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Übung zur Vorlesung
Mathematische Statistische Physik
Sommersemester 2008
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**Problem 27**
Calculate the upper critical dimension $d_u$ at which the gaussian fixed point becomes unstable to the appropriate perturbation, for:

(a) Tricritical point with

$$F[\varphi] = \int \frac{1}{2}(\nabla\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 + \frac{w}{6}\varphi^6 \,,$$

which occurs in Landau Theory when $r$ and $u$ both vanish.

(b) Dipolar Ising ferromagnet with

$$F[\varphi] = \int \frac{1}{2}r\varphi^2 + \frac{u}{4}\varphi^4 + \int_q \frac{1}{2}\left(Kq^2 + \mu\frac{q_\parallel^2}{q^2}\right)|\varphi(\vec{q})|^2 \,,$$

where $q_\parallel$ is the component of the wavevector parallel to the Ising axis.
(Hint: You will need to rescale lengths in the $\parallel$ and $\perp$ directions differently.)

How do the characteristic lengths $\xi_\parallel$ and $\xi_\perp$ diverge as $T \searrow T_c$ for $d > d_u$?

**Problem 28**
Ideal smectic-A crystals have liquid-like correlations in two dimensions and a solid-like periodic modulation of the density along the third direction. They can therefore be thought of as stacks of parallel planes along the $z$-axis separated by a distance $d$, such that the molecular density can be written in a Fourier series as

$$\rho(x) = \rho_0 + \sum_n [\langle\psi_n\rangle e^{in\vec{q_0}\cdot\vec{x}} + c.c.] \,,$$

with $\vec{q_0}$ along the $z$-axis, $\vec{q_0} = (2\pi/d)\hat{e}_z$. If the planes are not perfectly aligned along the $z$-axis, this can be taken into account by a function $u(\vec{x})$ such that

$$\rho(x) = \rho_0 + \sum_n [\langle\psi_n\rangle e^{in(\vec{q_0}\cdot\vec{x} - q_0 u(\vec{x}))} + c.c.] \,.$$

The effective free energy functional is supposed to be minimal if the planes are parallel and separated by the distance $d$. Any distortion will either increase the functional or leave it unaffected.

(a) Find the expression for $\nabla_\perp u$ $(= (\nabla_x u, \nabla_y u, 0))$, when $u(\vec{x})$ describes an infinitesimal rotation $\vec{q_0} \to \vec{q_0}' = \vec{q_0} + \delta\hat{\Omega} \times \vec{q_0}$.

(b) Use the fact that uniform rotations cannot not affect the effective free energy functional $F$ to show that there must be no $(\nabla_\perp u)^2$ term in $F$. Hence, argue that to lowest order it should have the form

$$F = \frac{1}{2}\int d^3x \left[B(\nabla_\| u)^2 + K_1(\nabla_\perp^2 u)^2\right] .$$

**Problem 29**

Consider a $d$-dimensional $X - Y$ ferromagnet with a small anisotropic term in the Hamiltonian $-g\cos(p\theta)$ with $p$ an even integer.

(a) What additional term (the lowest order one) does this imply in an effective free energy functional of the order parameter $\vec{\varphi}$ for $p = 2, 4$ and $6$? For $d > 4$, what are the eigenvalues $\lambda_{g,p}$ of these perturbations about the isotropic $X - Y$ critical fixed point?

(b) For $p = 2$ and $d = 3$, draw a schematic renormalization group flow diagram as a function of $g$ and the temperature $T$, showing all important fixed points, including one critical $X - Y$ fixed point and two critical Ising fixed points.

By integrating the recursion relations for $g$ and $\delta_0 \equiv \frac{T-T_{c0}}{T_{c0}}$ (where $T_{c0}$ is the transition temperature for $g = 0$) until $g(l) \sim 1$, derive a scaling form for the free energy density as a function of $\delta_0$ and $g$, $f_s \sim |\delta_0|^{2-\alpha_0}\Phi\left(\frac{\delta_0}{|g|^{1/\phi}}\right)$. Find the exponents $\alpha_0$ and $\phi$ in terms of the eigenvalues of the appropriate fixed point(s) in your RG flow diagram. Show that $\Phi(Y)$ will be singular for some value $Y_0$ of its argument. What does this imply about the shift in the transition temperature $T_c(g)$ away fro $T_{c0}$ for small $g$?

(c) For $g$ very small, sketch a plot of the logarithm of the singular part of the specific heat as a function of $\ln\delta$ ($\delta \equiv \frac{T-T_c(g)}{T_c(g)}$), labelling the regions and identifying the exponents in terms of the eigenvalues of the appropriate fixed points in part (b). What is the asymptotic critical behaviour for $g \neq 0$?
(Hint: Consider the singularity of the scaling function in (b) near $Y_0$ and analyse the form of $\Phi(Y)$ in various limits.)