SoSe 2007 6/15/07

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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 9

Tutorial 9.1 *Method of image charges*

A point charge Q in vacuum is located a distance d away from a plane interface separating a semi-infinite dielectric medium from the vacuum. For definiteness choose coordinates such that $z > 0$ corresponds to the vacuum and $z < 0$ to the dielectric medium.

a) Give reasons for the existence of a scalar electrostatic potential φ with $\vec{E} = -\vec{\nabla}\varphi$. Derive the corresponding Laplace equations on both sides of the interface and formulate the appropriate matching conditions.

b) Introduce appropriate image charges to solve the electrostatic problem. Use symmetry arguments to choose a promising Ansatz. Determine the electrostatic potential φ, the corresponding electric field $\vec{E}$ and the displacement field $\vec{D}$.

c) Show that the case of a metallic half space is included in the problem in a particular limit. Discuss the leading behavior of the fields far away from the point charge and distinguish the case of a dielectric and a metal.

d) Determine the induced surface charge σ^{ind} and calculate the interaction energy of the external point charge Q and the surface charge distribution. What force does the dielectric medium exert on the external charge?

Tutorial 9.2 *Analytic functions*

In two dimensions, the theory of analytic functions provides powerful methods to solve the Laplace equation.

a) Show that the real $\varphi(x,y)$ and imaginary part $\psi(x,y)$ of a holomorphic function,

$$f : \begin{cases} \mathbb{C} & \to \quad \mathbb{C} \\ z = x + \mathrm{i}y & \mapsto \quad f(z) = \varphi(x,y) + \mathrm{i}\psi(x,y)\,, \end{cases}$$

are harmonic in the euclidian plane $\mathbb{E}^2$. The functions φ and ψ are referred to as conjugate potentials.

b) Demonstrate that the field lines corresponding to the vector field $\vec{E} = -\vec{\nabla}\varphi$ coincide with lines of equal potential of $\psi(x,y) = \textit{const}$ and vice versa.

Problem 9.3 *Analytic functions – continued*

c) Identifying the complex plane $\mathbb{C}$ with the two-dimensional real plane $\mathbb{R}^2$, an analytic function $f : U \subset \mathbb{C} \to \mathbb{C}$, $z = x + \mathrm{i}y \mapsto f(z) = \varphi(x,y) + \mathrm{i}\psi(x,y)$ induces an angle-preserving map (conformal map) $(x,y) \mapsto (\varphi(x,y), \psi(x,y))$ of the real plane, i.e., the *Jacobian*

$$\mathcal{J} = \begin{pmatrix} \partial\varphi/\partial x & \partial\varphi/\partial y \\ \partial\psi/\partial x & \partial\psi/\partial y \end{pmatrix} \quad \text{fulfills} \quad \mathcal{J}^T \cdot \mathcal{J} = \Omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with some real function $\Omega = \Omega(x,y)$. Show that for an analytic function the Cauchy-Riemann equations guarantee this property.

d) Discuss and sketch the field lines and lines of equal potential of $\varphi(x,y) = \mathrm{Re}\, f(x + \mathrm{i}y)$ for

$$\text{(i)} \qquad f(z) = \ln z,$$

$$\text{(ii)} \qquad f(z) = \ln \frac{z-d}{z+d}, \qquad d \in \mathbb{R}.$$

To what physical problem of electrostatics do these potentials correspond to? Determine also the charges. Use elementary geometry to prove that in the second case the lines of equipotential constitute Apollonian circles.

Hint: The divergence theorem for a two-dimensional vector field $\vec{A}(\vec{x})$ and an area F reads

$$\int_F \vec{\nabla} \cdot \vec{A}(\vec{x})\, \mathrm{d}F = \int_{\partial F} \vec{A}(\vec{x}) \cdot \vec{n}\, \mathrm{d}\ell.$$

Problem 9.4 *Spherical harmonics*

A function $u : \mathbb{E}^3 \to \mathbb{C}$ operating on the three-dimensional euclidian space $\mathbb{E}^3$ is called homogeneous of degree $\ell \geq 0$ if $u(\lambda\vec{r}) = \lambda^\ell u(\vec{r})$ for scalars $\lambda \in \mathbb{R}$. The set of such homogeneous functions is denoted by H_ℓ and constitutes a vector space. The elements $u_\ell(\vec{r}) \in H_\ell$ are polynomials of the form

$$u_\ell(\vec{r}) = \sum_{\substack{ijk \\ (i+j+k)=\ell}} c_{ijk} x^i y^j z^k$$

with arbitrary coefficients $c_{ijk} \in \mathbb{C}$. The sum is restricted such that the total degree equals $i+j+k = \ell$.

a) Determine the dimension of the complex vector spaces H_ℓ using combinatorial arguments.

b) The Laplace operator restricted to H_ℓ acts as a linear mapping $\nabla^2 : H_\ell \to H_{\ell-2}$ for $\ell \geq 2$. The corresponding null space of the Laplace operator, $\mathsf{ker}(\nabla^2 \big| H_\ell) = \{u \in H_\ell : \nabla^2 u = 0\}$, consists of the homogeneous polynomials that are *harmonic*, i.e., they satisfy Laplace's equation. Determine the dimension of the null space, $\dim \mathsf{ker}(\nabla^2 \big| H_\ell)$, assuming the Laplace operator is onto (surjective), $\nabla^2(H_\ell) = H_{\ell-2}$.

c) Find an explicit basis for $\mathsf{ker}(\nabla^2)|H_\ell$ for the case of $\ell = 2$ and $\ell = 3$, i.e., of all quadratic and cubic harmonic polynomials.

Hint: Basically, you have to solve the equation $\nabla^2 u_\ell(\vec{r}) = 0$ for $u_\ell \in H_\ell$ yielding conditions on the coefficients c_{ijk}.

d) For a harmonic polynomial $u_\ell \in H_\ell$, the function $\mathrm{Y}_\ell(\vartheta, \phi) = u_\ell(\vec{x}/r)$ is referred to as a *spherical harmonic*. Here the cartesian coordinates are to be eliminated in favor of spherical coordinates $(x,y,z) = r(\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$. In particular, $r^\ell \mathrm{Y}_\ell(\vartheta,\phi)$ is a solution of the Laplace equation. Using the representation of the Laplace operator in spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\phi^2},$$

conclude that a spherical harmonic is an eigenfunction of the *angular momentum operator*,

$$\vec{\mathsf{L}}^2 \mathrm{Y}_\ell(\vartheta,\phi) := -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial \mathrm{Y}_\ell}{\partial\vartheta}\right) - \frac{1}{\sin^2\vartheta}\frac{\partial^2 \mathrm{Y}_\ell}{\partial\phi^2},$$

and determine the eigenvalue.

Problem 9.5 *AFM tip enhanced near-field optics*

The resolution of conventional light microscopy is limited due to diffraction by the wavelength λ of the light source and the numerical aperture. To image smaller structures a number of near-field optical methods has been introduced. In 'apertureless SNOM' (scanning near-field optical microscopy) a nanoscale tip enhances incident light in the immediate neighborhood of the tip apex.

As a minimal model we assume that the tip acts as an antenna that picks up the incident light. The reaction of the tip is considered as an oscillating dipole in front of an infinite half space of complex dielectric response $\varepsilon(\omega) = \varepsilon'(\omega) + \mathrm{i}\varepsilon''(\omega)$. Since the distance of the tip from the surface d is much smaller than the wavelength, $d \ll \lambda$ (near zone), retardation effects can be ignored, and a quasistatic approximation is appropriate.

Then, the problem reduces to an electostatic dipole $\vec{p}_\omega$ located a distance d from a dielectric material. The electric field is determined by

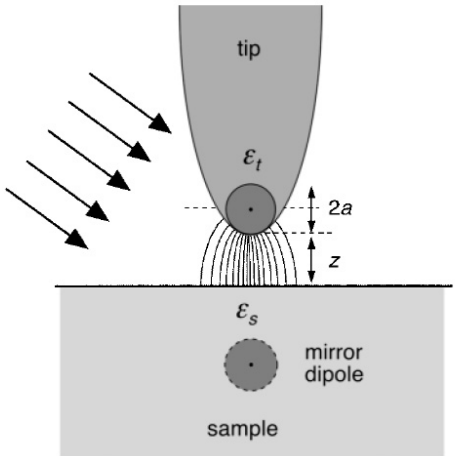
$$\vec{\nabla}\times\vec{E}_\omega(\vec{r}) = 0\,, \qquad \vec{\nabla}\cdot\vec{D}_\omega(\vec{r}) = 0\,,$$

together with the constitutive equations for matter, $D_\omega(\vec{r}) = \varepsilon(\omega)E_\omega(\vec{r})$ for $z < 0$, and vacuum $D_\omega(\vec{r}) = E_\omega(\vec{r})$ for $z > 0$.

a) Formulate appropriate conditions of continuity of the fields across the interface.

b) Determine the (quasi-)static electric potential $\varphi_\omega(\vec{r})$ by introducing appropriate image dipoles, and calculate the electric field $\vec{E}_\omega(\vec{r}) = -\vec{\nabla}\varphi_\omega(\vec{r})$.

c) Calculate the induced surface charge $\sigma_\omega(\vec{r})$ and distinguish the case where the dipole is perpendicular or parallel to the plane.



F. Keilmann, J. Electron Microsc. (Tokyo) **53**, 187 (2004).

d) For distances $d \ll r \ll \lambda$, the electric field in vacuum appears as a single dipole field originating from both the real and the induced dipole, $\vec{p}^{\,\rm eff}_\omega = \vec{p}_\omega + \vec{p}^{\,\rm ind}_\omega$. Since both dipoles are induced by the incident electromagnetic wave, the effective dipole moment is related to the field amplitude by an effective polarizability $\alpha_{\rm eff}$. The dipole at the tip reads $\vec{p}_\omega = \alpha\vec{E}^{\rm tot}_\omega$, where α is the polarizability of the tip, and $\vec{E}^{\rm tot}_\omega$ denotes the electric field of the incident wave $\vec{E}_\omega$ as well as the field of the mirror dipole. Show that if $\vec{E}_\omega$ is perpendicular to the plane, the effective dipole moment is given by

$$\vec{p}^{\,\rm eff}_\omega = \alpha^{\rm eff}\vec{E}_\omega\,, \qquad \alpha^{\rm eff} = \frac{\alpha(1+\beta)}{1 - a\beta/4d^3}$$

with the dielectric response function of the sample $\beta = [\varepsilon(\omega) - 1]/[\varepsilon(\omega) + 1]$.

Due date: Tuesday, 6/26/2007, at 9 p.m.