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Prof. Dr. E. Frey
Dr. T. Franosch

Lehrstuhl für Statistische Physik
Biologische Physik & Weiche Materie
Arnold-Sommerfeld-Zentrum für Theoretische Physik
Department für Physik

Ludwig Maximilians Universität

T II: Elektrodynamik

(Prof. E. Frey)

Problem set 5

Tutorial 5.1 *Wave equation in one dimension*

Consider the scalar field $u(x,t)$ that fulfills the one-dimensional wave equation in infinite space

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] u(x,t) = 0\,, \qquad -\infty < x < \infty.$$

The solution is completely specified by imposing initial conditions for the field, $u(x, t = 0) = F(x)$, as well as its time derivative, $\partial_t u(x, t = 0) = G(x)$.

a) Show, e.g. by a Fourier transform, that the most general solution of the wave equation was given by d'Alembert,

$$u(x,t) = u_+(x - ct) + u_-(x + ct)$$

with arbitrary functions $u_+(\cdot)$ and $u_-(\cdot)$. Then determine the solution $u(x,t)$ that fulfills the initial conditions.

b) Can you invent a generalization of the methods of images to solve the one-dimensional wave equation in a half space $x > 0$, with boundary condition $u(x = 0, t) = 0$?

Tutorial 5.2 *Drude-Hall model*

As an extension of Drude's theory of conductors consider the induced current density $\vec{j}^{(\mathrm{ind})}(\vec{r},t)$ in the presence of a constant and uniform external magnetic field $\vec{B} = B\hat{e}_z$. Motivate the constitutive equation

$$\partial_t \vec{j}^{(\mathrm{ind})}(\vec{r},t) + \frac{1}{\tau}\vec{j}^{(\mathrm{ind})}(\vec{r},t) - \frac{e}{m^* c}\vec{B}\times\vec{j}^{(\mathrm{ind})}(\vec{r},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{r},t)\,, \qquad \omega_p^2 = \frac{4\pi n e^2}{m^*}\,,$$

where m^* denotes the effective mass of the conduction electrons (charge $-e$), τ a characteristic relaxation time, and n the density of conduction electrons. The characteristic frequency ω_p is referred to as plasma frequency.

a) Perform a temporal Fourier transform, convention $\vec{E}(\vec{r},\omega) = \int e^{i\omega t}\vec{E}(\vec{r},t)\mathrm{d}t$. Show that the response becomes local in the frequency domain,

$$j_k^{(\mathrm{ind})}(\vec{r},\omega) = \sigma_{kl}(\omega)\cdot E_l(\vec{r},\omega),$$

and determine the dynamic magneto-conductivity tensor $\sigma_{kl}(\omega)$.

b) Specialize to d.c. fields, i.e. $\omega = 0$, and discuss the Hall resistivity.

Problem 5.3 *Spherical waves*

Consider solutions of the scalar three-dimensional wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t) = 0\,,$$

where the field is spherically symmetric, i.e. $\psi(\vec{r},t) = \psi(r,t)$ with $r = |\vec{r}|$.

a) Show that the substitution $\psi(r,t) = u(r,t)/r$ leads to a *one-dimensional* wave equation for $u(r,t)$. Impose appropriate boundary conditions on u such that the scalar field ψ remains finite at the origin.

b) Using d'Alembert's solution for the one-dimensional case, determine the spherical symmetric solution of the wave equation that fulfills the initial conditions

$$\psi(r,t=0) = F(r) \quad \text{and} \quad \frac{\partial}{\partial t}\psi(r,t=0) = G(r) \quad \text{for } r > 0.$$

Problem 5.4 *Nuclear magnetic resonance*

Nuclear magnetic resonance spectroscopy uses the magnetic moment of the nuclei of certain atoms to study physical, chemical, and biological properties of matter. The magnetization $\vec{M}$ due to the spin of the nuclei obeys the *Bloch* equations

$$\dot{\vec{M}}(t) = \gamma\vec{M}(t) \times \vec{H}(t) - \frac{1}{T_1}\left[\vec{M}(t) - \vec{M}_0\right].$$

Here the gyromagnetic ratio γ determines the frequency of the *Larmor precession.* The second term is a phenomenological damping term introducing a characteristic (energy) relaxation time T_1. Consider a strong d.c. field $\vec{H}_0$ aligning the magnetization $\vec{M}(t) = \vec{M}_0 \parallel \vec{H}_0$ in the static case. A small time-dependent field $\delta\vec{H}_\perp(t)$ is applied in addition to the d.c. field $\vec{H}_0$. The probing field $\delta\vec{H}_\perp(t)$ acts perpendicularly to $\vec{H}_0$ at all times. Since for positive gyromagnetic ratio, $\gamma > 0$, the Larmor precession is clockwise, the probing field shall rotate clockwise too.

a) Derive a constitutive equation for the induced magnetization $\delta\vec{M}(t) = \vec{M}(t) - \vec{M}_0$ to linear order in $\delta\vec{H}_\perp(t)$. Decompose the response into a component parallel and perpendicular to the static external field, $\delta\vec{M}(t) = \delta\vec{M}_\parallel(t) + \delta\vec{M}_\perp(t)$, and show that they fulfill

$$\delta\dot{\vec{M}}_\parallel(t) + \frac{1}{T_1}\delta\vec{M}_\parallel(t) = 0\,, \qquad \delta\dot{\vec{M}}_\perp(t) - \gamma\delta\vec{M}_\perp(t) \times \vec{H}_0 + \frac{1}{T_1}\delta\vec{M}_\perp(t) = \gamma\vec{M}_0 \times \delta\vec{H}_\perp(t)\,.$$

b) Discuss the free decay of the induced magnetization $\delta\vec{M}(t)$ in the absence of external driving, i.e., $\delta\vec{H}_\perp(t) \equiv 0$, for arbitrary initial condition $\delta\vec{M}(t=0)$.

Hint: It is favorable to complexify the transverse magnetization $\delta\vec{M}_\perp(t)$ as $\delta\mathcal{M}(t) = \delta M_x(t) + \mathrm{i}\delta M_y(t)$.

c) Derive the steady state response for a probing field rotating perpendicularly to the aligning field $\vec{H}_0$ at constant angular frequency, $\delta\vec{H}_\perp(t) = \delta H^\omega(\cos\omega t, -\sin\omega t, 0)$; here the z-axis has been chosen parallel to $\vec{H}_0$. Determine the complex susceptibility $\chi(\omega)$, sketch and discuss its real and imaginary parts, $\chi(\omega) = \chi'(\omega) + \mathrm{i}\chi''(\omega)$.

Hints: It is favorable to complexify by introducing $\delta\mathcal{H}(t) = \delta H_x(t) + \mathrm{i}\delta H_y(t)$ and similarly for the magnetization. The susceptibility is defined as $\chi(\omega) = \delta\mathcal{M}(t)/\delta\mathcal{H}(t)$, and the result is $\chi(\omega) = \mathrm{i}\gamma M_0/(-\mathrm{i}\omega + \mathrm{i}\omega_L + 1/T_1)$.

d) Determine the averaged power absorbed by the sample,

$$\mathcal{P}(\omega) = \overline{\delta\vec{H}_\perp(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{M}_\perp(t)},$$

where the average indicates a time average over many cycles.

e*) Assume that the strong aligning field is not totally uniform in space. This corresponds to small random local changes of the Larmor frequencies $\omega_L \to \omega_L + \Delta\omega_L$. For simplicity, assume for the probability distribution $p(\Delta\omega_L)$ of the local increments a Cauchy-Lorentz distribution,

$$p(\Delta\omega_L) = \frac{1}{\pi} \frac{T}{1 + (T\Delta\omega_L)^2}\,.$$

Verify that the probability distribution is normalized, $\int p(\Delta\omega_L)\,\mathrm{d}(\Delta\omega_L) = 1$. Determine the averaged complex susceptibility

$$\big\langle \chi(\omega) \big\rangle := \int \chi(\omega)\, p(\Delta\omega_L)\, \mathrm{d}(\Delta\omega_L)\,.$$

Can you interpret the additional damping?

Hint: Use the residue theorem.

Due date: Tuesday, 5/29/07, at 9 a.m.