The Shape of Spacetime

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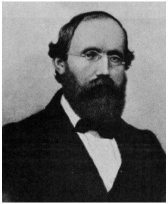
In this talk, I shall discuss the geometry of spacetime from the point of view of geometry.

I. Riemannian Geometry

The concept of geometry had gone through a radical change in the 19th century, thanks to the contributions of Gauss and Riemann. Riemann revolutionized our notions of space, freeing up mathematics in the process.

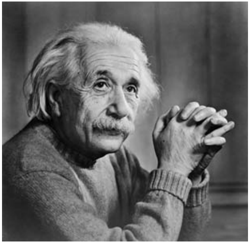
Gauss



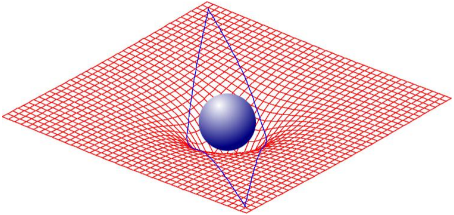
Riemann

Objects no longer had to be confined to the flat, linear space of Euclidean geometry. Riemann instead proposed a much more abstract conception of space — of any possible dimension — in which we could describe distance and curvature. In fact, one can develop a form of calculus that is especially suited to such an abstract space.

About 50 years later, Einstein realized that this kind of geometry, which involved curved spaces, was exactly what he needed to unify Newtonian gravity with special relativity. This insight culminated in his famous theory of general relativity.



Einstein



Curved Space-time

Geometry, topology and partial differential equations are the three major branches of mathematics that provide the major tools to understand space time. Here geometry includes subjects such as algebraic geometry and group representation theory. Along with intuitions based on physical principles, they gave beautiful descriptions of the universe.

A very important tool for understanding the interactions between topology and geometry comes from analysis, mostly based on partial differential equations. The power of linear elliptic equation can be seen from the application of Atiyah-Singer index formula. The large scale development of nonlinear theory started around the seventies, and the subject is called geometric analysis.

In the past 40 years, my research has been focus on understanding the relation between topology and curvature. And as such, I found out that my works have been somewhat interesting to my friends in the physics department.

II. General Relativity

We learned through special relativity that space and time should not be treated separately but should instead be merged together to form spacetime. Einstein struggled in his attempt to obtain a fundamental description of gravity. But he got some help from his friend Marcel Grossman, a mathematician, who told him of the work of other mathematicians, Riemann and Ricci.

Riemann provided the framework of abstract space, as well as the means for defining distance and curvature in such a space. Riemann thus supplied the background space or setting in which gravity, as Einstein formulated it, plays out.

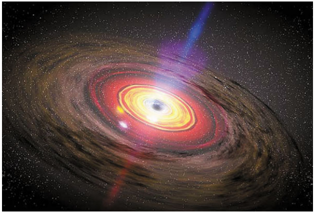
But Einstein also drew on the work of Ricci, who defined a special kind of curvature that could be used to describe the distribution of matter in spacetime. In fact, the Ricci curvature can be viewed as the trace of the curvature tensor. A remarkable feature of this curvature is that it satisfied the conservation law due to the identity of Bianchi. And it was exactly this conservation law that enabled Einstein to provide a geometric picture of gravity. Rather than considering gravity as an attractive force between massive objects, it could instead be thought of as the consequence of the curvature of spacetime due to the presence of massive objects.

Einstein commented on his work: "Since the gravitational field is determined by the configuration of masses and changes with it, the geometric structure of this space is also dependent on physical factors," he wrote. "Thus, according to this theory, space is–exactly as Riemann guessed–no longer absolute; its structure depends on physical influences. [Physical] geometry is no longer an isolated, self-contained science like the geometry of Euclid."

Reflecting on his accomplishment, Einstein wrote, "In the light of the knowledge attained, the happy achievement seems almost a matter of course, and any intelligent student can grasp it without too much trouble. But the years of anxious searching in the dark, with their intense longing, their alternations of confidence and exhaustion, and the final emergence into the light–only those who have experienced it can understand that."

Einstein's struggle to understand gravity is remarkable and his success in this area even more so. One thing that is resoundingly apparent is the critical contribution of Riemannian geometry to that effort.

When I looked at the equations of Einstein more than a half century later, I was intrigued by the fact that matter only controls part of the curvature of spacetime. I wondered whether we could construct a spacetime that is a vacuum, and thus has no matter, yet its curvature is still pronounced, meaning that its gravity would be nonzero. Well, the famous Schwarzschild solution to Einstein's equations is such an example. This solution applies to a non-spinning black hole — a vacuum that, curiously, has mass owing to its extreme gravity. But that solution admits a singular point, or singularity - a place where the laws of physics break down.



Black hole

I became interested in a different situation — a smooth space, without a singularity, that was compact and closed, unlike the open, extended space of the Schwarzschild solution. The question was: Could there be a compact space that contained no matter — a closed vacuum universe, in other words - whose force of gravity was nontrivial? I was obsessed with this question and believed that such a space could not exist. If I could prove that, I was sure that it would be an elegant theorem in geometry.

III. Calabi Conjecture

When I started thinking about this in the early 1970s, I did not realize that the geometer Eugenio Calabi had posed almost the exact same question. Calabi framed the problem in fairly complicated mathematical language — involving difficult concepts like Kähler manifolds, Ricci curvature, and Chern classes — that ostensibly had nothing to do with physics. Yet his abstract conjecture could also be framed in terms of Einstein's theory of general relativity. The additional information that he put in is that the space should admit some kind of internal symmetry called supersymmetry — a term coined by physicists. In that context, the question that I asked on Einstein's equation translated to: Can there be gravity, or the curving of space, in a closed vacuum — a compact supersymmetric space that has no matter?

For about three years, my friends and I tried to prove that the class of spaces proposed by Calabi could not exist. We, along with many others, considered them to be "too good to be true." We were skeptical not only because the conjecture argued for the existence of a closed vacuum with gravity but also because it implied that there was a systematic way of constructing many such examples. Despite the reasons we had for finding Calabi's argument dubious, try as we might, we could not prove that such spaces do not exist.

I announced a counterexample in 1973 in a big geometry conference in Stanford. Then a serious gap was found. I tried hard to come up with a new argument, working for two weeks straight with practically no sleep, pushing myself to the brink of collapse. Each time I found a possible counterexample, I soon found a subtle reason as to why it could not work. After many such abortive attempts, I concluded that the conjecture must be correct after all. Once I made up my mind, I switched gears completely, putting all my energies into proving it right. I finally did so, several years later, in 1976.

I should say that at the same Stanford conference, the physicist Robert Geroch gave a talk on an important question in general relativity called the positive mass conjecture, which holds that the total mass or energy in any closed physical system must be positive. Schoen and I eventually proved this conjecture after some painful contortions (involving minimal surfaces) and a lot of hard work. The conjecture shows that spacetime, as is described in general relativity, will not collapse because of negative mass.

The experience led me to think more about general relativity, and we proved some theorems about formations of black holes and clarified the concept of mass of a gravitational system which is not necessarily isolated.

For example, we proved, in a rigorous manner, that for a general spacetime, if matter density $\rho \geq \frac{4\pi^2}{3r^2}$ in a region of (suitably defined) radius$\leq r$, the black hole must form.

The understanding of the quasi-local mass in general relativity is actually important for the understanding of the geometry of spacetime. I found a satisfactory answer only recently with Mu-Tao Wang. It depends on the work of Schoen-Yau, Witten and Liu-Yau.

My favorable interactions with general relativists also made me more open to collaborating with physicists in the development of string theory, although that didn't come until several years later.

In my proof of the Calabi conjecture, I found a general mechanism to construct spaces satisfying Calabi's equations, which are now called Calabi-Yau spaces. I had a strong sense that I had somehow stumbled onto a beautiful piece of mathematics. And as such, I felt it must be relevant to physics and to our deepest understanding of nature. However, I did not know exactly where these ideas might fit in, as I didn't know much physics at the time.

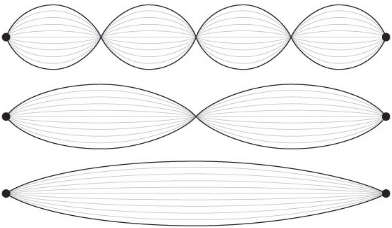
IV. String Theory

In 1984 I received phones calls from two physicists, Gary Horowitz and Andy Strominger. They were excited about a model for describing the vacuum state of the universe, based on a new theory called string theory.

Gary Horowitz

Andy Strominger

String theory is built on the assumption that particles, at their most basic level, are made of vibrating bits of strings—and exceedingly tiny strings at that. In order for the theory to be consistent with quantum mechanics (at least in some versions of string theory), spacetime requires a certain symmetry built into it called supersymmetry. Spacetime is also assumed to be ten-dimensional.



Vibrating strings

Horowitz and Strominger became interested in the multidimensional spaces whose existence I proved, mathematically, in my confirmation of the Calabi conjecture. They believed that these spaces might play an important role in string theory, as they seemed to be endowed with the right kind of supersymmetry — a property deemed essential to the theories they were working on. They asked me if their assessment of the situation was correct and, to their delight, I told them that it was.

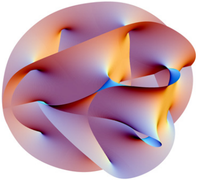
Then I got a phone call from Edward Witten whom I'd met in Princeton the year before. Witten believed that this was one of the most exciting eras in theoretical physics, just like the time when quantum mechanics was being developed.

Witten

Witten was now collaborating with Candelas, Horowitz, and Strominger, trying to figure out the shape, or geometry, of the six "extra" dimensions of string theory. The physicists proposed that these six dimensions were curled up into a miniscule space, which they called Calabi-Yau space — part of the same family of spaces originally proposed by Calabi and later proved by me.

With Candelas, 2001

String theory, again, assumes that spacetime has 10 dimensions overall. The three large spatial dimensions that we're familiar with, plus time, make up the four-dimensional spacetime of Einstein's theory. But there are also six additional dimensions hidden away in Calabi-Yau space, and this invisible space exists at every point in "real space", according to string theory, even though we can't see it.



Calabi-Yau space

The existence of this extra-dimensional space is fantastic on its own, but string theory goes much farther. It says that the exact shape, or geometry, of Calabi-Yau space dictates the properties of our universe and the kind of physics we see. The shape of Calabi-Yau space — or the "shape of inner space," as we put it in my recent book with Steve Nadis - determines the kinds of particles that exist, their masses, and the ways in which they interact.

Analyzing the spectrum of the Dirac operator of the ten-dimensional spacetime would then reveal the variety of particles that we might observe. Based on the principle of separation of variables on this ten-dimensional spacetime, which is the product of the four-dimensional spacetime with the six-dimensional Calabi-Yau space, we know that part of the spectrum is determined by the Calabi-Yau space. Particles with nonzero mass will be extremely massive if the diameter of the Calabi-Yau space is very small. We do not expect to observe any of these particles, as they would only appear at incredibly high energies.

But particles with zero mass are potentially observable and can be calculated from the topology of the Calabi-Yau space. This gives you an idea of why the topology of this tiny, six-dimensional space could play an important role in physics.

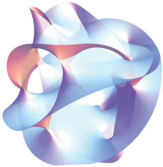
While Einstein had said the phenomenon of gravity is really a manifestation of geometry, string theorists boldly proclaimed that the physics of our universe is a consequence of the geometry of Calabi-Yau space. That's why string theorists were so anxious to figure out the precise shape of this six-dimensional space — a problem we're still working on today.

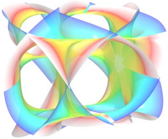
Witten was eager to learn more about Calabi-Yau spaces. He flew from Princeton to San Diego to talk with me about how to construct them. He also wanted to know how many Calabi-Yau spaces there were for physicists to choose among. Initially, physicists thought there might only be a few examples — a few basic topologies — which made the goal of determining the "internal" shape of our universe seem a lot more manageable. But we soon realized there were many more examples of Calabi-Yau spaces–many more possible topologies - than were originally anticipated. In the early 1980s, I guessed that there were tens of thousands of these spaces, and that number has grown considerably since then.

The task of figuring out the shape of inner space suddenly seemed more daunting, and perhaps even hopeless if the number of possibilities turned out to be infinite. The latter question has yet to be settled, although I have always thought that the number of Calabi-Yau's of any dimension is finite.

The excitement over Calabi-Yau spaces started in 1984, when physicists first began to see how these complex geometries might fit into their new theories. That enthusiasm kept up for a few years, before waning. But interest in Calabi-Yau spaces picked up again in the late 1980s, when Brian Greene, Ronen Plesser, Philip Candelas, and others began exploring the notion of "mirror symmetry."

The basic idea here was that two different Calabi-Yau spaces, which had different topologies and seemed to have nothing in common, nevertheless gave rise to the same physics. This established a previously unknown kinship between so-called mirror pairs of Calabi-Yau's.





The connection between mirror manifolds, which was uncovered through physics, proved to be extremely powerful in the hands of mathematicians. When they were stumped trying to solve a problem involving one Calabi-Yau space, they could try solving the same problem on its mirror pair. On many occasions, this approach was successful. As a result, mathematical problems of counting curves that had defied resolution, sometimes for as long as a century, were now being solved. And a branch of mathematics called enumerative geometry was suddenly rejuvenated. These advances gave mathematicians greater respect for physicists, as well as greater respect for string theory itself.

Mirror symmetry is an important example of what we call a duality. It sheds light on the deep geometry of Calabi-Yau space. It has also helped us solve some very difficult questions of counting rational curves of various degrees on the quintic, a Calabi-Yau manifold.

This problem, named after the German mathematician Hermann Schubert, dates back to the 19th century. Schubert showed that the number of degree one rational curves on a quintic is 2,875. In 1986, Sheldon Katz found that there are 609,250 degree two curves. Then around 1989, two Norwegian mathematicians Geir Ellingsrud and Stein Strømme found that number of degree three curves – based on algebraic geometry techniques – was 2,683,549,425.

Relying on a string theory approach, a group of physicists, led by Candelas, arrived at a different number, 317,206,375. The physicists, however, had used a formula that, up to then, had not been motivated by mathematical principles. As such, rigorous justification of that formula still awaited confirmation by mathematicians.

In January of 1990, I organized the first major meeting between string theorists and mathematicians at the urging of Isadore Singer. The event took place at the Mathematical Sciences Research Institute MSRI) in Berkeley. At this meeting there was a somewhat tense debate regarding who was right, Ellingsrud and Strømme or the Candelas team. The discrepancy between the two camps lasted a few months until the mathematicians discovered a mistake in their computer code. After they corrected that error, their number agreed perfectly with that put forth by the physicists. And ever since then, mathematicians have begun to appreciate the depth of the insight provided by the string theorists.

The episode also provided firm evidence that mirror symmetry had a mathematical basis. It took several years but by the mid-to-late 1990s a rigorous mathematics proof of mirror symmetry-and a validation of the Candelas, et al. formula — was finally achieved independently by Givental and Lian-Liu-Yau.

The discussion on mirror symmetry is on the tree level of the perturbation as only curves of genus zero are considered. We shall now discuss the higher loop contributions.

Given a Calabi-Yau space, one can associate a sigma model by studying the space of maps from surfaces into the manifold (the surfaces are the traces of the string vibration). One can associate two different kind of super conformal field theories (a scalar invariant quantum field theory that admits supersymmetries) to the sigma model by certain twisting. One is called A model and the correlation functions is related to the counting of algebraic curves in the Calabi-Yau manifold. The algebraic curves appear as instantons from the sigma model. They can be considered as quantum corrections to the classical correlation functions. In mathematics, they contribute to the Gromov-Witten invariants. This is reasonably understood.

The other one is called B model and is proposed by Bershadsky-Cecotti-Ooguri-Vafa by suitable quantization of the Kodaira-Spencer theory of gravity. The higher genus contribution of the partition function of the A model of a Calabi-Yau manifold is supposed to be the same as the partition function of the B model of the Calabi-Yau manifold which is the mirror image of the first one.

Kevin Costello and my student Si Li recently formulated a mathematical theory of perturbative quantization of Kodaira-Spencer gauge theory. They have been successful in carrying out the theory when the Calabi-Yau space is an elliptic curve. And in this case, Si Li was able to verify the theory of mirror symmetry to be valid for all genus and hence for all loops for the corresponding quantum field theory.

Mirror symmetry is a mysterious concept. However in 1996, Strominger-Yau-Zaslow was able to reinterpret mirror symmetry, based on the newly developed brane theory, in terms of T-duality. Roughly speaking, we decompose the Calabi-Yau manifold by two subclasses of three dimensional submanifolds. One set will consist of 3-torus: $S^1 \times S^1 \times S^1$. If we perform an operation similar to

$$r \rightarrow \frac{1}{r}$$

on the set of torus, we shall obtain the mirror manifolds.

The SYZ construction is compatible with a proposal of Kontsevich on mapping the derived category of one manifold into the Fukaya category of the mirror manifold.
It has generated a great deal of important activities in the mathematics community. The predictions look very good.
Gross-Siebert is giving a formulation of corresponding constructions in algebraic geometry. The subject is taking on its own life in mathematics.

V. Conclusion

In the past thirty years, some extremely intriguing, as well as powerful, mathematics has been inspired by string theory. Mathematical formulae developed through this connection have proved to be correct, and will always remain so, regardless of the scientific validity of string theory. Although it is empirically unproven, string theory now stands as the only consistent theory that unifies the different forces. And it is beautiful. Moreover, the effort to unify the different forces of nature has unexpectedly led to the unification of different areas mathematics that at one time seemed unrelated.

We still don't know what the final word will be for string theory. However, in the past two thousand years, the concept of geometry has evolved over several important stages to the current state of modern geometry. Each time geometry has been transformed in a major way, the new version has incorporated our improved understanding of nature arrived at through advances in theoretical physics. It seems likely that we shall witness another major development in the 21st century, the advent of quantum geometry — a geometry that can incorporate quantum physics in the small and general relativity in the large.

The fact that abstract mathematics can reveal so much about nature is something I find both mysterious and fascinating.

In the case of string theory, geometry and physics have come together to produce some beautiful mathematics, as well as some very intriguing physics. The mathematics is so beautiful, in fact, and it has branched out into so many different areas, that it makes you wonder whether the physicists might be onto something after all.

The story is still unfolding, to be sure. I consider myself lucky to have been part of it and hope to stay involved in this effort for as long as I can contribute.