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Prof. Dr. E. Frey
Dr. T. Franosch

Lehrstuhl für Statistische Physik
Biologische Physik & Weiche Materie
Arnold-Sommerfeld-Zentrum für Theoretische Physik
Department für Physik

Ludwig Maximilians Universität

T II: Elektrodynamik
(Prof. E. Frey)

Problem set 3

Tutorial 3.1 *Infinitely long wire*

Consider an infinitely long straight wire of circular cross section πR^2 carrying a constant current I. The current density $\vec{j}(\vec{x})$ is distributed uniformly in the wire.

a) Determine the magnetic field $\vec{B}(\vec{x})$ inside and outside of the wire. Discuss the field lines.

b) Construct an appropriate vector potential $\vec{A}(\vec{x})$ and discuss the corresponding field lines.

Hint: It is favorable to use cylindrical coordinates and there the curl operation reads

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\varphi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial \varphi}\right)\hat{e}_z \,.$$

Tutorial 3.2 *Dielectric cylinder*

Consider an infinitely long cylinder of circular cross section of radius R, filled with a dielectric medium of dielectric constant ε. The cylinder is placed in a homogeneous electric field $\vec{E}_{\text{ext}}$ perpendicular to the axis of the cylinder; this field induces a polarization of the cylinder. Calculate and discuss the resulting electrostatic potential $\varphi(\vec{x})$ and field $\vec{E}(\vec{x})$.

Hint: Show that the (effectively two-dimensional) potential

$$\varphi(\vec{x}) = \begin{cases} -\vec{E}_{\text{ext}} \cdot \vec{x} + \vec{p} \cdot \vec{x}/r^2 & \text{outside the cylinder } (r = \sqrt{x^2+y^2} > R), \\ -\vec{E}_{\text{in}} \cdot \vec{x} + const & \text{inside,} \end{cases}$$

fulfills the Laplace equation. Use the boundary conditions at the interface for the normal part of $\vec{D}(\vec{x})$ and the tangential part of $\vec{E}(\vec{x})$ to determine the remaining constants. Symmetry considerations are useful to argue on the orientation of the vectors $\vec{E}_{\text{in}}$ and $\vec{p}$. In the course of the lecture, you will learn why the present form of the potential is the only solution to this problem.

Problem 3.3 *Charged rod*

An infinitely thin, straight rod of length L carries a charge Q homogeneously distributed along the rod.

a) Integrate the Coulomb solution,

$$\varphi(\vec{x}) = \int \frac{\mathrm{d}q(\vec{y})}{|\vec{x} - \vec{y}|} \,,$$

and determine the electrostatic potential $\varphi(\vec{x})$. *Answer:*

$$\varphi(x,y,z) = \lambda \ln \frac{\sqrt{(z-L/2)^2+r^2}-(z-L/2)}{\sqrt{(z+L/2)^2+r^2}-(z+L/2)}\,, \qquad r^2 = x^2+y^2\,.$$

Show that the equipotential surfaces are ellipses of revolution.

b) Discuss the leading behavior of the potential far away from the rod. What determines the leading correction?

c) Expand the potential close to the center of the rod and compare with an infinitely long rod carrying a charge per unit length λ. Determine the electric field for this case by applying Gauß's law in its integral form.

Problem 3.4 *Polarization and magnetization*

a) For a static polarization field $\vec{P}(\vec{r})$ that vanishes sufficiently rapidly at infinity, an electrostatic potential $\varphi(\vec{r})$ is given by

$$\varphi(\vec{r}) = -\vec{\nabla}_r \cdot \int \frac{\vec{P}(\vec{R})}{|\vec{r}-\vec{R}|}\,\mathrm{d}^3\vec{R}\,.$$

Argue that this expression indeed represents a solution of Poisson's equation,

$$-\nabla^2\varphi = 4\pi\rho^{(\mathrm{ind})} = -4\pi\,\mathrm{div}\,\vec{P}.$$

b) Use the preceding result to calculate the electrostatic potential $\varphi(\vec{r})$ corresponding to a sphere of homogeneous polarization, $\vec{P} = \mathit{const}$. Determine also the electric field and sketch the field lines.

c) Similarly, for a static magnetization field $\vec{M}(\vec{r})$ a solution of the magnetostatic problem is provided in terms of the vector potential

$$\vec{A}(\vec{r}) = \vec{\nabla}_r \times \int \frac{\vec{M}(\vec{R})}{|\vec{r}-\vec{R}|}\,\mathrm{d}^3\vec{R}\,.$$

Corroborate again that the preceding formula constitutes a solution of

$$-\nabla^2\vec{A} = 4\pi\vec{j}^{(\mathrm{ind})}/c = 4\pi\,\mathrm{curl}\,\vec{M}.$$

d) Determine a vector potential for a homogeneously magnetized sphere, $\vec{M} = \mathit{const}$, and calculate the magnetic field.

e) Argue that the magnetic fields arising due to a static magnetization field $\vec{M}$ can be expressed in terms of a scalar magnetostatic potential $\varphi_M(\vec{r})$ by $\vec{H} = -\vec{\nabla}\varphi_M$. Determine the field equation for φ_M that contains $\vec{M}$ as source terms. Compare the polarized with the magnetized sphere.

Problem 3.5 *Angular momentum conservation law*

The angular momentum density of the electromagnetic field is defined by the antisymmetric tensor field

$$L_{ij}(\vec{x},t) = \frac{1}{c^2}(x_iS_j - x_jS_i)\,,$$

where $\vec{S}$ denotes the Poynting vector.

a) Employ the momentum balance law to construct a local balance law for the angular momentum density of the form

$$\partial_t L_{ij} + \nabla_k M_{ijk} = -D_{ij}\,.$$

Determine the angular moment current tensor M_{ijk} as well as the mechanical torque tensor D_{ij}. Rewrite the balance law in terms of the pseudo-vector field

$$L_i(\vec{x}, t) = \frac{1}{2}\varepsilon_{ijk} L_{jk} \,,$$

and suitable M_{ik} and D_i.

b) Formulate the angular momentum conservation law in integral form, for $\mathcal{L}_i = \int_V L_i \, \mathrm{d}V$.

c) Demonstrate that in the radiation gauge, i.e., $\varphi_s = 0$, the angular momentum of the field can be decomposed, $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_B$, in a 'spin' part

$$\mathcal{L}_S = \frac{1}{4\pi c^2} \int_V \vec{A} \times \dot{\vec{A}} \, \mathrm{d}V \,,$$

and an 'orbital' part $\mathcal{L}_B$ that depends explicitly on the point of reference of the coordinate system.

Due date: 5/15/07