q)-\omega_k)\lambda_l(B) \\
& \times\Big(:\mathbf{s}_{pq}\cdot\mathbf{S}:(:\mathbf{\Phi}^\dagger_l\cdot\mathbf{\Phi}_k+\mathbf{\Phi}_k\cdot\mathbf{\Phi}_l) \\
& -((\mathbf{S}\cdot\mathbf{\Phi}_k)(:\mathbf{s}_{pq}:\cdot\mathbf{\Phi}_l)+(:\mathbf{s}_{pq}:_f\cdot\mathbf{\Phi}^\dagger_l)(\mathbf{S}\cdot\mathbf{\Phi}_k:_b))\Big) \\
& -\sum_{pqk}\lambda_k(B)L_{pqk}(B)((\epsilon_p-\epsilon_q)-\omega_k) \\
& \times(1+2n_b(k))(:\mathbf{s}_{pq}\cdot\mathbf{S}:) \qquad \text{(B.5)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^L_{new}(B), H^f_{int}(B)] &= -\sum_{pqmnk} L_{mnk}(B)((\epsilon_m-\epsilon_n)-\omega_k)J(p,q)(B) \\
&\quad \times((:\mathbf{s}_{mn}\cdot\mathbf{S})(\mathbf{s}_{pq}\cdot\mathbf{\Phi}_k:)-:(\mathbf{\Phi}_k\cdot\mathbf{S})(\mathbf{s}_{mn}\cdot\mathbf{s}_{pq}:)) \\
&\quad +\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} L_{pmk}(B)((\epsilon_p-\epsilon_m)-\omega_k)J(m,q)(B) \\
&\quad \times(1-2n_f(m)):c^\dagger_{p\alpha}c_{q\alpha}:\mathbf{\Phi}_k\cdot\mathbf{S} \\
&\quad +\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} L_{mqk}(B)((\epsilon_m-\epsilon_q)-\omega_k)J(p,m)(B) \\
&\quad \times(1-2n_f(m)):c^\dagger_{p\alpha}c_{q\alpha}:\dot{\mathbf{\Phi}}_k\mathbf{S} \\
&\quad +\frac{i}{4}\sum_{pqmk} L_{pmk}(B)((\epsilon_p-\epsilon_m)-\omega_k)J(m,q)(B) \\
&\quad \times(1-2n_f(m))\mathbf{\Phi}_k\cdot:\mathbf{S}\times\mathbf{s}_{pq}: \\
&\quad -\frac{i}{4}\sum_{pqmk} L_{mqk}(B)((\epsilon_m-\epsilon_q)-\omega_k)J(p,m)(B) \\
&\quad \times(1-2n_f(m))\dot{\mathbf{\Phi}}_k:\mathbf{S}\times\mathbf{s}_{pq}: \\
&\quad +\frac{1}{2}\sum_{pqk} L_{qpk}(B)((\epsilon_q-\epsilon_p)-\omega_k)J(q,p)(B) \\
&\quad \times(n_f(p)+n_f(q)-2n_f(p)n_f(q))\mathbf{\Phi}_k\cdot\mathbf{S} \\
&\quad +\frac{i}{4}\sum_{pqmk} L_{pmk}(B)((\epsilon_p-\epsilon_m)-\omega_k)J(m,q)(B)\dot{\mathbf{\Phi}}_k:\mathbf{s}_{pq}: \\
&\quad +\frac{i}{4}\sum_{pqmk} L_{mqk}(B)((\epsilon_m-\epsilon_q)-\omega_k)J(p,m)(B)\mathbf{\Phi}_k\cdot:\mathbf{s}_{pq}: \qquad \text{(B.6)}
\end{aligned}
$$

and

$$
[\eta^\mu_{new}(B), H^f_0(B)] = 0 \qquad \text{(B.7)}
$$

and

$$
[\eta^\mu_{new}(B), H^f_{int}(B)] = -\sum_{klpq}(\omega_k-\omega_l)\mu_{kl}(B)J(p,q)(B)(\mathbf{\Phi}^\dagger_k\times\mathbf{\Phi}_l)\cdot(:\mathbf{s}_{pq}\times\mathbf{S}) \qquad \text{(B.8)}
$$

and

$$
[\eta^\mu_{new}(B), H^b_0(B)] = -i\sum_{kl}(\omega_k-\omega_l)^2\mu_{kl}(B)\mathbf{S}\cdot(\mathbf{\Phi}_k\times\mathbf{\Phi}_l) \qquad \text{(B.9)}
$$

and

$$
\begin{aligned}
[\eta^{\mu}_{new}(B), H^{b}_{int}(B)] \;=\; & \sum_{klm}(\omega_k-\omega_l)\mu_{kl}(B)\lambda_m(B) \\
& \times(:\mathbf{\Phi}^{\dagger}_m\cdot\mathbf{\Phi}_l\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S}- :\mathbf{\Phi}^{\dagger}_m\cdot\mathbf{\Phi}^{\dagger}_k\mathbf{\Phi}_l\cdot\mathbf{S}) \\
& +\sum_{klm}(\omega_k-\omega_l)\mu_{kl}(B)\lambda_m(B) \\
& \times(:\mathbf{\Phi}_m\cdot\mathbf{\Phi}_l\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S}- :\mathbf{\Phi}_m\cdot\mathbf{\Phi}^{\dagger}_k\mathbf{\Phi}_k\cdot\mathbf{S}) \\
& +\sum_{kl}(\omega_k-\omega_l)\mu_{kl}(B)\lambda_l(B)(1+2n_b(l))\mathbf{S}\cdot(\mathbf{\Phi}^{\dagger}_l+\mathbf{\Phi}_k) \qquad \text{(B.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta^{f}_{0}(B), H^{K}_{new}(B)] \;=\; & \sum_{pqmnk}K_{mnk}(B)(\epsilon_p-\epsilon_q)J(p,q)(B) \\
& ((:\mathbf{s}_{mn}\cdot\mathbf{S})(\mathbf{s}_{pq}\cdot\mathbf{\Phi}^{\dagger}_k:)- :(\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S})(\mathbf{s}_{mn}\cdot\mathbf{s}_{pq}:)) \\
& -\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk}K_{pmk}(B)(\epsilon_m-\epsilon_q)J(m,q)(B) \\
& \times(1-2n_f(m)):c^{\dagger}_{p\alpha}c_{q\alpha}:\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S} \\
& -\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk}K_{mqk}(B)(\epsilon_p-\epsilon_m)J(p,m)(B) \\
& \times(1-2n_f(m)):c^{\dagger}_{p\alpha}c_{q\alpha}:\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S} \\
& -\frac{i}{4}\sum_{pqmk}K_{pmk}(B)(\epsilon_m-\epsilon_q)J(m,q)(B) \\
& \times(1-2n_f(m))\mathbf{\Phi}^{\dagger}_k\cdot:\mathbf{S}\times\mathbf{s}_{pq}: \\
& +\frac{i}{4}\sum_{pqmk}K_{mqk}(B)(\epsilon_m-\epsilon_q)J(p,m)(B) \\
& \times(1-2n_f(m))\mathbf{\Phi}^{\dagger}_k\cdot:\mathbf{S}\times\mathbf{s}_{pq}: \\
& -\frac{1}{2}\sum_{pqk}K_{qpk}(B)((\epsilon_q-\epsilon_p)+\omega_k)J(p,q)(B) \\
& \times(n_f(p)+n_f(q)-2n_f(p)n_f(q))\mathbf{\Phi}^{\dagger}_k\cdot\mathbf{S} \\
& -\frac{i}{4}\sum_{pqmk}K_{mpk}(B)(\epsilon_p-\epsilon_q)J(p,q)(B)\mathbf{\Phi}^{\dagger}_k\cdot:\mathbf{s}_{mq}: \\
& -\frac{i}{4}\sum_{pqmk}K_{qmk}(B)(\epsilon_p-\epsilon_q)+\omega_k)J(p,q)(B)\mathbf{\Phi}^{\dagger}_k\cdot:\mathbf{s}_{pm}: \qquad \text{(B.11)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^f(B), H_{new}^L(B)] \;=\; & \sum_{pqmnk} L_{mnk}(B)(\epsilon_p-\epsilon_q)J(p,q)(B) \\
& ((:\mathbf{s}_{mn}\cdot\mathbf{S})(\mathbf{s}_{pq}\cdot\mathbf{\Phi}_k:) - :(\mathbf{\Phi}_k\cdot\mathbf{S})(\mathbf{s}_{mn}\cdot\mathbf{s}_{pq}:)) \\
& -\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} L_{pmk}(B)(\epsilon_m-\epsilon_q)J(m,q)(B) \\
& \times(1-2n_f(m)) : c_{p\alpha}^\dagger c_{q\alpha} : \mathbf{\Phi}_k\cdot\mathbf{S} \\
& -\frac{1}{4}\sum_{\alpha=1}^{2}\sum_{pqmk} L_{mqk}(B)(\epsilon_p-\epsilon_m)J(p,m)(B) \\
& \times(1-2n_f(m)) : c_{p\alpha}^\dagger c_{q\alpha} : \mathbf{\Phi}_k\cdot\mathbf{S} \\
& -\frac{i}{4}\sum_{pqmk} L_{pmk}(B)(\epsilon_m-\epsilon_q)J(m,q)(B) \\
& \times(1-2n_f(m))\mathbf{\Phi}_k\cdot : \mathbf{S}\times\mathbf{s}_{pq} : \\
& +\frac{i}{4}\sum_{pqmk} L_{mqk}(B)(\epsilon_m-\epsilon_q)J(p,m)(B) \\
& \times(1-2n_f(m))\mathbf{\Phi}_k\cdot : \mathbf{S}\times\mathbf{s}_{pq} : \\
& -\frac{1}{2}\sum_{pqk} L_{qpk}(B)((\epsilon_q-\epsilon_p)+\omega_k)J(p,q)(B) \\
& \times(n_f(p)+n_f(q)-2n_f(p)n_f(q))\mathbf{\Phi}_k\cdot : \mathbf{S} \\
& -\frac{i}{4}\sum_{pqmk} L_{mpk}(B)(\epsilon_p-\epsilon_q)J(p,q)(B)\mathbf{\Phi}_k\cdot\mathbf{s}_{mq} : \\
& -\frac{i}{4}\sum_{pqmk} L_{qmk}(B)(\epsilon_p-\epsilon_q)+\omega_k)J(p,q)(B)\mathbf{\Phi}_k\cdot\mathbf{s}_{pm} : \qquad \text{(B.12)}
\end{aligned}
$$

and

$$
[\eta_0^f(B), H_{new}^\mu(B)] = -\sum_{pqkl}(\epsilon_p-\epsilon_q)\mu_{kl}(B)J(p,q)(B) : \mathbf{s}_{pq} : \cdot : \mathbf{\Phi}_k^\dagger\times\mathbf{\Phi}_l : \times\mathbf{S} \qquad \text{(B.13)}
$$

and

$$
\begin{aligned}
[\eta_0^b(B), H_{new}^K(B)] \;=\; & \sum_{pqkl} K_{pqk}(B)\omega_l\lambda_l(B) \\
& \times\Big(:\mathbf{s}_{pq}\cdot\mathbf{S}:(\mathbf{\Phi}_k^\dagger\cdot\mathbf{\Phi}_l^\dagger - :\mathbf{\Phi}_k^\dagger\cdot\mathbf{\Phi}_l:) \\
& -(\mathbf{S}\cdot\mathbf{\Phi}_k^\dagger : \mathbf{s}_{pq} : \cdot\mathbf{\Phi}_l^\dagger - : \mathbf{S}\cdot\mathbf{\Phi}_k^\dagger : \mathbf{s}_{pq} :_f \cdot\mathbf{\Phi}_l :_b +)\Big) \\
& -\sum_{pqk}\lambda_k(B)K_{pqk}(B)\omega_k(1+2n_b(k)) : \mathbf{s}_{pq}\cdot\mathbf{S} : \qquad \text{(B.14)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\eta_0^b(B), H_{new}^L(B)] \;=\; & \sum_{pqkl} L_{pqk}(B)\omega_l\lambda_l(B) \\
& \Big(:\mathbf{s}_{pq}\cdot\mathbf{S}:(:\mathbf{\Phi}_l^\dagger\cdot\mathbf{\Phi}_k-\mathbf{\Phi}_k\cdot\mathbf{\Phi}_l) \\
& -((\mathbf{S}\cdot\mathbf{\Phi}_k)(:\mathbf{s}_{pq}:\cdot\mathbf{\Phi}_l)-(:\mathbf{s}_{pq}:_f\cdot\mathbf{\Phi}_l^\dagger)(\mathbf{S}\cdot\mathbf{\Phi}_k:_b))\Big) \\
& +\sum_{pqk}\lambda_k(B)L_{pqk}(B)\omega_k(1+2n_b(k)):\mathbf{s}_{pq}\cdot\mathbf{S}:
\end{aligned}
\tag{B.15}
$$

and

$$
\begin{aligned}
[\eta_0^b(B), H_{new}^\mu(B)] \;=\; & \sum_{klm}\omega_m\lambda_m(B)\mu_{kl}(B) \\
& \Big(:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_k^\dagger\mathbf{\Phi}_l\cdot\mathbf{S}:-:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_l\mathbf{\Phi}_k^\dagger\cdot\mathbf{S}: \\
& -:\mathbf{\Phi}_m\cdot\mathbf{\Phi}_k^\dagger\mathbf{\Phi}_l\cdot\mathbf{S}:+:\mathbf{\Phi}_m\cdot\mathbf{\Phi}_l\mathbf{\Phi}_k^\dagger\cdot\mathbf{S}:\Big) \\
& -\sum_{kl}\omega_l\lambda_l(B)\mu_{kl}(B)(1+2n_b(l))(\mathbf{\Phi}_k^\dagger+\mathbf{\Phi}_k)\cdot\mathbf{S}
\end{aligned}
\tag{B.16}
$$

and

$$
[\eta_{new}^\Psi(B), H_0^b(B)] \;=\; -i\sum_{kl}(\omega_k+\omega_l)^2\Psi_{kl}(B)\mathbf{S}\cdot(\mathbf{\Phi}_k^\dagger\times\mathbf{\Phi}_l^\dagger-\mathbf{\Phi}_k\times\mathbf{\Phi}) \tag{B.17}
$$

and

$$
\begin{aligned}
[\eta_{new}^\Psi(B), H_{int}^b(B)] \;=\; & \sum_{klm}(\omega_k+\omega_l)\Psi_{kl}(B)\lambda_m(B) \\
& \big(:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_l\mathbf{\Phi}_k\cdot\mathbf{S}:-:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_k\mathbf{\Phi}_l\cdot\mathbf{S}:\big) \\
& +\sum_{kl}(\omega_k+\omega_l)\lambda_l(B)(\Psi_{kl}(B)-\Psi_{lk}(B)) \\
& \times(1+2n_b(l))\mathbf{S}\cdot(\mathbf{\Phi}_k^\dagger+\mathbf{\Phi})
\end{aligned}
\tag{B.18}
$$

and

$$
\begin{aligned}
[\eta_0^b(B), H_{new}^\Psi(B)] \;=\; & \sum_{klm}\omega_m\Psi_{kl}(B)\lambda_m(B) \\
& \big(:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_l\mathbf{\Phi}_k\cdot\mathbf{S}:-:\mathbf{\Phi}_m^\dagger\cdot\mathbf{\Phi}_k\mathbf{\Phi}_l\cdot\mathbf{S}:\big) \\
& +\sum_{kl}\omega_l\lambda_l(B)(\Psi_{kl}(B)-\Psi_{lk}(B)) \\
& \times(1+2n_b(l))\mathbf{S}\cdot(\mathbf{\Phi}_k^\dagger+\mathbf{\Phi})
\end{aligned}
\tag{B.19}
$$

and

$$[\eta^{\Psi}_{new}(B), H_0(B)] = 0 \tag{B.20}$$

and

$$\begin{aligned}[\eta^{\Psi}_{new}(B), H^f_{int}(B)] &= -\sum_{klpq}(\Psi_{kl}(B) - \Psi_{lk}(B))(\omega_l + \omega_k)J(p,q)(B) \\ &\quad \times \Big((: \mathbf{s}_{pq} : \cdot \mathbf{\Phi}^{\dagger}_k)(\mathbf{\Phi}^{\dagger}_l \cdot \mathbf{S}) + (: \mathbf{s}_{pq} : \cdot \mathbf{\Phi}_k)(\mathbf{\Phi}_l \cdot \mathbf{S})\Big)\end{aligned} \tag{B.21}$$

and

$$\begin{aligned}[\eta^{f}_{0}(B), H^{\Psi}_{new}(B)] &= \sum_{klpq}(\Psi_{kl}(B) - \Psi_{lk}(B))(\epsilon_p - \epsilon_q)J(p,q)(B) \\ &\quad \times \Big((: \mathbf{s}_{pq} : \cdot \mathbf{\Phi}^{\dagger}_k)(\mathbf{\Phi}^{\dagger}_l \cdot \mathbf{S}) + (: \mathbf{s}_{pq} : \cdot \mathbf{\Phi}_k)(\mathbf{\Phi}_l \cdot \mathbf{S})\Big)\end{aligned} \tag{B.22}$$

.

Appendix C

The $\mathcal{T}$-matrix

C.1 The Correlation Function

The $\mathcal{T}$-matrix is given by the following expression

$$Im(\hat{T}_\sigma(\omega)) = Im\left(-i\int_\infty^{-\infty} dt \quad \Theta(t)\langle\{O_\sigma(t), O_\sigma^\dagger(0)\}\rangle e^{i\omega t}\right), \tag{C.1}$$

where σ denotes the spin degrees of freedom. The time evolution of the operator $O_\sigma(t)$ can be expressed in the Heisenberg representation, thus the expectation value reads

$$\langle\{O_\sigma(B=\infty,t), O_\sigma(B=\infty,t=0)^\dagger\}\rangle = \langle e^{iH_0t}O_\sigma e^{-iH_0t}O_\sigma^\dagger + O_\sigma^\dagger e^{iH_0t}O_\sigma e^{-iH_0t}\rangle, \tag{C.2}$$

and with the help of the operator relation

$$e^A B = Be^{A+D} \quad [A,B] = DB, [A,D] = [B,D] = 0, \tag{C.3}$$

the expectation value can be calculated. Recall, the free part of the Hamiltonian is

$$H_0 = \sum_{p\alpha} \epsilon_p : c_{p\alpha}^\dagger c_{p\alpha} : + \sum_k \omega_k : \mathbf{\Phi}_k^\dagger \mathbf{\Phi}_k :, \tag{C.4}$$

and furthermore, the necessary commutator relations are

$$[H_0, c_{q\alpha}^\dagger \mathbf{S}\cdot\frac{\sigma_{\alpha\sigma}}{2}] = \epsilon_q c_{q\alpha}^\dagger \mathbf{S}\cdot\frac{\sigma_{\alpha\sigma}}{2} \tag{C.5}$$

and

$$[H_0^f, (\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)\cdot c_{q\beta}^\dagger(\frac{\sigma_{\beta\sigma}}{2}\times\mathbf{S})] = \epsilon_q(\mathbf{\Phi}_k^\dagger - \mathbf{\Phi}_k)\cdot c_{q\beta}^\dagger(\frac{\sigma_{\beta\sigma}}{2}\times\mathbf{S}) \tag{C.6}$$

and

$$[H_0, \mathbf{\Phi}_k^\dagger \cdot c_{q\beta}^\dagger(\frac{\sigma_{\beta\sigma}}{2}\times\mathbf{S})] = \omega_k(\mathbf{\Phi}_k^\dagger \cdot c_{q\beta}^\dagger(\frac{\sigma_{\beta\sigma}}{2}\times\mathbf{S}) \tag{C.7}$$

and

$$[H_0, -\mathbf{\Phi}_k \cdot c^\dagger_{q\beta}(\frac{\sigma_{\beta\sigma}}{2} \times \mathbf{S})] = (-\omega_k) - \mathbf{\Phi}_k \cdot c^\dagger_{q\beta}(\frac{\sigma_{\beta\sigma}}{2} \times \mathbf{S}). \tag{C.8}$$

After some straightforward manipulations we obtain an expression for the expectation value

$$\begin{aligned}
\langle\{O_\sigma(B=\infty,t) \quad , \quad & O_\sigma(B=\infty,t=0)^\dagger\} = \\
&= \langle e^{iH_0t}(\sum_p T_p c^\dagger_{p\alpha}\mathbf{S}\cdot\frac{\sigma_{\alpha\sigma}}{2} + i\sum_{pk} P_{pk}(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot c^\dagger_{p\alpha}(\frac{\sigma_{\alpha\sigma}}{2}\times\mathbf{S}))e^{-iH_0t} \\
&\quad \times(\sum_q T_q c^\dagger_{q\beta}\mathbf{S}\cdot\frac{\sigma_{\beta\sigma}}{2} + i\sum_{ql} P_{ql}(\mathbf{\Phi}^\dagger_l - \mathbf{\Phi}_l)\cdot c^\dagger_{q\beta}(\frac{\sigma_\sigma\beta}{2}\times\mathbf{S})) \\
&\quad +(\sum_q T_q c^\dagger_{q\beta}\mathbf{S}\cdot\frac{\sigma_{\beta\sigma}}{2} + i\sum_{ql} P_{ql}(\mathbf{\Phi}^\dagger_l - \mathbf{\Phi}_l)\cdot c^\dagger_{q\beta}(\frac{\sigma_\sigma\beta}{2}\times\mathbf{S})) \\
&\quad \times e^{iH_0t}(\sum_p T_p c^\dagger_{p\alpha}\mathbf{S}\cdot\frac{\sigma_{\alpha\sigma}}{2} + i\sum_{pk} P_{pk}(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot c^\dagger_{p\alpha}(\frac{\sigma_{\alpha\sigma}}{2}\times\mathbf{S}))e^{-iH_0t}\rangle \\
&= \frac{3}{16}\sum_p T_p^2 e^{i\epsilon_p t} + \frac{3}{8}\sum_{pk} P^2_{pk}(1-n_f(p)+n_b(k))e^{i(\epsilon_p+\omega_k)t} \\
&\quad +\frac{3}{8}\sum_{pk} P^2_{pk}(n_f(p)+n_b(k))e^{-i(\omega_k-\epsilon_p)t}.
\end{aligned} \tag{C.9}$$

Therefore we get for the $\mathcal{T}$-matrix

$$\begin{aligned}
Im(\hat{T}_\sigma(\omega)) \quad = \quad & Im\left(-\frac{3i}{16}\sum_p T_p^2\int_{-\infty}^{\infty} dt \quad \Theta(t)e^{i(\omega+\epsilon_p)t}\right. \\
& -\frac{3i}{8}\sum_{pk} P^2_{pk}(1-n_f(p)+n_b(k))\int_{-\infty}^{\infty} dt \quad \Theta(t)e^{i(\omega+(\epsilon_p+\omega_k))t} \\
& \left.-\frac{3i}{8}\sum_{pk} P^2_{pk}(n_f(p)+n_b(k))\int_{-\infty}^{\infty} dt \quad \Theta(t)e^{i(\omega+(+\epsilon_p-\omega_k))t}\right).
\end{aligned} \tag{C.10}$$

The appearing time integral can be evaluated by using the following relation

$$Im(-i\int_{-\infty}^{\infty} dt \quad \Theta(t)e^{i(\alpha+i\eta)t}) = Im(\frac{1}{\alpha+i\eta}) = Im(\frac{\alpha-i\eta}{\alpha^2+\eta^2}) = -\frac{\eta}{\alpha^2+\eta^2}, \tag{C.11}$$

by taking the limit $\eta \to 0$ we find a representation of the delta function

$$\underset{\eta\to 0}{\text{limes}}\left(-\frac{\eta}{\alpha^2+\eta^2}\right) = -\pi\delta(\alpha). \tag{C.12}$$

C.2 The O operator

The operator $O(B)$ and the generator $\eta(B)$ in terms of creation and annihilation operators were given in chapter 4. Now we want to calculate the flow of the operator and therefore we have to calculate

$$[\eta^f + \eta^b, O^f_\sigma + O^b_\sigma]. \tag{C.13}$$

The commutators are

$$\begin{aligned}
[\eta^f, O^f_\sigma] &= i\sum_{pqm}\sum_{\alpha\beta\gamma}(\epsilon_p - \epsilon_q)J(p,q)T_m : c^\dagger_{p\alpha}c_{q\beta}c^\dagger_{m\gamma} : \mathbf{S}\cdot(\frac{\sigma_{\alpha\beta}}{2}\times\frac{\sigma_{\gamma\sigma}}{2}) \\
&\quad +\frac{1}{4}\sum_{pq}\sum_{\alpha\beta}(\epsilon_p - \epsilon_q)J(p,q)T_q(\frac{\sigma_{\alpha\beta}}{2}\cdot\frac{\sigma_{\beta\sigma}}{2})c^\dagger_{p\alpha} \\
&\quad -\sum_{pq}\sum_{\alpha}(\epsilon_p - \epsilon_q)J(p,q)T_q(1-2n_f(q))c^\dagger_{p\alpha}\frac{\sigma_{\alpha\sigma}}{2}\cdot\mathbf{S}
\end{aligned} \tag{C.14}$$

and

$$\begin{aligned}
[\eta^f, O^b] &= -\sum_{pqmk}\sum_{\alpha\beta\gamma}(\epsilon_p - \epsilon_q)J(p,q)P_{km}(\frac{\sigma_{\alpha\beta}}{2}\cdot\frac{\sigma_{\gamma\sigma}}{2})((\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot\mathbf{S}) : c^\dagger_{p\alpha}c_{q\beta}c^\dagger_{m\gamma} : \\
&\quad +\sum_{pqmk}\sum_{\alpha\beta\gamma}(\epsilon_p - \epsilon_q)J(p,q)P_{km}(\frac{\sigma_{\gamma\sigma}}{2}\cdot\mathbf{S})(\frac{\sigma_{\alpha\beta}}{2}\cdot(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)) : c^\dagger_{p\alpha}c_{q\beta}c^\dagger_{m\gamma} : \\
&\quad +\frac{i}{4}\sum_{pqk}\sum_{\alpha\beta}(\epsilon_p - \epsilon_q)J(p,q)P_{kq}(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot(\frac{\sigma_{\alpha\beta}}{2}\times\frac{\sigma_{\beta\sigma}}{2})c^\dagger_{p\alpha} \\
&\quad -\frac{1}{4}\sum_{pqk}(\epsilon_p - \epsilon_q)J(p,q)P_{kq}(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot\mathbf{S}c^\dagger_{p\sigma}(1-2n_f(q)) \\
&\quad -\frac{i}{2}\sum_{pqk}\sum_{\alpha}(\epsilon_p - \epsilon_q)J(p,q)P_{km}(1-2n_f(q))(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot(c^\dagger_{p\alpha}\sigma_{\alpha\sigma}\times\mathbf{S})
\end{aligned} \tag{C.15}$$

and

$$\begin{aligned}
[\eta^b, O^b_\sigma] &= \sum_{pkl}\sum_{\alpha}\omega_k\lambda_k P_{lp}\varepsilon_{dbc}\varepsilon_{daf}S^f c^\dagger_{p\alpha}\sigma^c_{\alpha\sigma} \\
&\quad \times\left(\Phi^{\dagger a}_k\Phi^{\dagger b}_l + \Phi^a_k\Phi^b_l + : \Phi^a_k\Phi^{\dagger b}_l : + : \Phi^{\dagger a}_k\Phi^b_l :\right) \\
&\quad -2\sum_{kp}\sum_{\alpha}\omega_k\lambda_k P_{lp}(1+2n_b(k))c^\dagger_{p\alpha}\frac{\sigma_{\alpha\sigma}}{2}\cdot\mathbf{S}
\end{aligned} \tag{C.16}$$

and

$$[\eta^b, O^f] = i\sum_{kp}\sum_{\alpha}\lambda_k\omega_k T_p(\mathbf{\Phi}^\dagger_k - \mathbf{\Phi}_k)\cdot(c^\dagger_{p\alpha}\sigma_{\alpha\sigma}\times\mathbf{S}). \tag{C.17}$$

C.3 Flow equation

We just take the back coupling terms, while the newly generated terms are of higher order, and the flow equations read

$$\begin{aligned}\frac{dT_p(B)}{dB} &= -\sum_q(\epsilon_p-\epsilon_q)J(p,q)(B)T_q(B)(1-2n_f(q)) \\ &\quad -2\sum_k \omega_k\lambda_k P_{kp}(B)(1+2n_b(k))\end{aligned} \tag{C.18}$$

and

$$\frac{dP_{kp}(B)}{dB} = \lambda_k\omega_k T_p(B) - \frac{1}{2}\sum_q(\epsilon_p-\epsilon_q)J(p,q)(B)P_{kq}(B)(1-2n_f(q)). \tag{C.19}$$

Up to leading order we can neglect the second term in Eqs.(C.19). Since we are interested in the behaviour of the $\mathcal{T}$-matrix at the unstable fixed point we replace the the Kondo coupling and the bosonic coupling by its fixed point value and solve the above flow equations with the initial conditions $T_p(B=0)=J$ and $P_{kp}(B=0)=0$. Now, we integrate Eqs(C.19) and obtain the following expression

$$P_{kp}(B) = \sqrt{\frac{1}{2}\frac{\varepsilon\omega_k^{1-\varepsilon}}{2^\varepsilon\Gamma(\frac{1}{2}(4-\varepsilon))}}\omega_k\int_0^B dB' \quad T_p(B')B'^{-\frac{\varepsilon}{4}}e^{B'\omega_k^2}, \tag{C.20}$$

inserting this into Eqs.(C.18), we end up with a differential equation only for $T_p(B)$

$$\begin{aligned}\frac{dT_p(B)}{dB} &= +\frac{1}{\rho_0}\sum_q(\epsilon_q-\epsilon_p)\frac{\varepsilon}{2}e^{-B(\epsilon_q-\epsilon_p)^2}T_q(B)(1-2n(q)) \\ &\quad -\sum_k\frac{\varepsilon\omega_k^{3-\varepsilon}}{2^\varepsilon\Gamma(\frac{1}{2}(4-\varepsilon))}B^{-\frac{\varepsilon}{4}}e^{-B\omega_k^2}(1+2n(k)) \\ &\quad \times\int_0^B dB'T_p(B')B'^{-\frac{\varepsilon}{4}}e^{B'\omega_k^2}.\end{aligned} \tag{C.21}$$

We turn the sum over all momenta into an integral, as is has been done before, and make use of the bosonic spectral function. After some straightforward manipulations we end up with the following differential equation for the coupling

$$\boxed{\frac{dT_p(B)}{dB} = \varepsilon\frac{T_p(B)}{2B}\left(e^{-B\epsilon_p^2}-\frac{1}{2}\right)}. \tag{C.22}$$

In the following we solve this differential equations in its limiting cases, namely for a comparable small flow and a large flow, always compared to the energy above the fermi

sea. The first case is $B\epsilon_p^2 \ll 1$, this means that we can approximate the e-function as $(e^{B\epsilon_p^2} = 1)$ and the differential equation reduces to

$$\frac{dT_p}{dB} = \frac{\varepsilon}{4}\frac{T_p(B)}{B}, \tag{C.23}$$

with the initial condition $T_p(B = D^{-2}) = J$. The solution reads

$$T_p^{(1)}(B) = J(BD^2)^{\frac{\varepsilon}{4}} \quad \text{if} \quad D^{-2} < B < \epsilon_p^{-2}. \tag{C.24}$$

On the other hand, we take the limit $B\epsilon_p^2 \gg 1$ implying that $e^{-B\epsilon_p^2} = 0$. The simplified differential equation takes the form

$$\frac{dT_p}{dB} = -\frac{\varepsilon}{4}\frac{T_p(B)}{B}, \tag{C.25}$$

at this point the initial condition is given by the former solution of T_p evaluated at $B = \epsilon_p^{-2}$,namely $T_p(B = \epsilon_p^{-2}) = J(\frac{D}{\epsilon_p})^{\frac{\varepsilon}{2}}$. After integrating Eqs.(C.25) we obtain

$$T_p^{(2)}(B) = J(\frac{D}{\epsilon_p^2})^{\frac{\varepsilon}{2}} B^{-\frac{\varepsilon}{4}}. \tag{C.26}$$

Then this is inserted into Eqs.(C.20) leading to

$$\begin{aligned} P_{kp}(B) &= \sqrt{\frac{1}{2}\frac{\varepsilon\omega_k^{1-\varepsilon}}{2^\varepsilon\Gamma(\frac{1}{2}(4-\varepsilon))}}\omega_k \\ &\quad \times \left(\int_{D^{-2}}^{\epsilon_p^{-2}} dB' T_p^{(1)}(B') B'^{-\frac{\varepsilon}{4}} e^{-B\omega_k^2} + \int_{\epsilon_p^{-2}}^{B} dB' T_p^{(2)}(B') B'^{-\frac{\varepsilon}{4}} e^{-B\omega_k^2}\right). \end{aligned} \tag{C.27}$$

Due to clarity reasons we calculate both integrals independently, starting with the first one

$$\begin{aligned} \int_{D^{-2}}^{\epsilon_p^{-2}} dB' T_p^{(1)}(B') B'^{-\frac{\varepsilon}{4}} e^{-B\omega_k^2} &= JD^{\frac{\varepsilon}{2}} \int_{D^{-2}}^{\epsilon_p^{-2}} dB' e^{-B\omega_k^2} \\ &= J\frac{D^{\frac{\varepsilon}{2}}}{\omega_k^2}\left(e^{-(\frac{\omega_k}{D})^2} - e^{-(\frac{\omega_k}{\epsilon_p})^2}\right). \end{aligned} \tag{C.28}$$

The second one, however, is a little bit more tricky and is given by

$$\int_{D^{-2}}^{\epsilon_p^{-2}} dB' T_p^{(2)}(B') B'^{-\frac{\varepsilon}{4}} e^{-B\omega_k^2} = J\frac{D^{\frac{\varepsilon}{2}}}{\epsilon_p^\varepsilon} \int_{\epsilon_p^{-2}}^{B} dB' B'^{-\frac{\varepsilon}{2}} e^{-B'\omega_k^2}. \tag{C.29}$$

In addition we can make the approximation $B\omega_k^2 \gg 1$ which sends the upper integral boundary technically to infinity, and gives us

$$J\frac{D^{\frac{\varepsilon}{2}}}{\epsilon_p^\varepsilon}\omega_k^{\varepsilon-2}\Gamma(1-\frac{\varepsilon}{2}, \left(\frac{\omega_k}{\epsilon_p}\right)^2). \tag{C.30}$$

Collecting all the terms leads to the following expression

$$\begin{aligned} P_{kp}(B) &= J\sqrt{\frac{1}{2}\frac{\varepsilon\omega_k^{1-\varepsilon}}{2^\varepsilon\Gamma(\frac{1}{2}(4-\varepsilon))}}\frac{D^{\frac{\varepsilon}{2}}}{\omega_k} \\ &\left(e^{-(\frac{\omega_k}{D})^2} - e^{-(\frac{\omega_k}{\epsilon_p})^2} + \frac{\omega_k^\varepsilon}{\epsilon_p^\varepsilon}\Gamma(1-\frac{\varepsilon}{2},\left(\frac{\omega_k}{\epsilon_p}\right)^2)\right). \end{aligned} \tag{C.31}$$

Going back to Eqs.(C.10) we can make a few preliminary thoughts, which will simplify the expression of the $\mathcal{T}$-matrix, namely integrating out the time dependence in the second term of Eqs.(C.10) yields to a delta function $\delta(\omega+\epsilon_p+\omega_k)$, therefore the only contributions to the sum over ϵ_p are, if $\omega+\epsilon_p+\omega_k=0$, so ϵ_p runs only over purely negative terms, since ω and $\omega_k>0$. Although, the expression $1-n_p$ is strictly zero and the whole term does not make a contribution to the $\mathcal{T}$-matrix. However, by the very same reasoning the third term does make a contribution, but the delta function put some restriction to the summation over ω_k.

Inserting Eqs.(C.31) into the simplified Eqs.(C.10), the $\mathcal{T}$-matrix boils down to

$$\begin{aligned} Im(\hat{T}_\sigma(\omega)) &= \frac{3}{8}\sum_{\omega_k=0}^{\omega} P^2_{\omega_k,\omega_k-\omega} \\ &= \frac{3\varepsilon^3 D^\varepsilon}{64\rho_0^2}\sum_{\omega_k=0}^{\omega}\frac{1}{\omega_k^{2+2\varepsilon}}\Big(e^{-(\frac{\omega_k}{D})^2} - e^{-(\frac{\omega_k}{(\omega_k-\omega)})^2} \\ &\qquad + \frac{\omega_k^\varepsilon}{(\omega_k-\omega)^\varepsilon}\Gamma(1-\frac{\varepsilon}{2},\left(\frac{\omega_k}{(\omega_k-\omega)}\right)^2)\Big)^2. \end{aligned} \tag{C.32}$$

Remember that we can change the sum over ω_k into an integral by inserting the density of states. After taking the square we basically have to calculate three different types of integrals, beforehand the behaviour of the integrals at their boundaries have to be investigated in order to ensure the convergence of the integrals:

1. Term

$$\int_0^\omega d\tilde{\omega}^{1+\varepsilon}\left(1-e^{-\left(\frac{\tilde{\omega}}{\tilde{\omega}-\omega}\right)^2}\right)^2 \tag{C.33}$$

has no contribution if $\tilde{\omega}\to 0$, the main contribution comes from the upper part of the boundary so we effectively have to calculate

$$\int_0^\omega d\tilde{\omega}\frac{1}{\tilde{\omega}^{1+\varepsilon}} \approx \frac{1}{\varepsilon}\omega^{-\varepsilon}. \tag{C.34}$$

2. Term

$$\int_0^\omega d\tilde{\omega}\frac{1}{\tilde{\omega}^{1-\varepsilon}}\frac{1}{(\tilde{\omega}-\omega)^{2\varepsilon}}(\Gamma(1-\frac{\varepsilon}{2},\left(\frac{\tilde{\omega}}{(\tilde{\omega}-\omega)}\right)^2))^2 \tag{C.35}$$

goes to zero at the upper bound since the extended Gamma function vanishes sufficiently fast and is well behaved at the lower bound, which, indeed accounts for almost the whole integral, so the above expression reduces to

$$\int_0^\omega d\tilde{\omega}\frac{1}{\tilde{\omega}^{1-\varepsilon}}\frac{1}{(\tilde{\omega}-\omega)^{2\varepsilon}} \approx (\omega)^{-2\varepsilon}\int_0^\omega d\tilde{\omega}\frac{1}{\tilde{\omega}^{1-\varepsilon}} \approx \frac{1}{\varepsilon}\omega^{-\varepsilon} \tag{C.36}$$

3. Term

$$2\int_0^\omega d\tilde{\omega}\frac{1}{\tilde{\omega}^{1+\varepsilon}}\frac{1}{(\tilde{\omega}-\omega)^{\varepsilon}}\left(1-e^{-\left(\frac{\tilde{\omega}}{\tilde{\omega}-\omega}\right)^2}\right)\Gamma(1-\frac{\varepsilon}{2},\left(\frac{\tilde{\omega}}{(\tilde{\omega}-\omega)}\right)^2) \tag{C.37}$$

has no significant contribution from the upper as well as the lower bound, because at both boundaries the function tends to zero, which means in this case that we can neglect the value of this integral in comparison to the others.

Finally, we are in the position to calculate Eqs.(C.10) at least its approximate ω-dependence

$$\boxed{Im(\hat{T}_\sigma(\omega)) \approx \varepsilon^2\omega^{-\varepsilon}}. \tag{C.38}$$

Although, at the present stage this expression is in discussion and we do not claim the absolute correctness of it, since certain arguments still have to be checked and possible false conclusions to be sorted out. So we would like to request the reader, not to take this result for granted.

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